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# Comparison between New Iterative Method and Homotopy Perturbation Method for Solving Fractional Derivative Integro-Differential Equations

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**KEY WORDS**

New iterative method; Homotopy perturbation method; Integro-differential equations of fractional derivative order; Caputo fractional derivative.

**ABSTRACT**

In this work, we implement relatively new analytical techniques, the new iterative method (NIM) and homotopy perturbation method (HPM), for solving linear and nonlinear integro-differential equations of fractional derivative order. The fractional derivatives are described in the Caputo sense. The two methods in applied mathematics can be used as alternative methods for obtaining analytical and approximate solutions for different types of fractional differential and integro-differential equations. In these schemes, the solution takes the form of a convergent series with easily computable components. Numerical results show that the two approaches are easy to implement and accurate when applied to integro-differential equations of fractional derivative order.

## 1. Introduction

In the past decades, both mathematicians and physicists have devoted considerable effort to fined robust and stable numerical and analytical methods for solving fractional differential and integro-differential equations of physical interest. Numerical and analytical methods have included finite difference method [1-3], Adomian decomposition method [4-8], variational iteration method [9-12], homotopy perturbation method (HPM) [13-

16], generalized differential transform method [17-20], homotopy analysis method [21-23] and new iterative method (NIM) [24-30, 44, 45]. Among them, the HPM and the NIM [13-16, 24-30, 44, 45] provides an effective procedure for explicit and numerical solutions of a wide and general class of differential and integro-differential systems representing real physical problems. In both methods, the validity of them is independent of whether or not there exist small parameters in the considered equation.

The motivation of this work is to extend the analysis of both the NIM pro- posed by Gejji-Jafari [24-32] and the HPM proposed by He [13-16] with a reliable algorithms to solve linear and nonlinear integro-differential equations with fractional derivative order defined as follows: nonlinear integro-differential equations with fractional derivative order defined as follows:

$$y^{(\alpha)}(x)+f(x)y(x)+ +\int_a^b w(x,t)y^{(q)}(t)y^{(m)}(t)\,dt=g(x) \quad (1a)$$

$$\frac{d^k y}{dx^k}=h_k,\ k=0,1,2,...,n-1;\ h_k\in R \quad (1b)$$

where $h_k$, $k=0, 1, 2, ..., n-1$ are real constants, $q$, $m$, $n$ are integers and $\alpha$ is fraction with $q\le m\le n-1<\alpha\le n$. In (1) the functions $f$, $g$ and $w$ are given solution. The obtained results shown that these methods with the modification algorithms are very simple and effective.

For considering some properties of the fractional order differential operator, consider the nonlinear differential equation:

$$D_x^\alpha y(x)=f(y, y', ..., y^{(n-1)}),\ x>0 \quad (2)$$

where $n-1<\alpha\le n$, $f$ is a nonlinear function and $D^\alpha$ denotes the differential operator in the sense of Caputo [33], defined by:

$$D_x^\alpha f(x)=I_x^{n-\alpha}D_x^n f(x). \quad (3)$$

Here $D_x^\alpha$ is the usual differential operator of order $\alpha$ and $I_x^\alpha$ is the Riemann-Liouvil integral operator of order $\alpha>0$, defined by:

$$I_x^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-\xi)^{\alpha-1}f(\xi)\,d\xi,\quad x>0. \quad (4)$$

Properties of the operators $I_x^\alpha$ and $D_x^\alpha$ can be found in [34, 35], we mention only the following, for $f\le C_\mu$, $\mu\ge-1$, $\beta\ge0$ and $v>-1$:

1- $I_x^\alpha I_x^\beta f(x)=I_x^{\alpha+\beta}f(x) =I_x^\beta I_x^\alpha f(x)$,

2- $I_x^\alpha x^v=\dfrac{\Gamma(v+1)}{\Gamma(v+1+\alpha)}x^{v+\alpha}$,

3- $D_x^\alpha x^v=\dfrac{\Gamma(v+1)}{\Gamma(v+1-\alpha)}x^{v-\alpha}$.

## 2. Analysis of the Methods.

In this section, we discuss the analysis and the algorithms of the two considered methods.

### 2.1. New Iterative Method.

Consider the following general functional equation [24-32]:

$$y(x)=f(x)+N(y(x)), \quad (5)$$

where $N$ is a nonlinear operator from a Banach space $B\to B$ and $f$ is a known function (element) of the Banach space $B$. We are looking for a solution $y$ of Equation (5) having the series form:

$$y(x)=\sum_{i=0}^{\infty}y_i(x). \quad (6)$$

The nonlinear operator $N$ can be decomposed as:

$$N\left(\sum_{i=0}^{\infty}y_i\right)=N(y_0)$$

$$+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_i\right)-N\left(\sum_{j=0}^{i-1} y_i\right)\right\}. \quad (7)$$

From Equations (6) and (7), Equation (5) is equivalent to:

$$\sum_{i=0}^{\infty} y_i = f + N(y_0)$$

$$+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} y_i\right)-N\left(\sum_{j=0}^{i-1} y_i\right)\right\}. \quad (8)$$

We define the recurrence relation:

$$\begin{cases} y_0 = f\,, \\ y_1 = N(y_0), \\ y_{n+1} = N(y_0 + y_1 + \ldots + y_n) \\ \qquad -N(y_0 + y_1 + \ldots y_{n-1}), \\ n = 1, 2, \ldots. \end{cases} \quad (9)$$

Then

$$(y_1 + y_2 + \ldots + y_{n+1}) = N(y_0 + y_1 + \ldots + y_n),$$
$$n = 1, 2, \ldots, \quad (10)$$

and

$$y = \sum_{i=0}^{\infty} y_i\,. \quad (11)$$

Then $n$-term approximate solution of Equations (5) and (6) is given by $y(x) = \sum_{i=0}^{n-1} y_i$.

**Remark 1.** If $N$ is a contraction, i.e., $\| N(x) - N(y) \| \le k \| x - y \|, \quad 0 < k < 1$, then:

$$\| y_{n+1} \| \le k^{n+1} \| y_0 \|, \; n = 0, 1, 2, \ldots.$$

**Proof.** From Equation (9), we have:

$$y_0 = f\,,$$
$$\| y_1 \| = \| N(y_0) \| \le k \| y_0 \|,$$
$$\| y_2 \| = \| N(y_0 + y_1)$$
$$-N(y_0) \| \le k \| y_1 \| \le k^2 \| y_0 \|,$$
$$\| y_3 \| = \| N(y_0 + y_1 + y_2)$$
$$-N(y_0 + y_1) \| \le k \| y_2 \|$$
$$\le k^3 \| y_0 \|,$$
$$\vdots$$
$$\| y_{n+1} \| = \| N(y_0 + \ldots + y_n)$$
$$-N(y_0 + \ldots + y_{n-1}) \|$$
$$\le k \| y_n \| \le k^{n+1} \| y_0 \|,$$

$n = 0, 1, \ldots,$ and the series $\sum_{i=0}^{\infty} y_i$ absolutely and uniformly converges to a solution of Equation (5) [36], which is unique in vies of the Banach fixed point theorem [37].

### 2.2. The Reliable Algorithm.

After the above presentation of the new iterative method, we present a reliable approach of this method. This new modification can be implemented for integer order and fractional order linear and nonlinear integro-differential equations. To illustrate the basic idea of the new algorithm, we consider the nonlinear integro-differential equation with fractional derivative order (1) defined as follows:

$$D_x^{\alpha} y(x) + f(x) y(x)$$
$$+\int_a^b w(x,t) y^{(q)}(t) y^{(m)}(t)\, dt = g(x) \quad (12a)$$

with the initial conditions:

$$\frac{d^k y(0)}{dx^k} = h_k\,,\; k = 0, 1, 2, \ldots, n-1;\; h_k \in R. \quad (12b)$$

In view of the new iterative method, the above nonlinear integro-differential equation (12) is equivalent to the nonlinear integral equation:

$$y(x) = I_x^{\alpha}[g(x)] - I_x^{\alpha}\left[ f(x) y(x) \right.$$
$$\left. +\int_a^b w(x,t) y^{(q)}(t) y^{(m)}(t)\, dt \right]$$
$$= f - N(y), \quad (13)$$

where

$$f = I_x^{\alpha}[g(x)], \quad (14a)$$

and $I_x^{\alpha}$ is a fractional order integral operator with respect to $x$.

**Remark 2.** When the general functional Equation (5) is linear, the recurrence relation (9) can be simplified in the form:

$$\begin{cases} y_0 = f\,, \\ y_{n+1} = N\,(y_n\,), \quad n = 0, 1, 2, \dots \end{cases} \qquad (15)$$

**Proof.** From the properties of integration and by using equations (9), (14b), we have:

$$\begin{aligned} y_{n+1} &= N\,(y_0 + \dots + y_{n-1} + y_n\,) \\ &\quad - N\,(y_0 + \dots + y_{n-1}\,) \\ &= I_x^{\alpha}\,[y_0 + \dots + y_{n-1} + y_n\,] \\ &\quad - I_x^{\alpha}\,[y_0 + \dots + y_{n-1}\,] \\ &= I_x^{\alpha}\,[y_0\,] + \dots + I_x^{\alpha}\,[y_{n-1}\,] + I_x^{\alpha}\,[y_n\,] \\ &\quad - I_x^{\alpha}\,[y_0\,] - \dots - I_x^{\alpha}\,[y_{n-1}\,] \\ &= I_x^{\alpha}[y_n\,] = N\,(y_n\,),\; n = 0, 1, 2, \dots \end{aligned}$$

The convergence of the NIM has been proved in [31, 32]. We get the solution of Equation (13) by employing the recurrence relation (9) or (15).

## 2.3. Homotopy Perturbation Method.

In this subsection the basic ideas of the HPM are introduced [13-16].

To achieve our goal m we consider the following nonlinear differential equation:

$$L\,(y\,) + N\,(y\,) = \Psi\,(x\,),\; x \in \Omega, \qquad (16a)$$

with the boundary conditions:

$$B\left(u, \frac{d\,y}{d\,x}\right) = 0,\; x \in \Gamma, \qquad (16b)$$

where $L$ is a linear operator, $N$ is a nonlinear operator, $B$ is a boundary operator, $\Psi\,(x\,)$ is a Known analytic function and $\Gamma$ is the boundary of the domain $\Omega$.

By the homotopy technique [13-16], He construct a homotopy $v\,(x\,,\,p\,) : \Omega \times [0, 1] \to R$ which satisfies:

$$\begin{aligned} H(v, p) &= (1-p)[L(v\,) - L(y_0)] + p[L(v\,) \\ &\quad + N\,(v\,) - \Psi\,(x\,)] = 0, \end{aligned} \qquad (17a)$$

or

$$\begin{aligned} H\,(v\,,\,p\,) &= L\,(v\,) - L\,(y_0\,) + p\,L\,(y_0\,) \\ &\quad + p[N\,(v\,) - \Psi(x\,)] = 0, \end{aligned} \qquad (17b)$$

where $x \in \Omega,\; p \in [0, 1]$ is an impeding parameter and $y_0$ is an initial approximation which satisfies the boundary conditions. Obviously, from (17), we have

$$\begin{aligned} &H\,(v\,,\,0) = L\,(v\,) - L\,(y_0\,), \\ &H\,(v\,,\,1) = L\,(v\,) + N\,(v\,) - \Psi\,(x\,) = 0, \end{aligned} \quad . \, (18)$$

The change process of $p$ from zero to unity is just that of $v\,(x\,,\,p\,)$ from $y_0$ to $y$. In topology, this is called deformation, $L\,(v\,) - L\,(y_0\,)$ and $L\,(v\,) + N\,(v\,) - \Psi(x\,)$ are called homotopic. The basic assumption is that the solution of Equation (17) ca be expressed as a power series in $p$:

$$v = v_0 + p\,v_1 + p^2\,v_2 + \dots\,. \qquad (19)$$

Setting $p = 1$, the approximate solution of Equation (16) is given by:

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots\,. \qquad (20)$$

The convergence of the series (20) has been proved in [38, 39]. For the author you can see [40-43].

## 2.4. The Reliable Algorithm.

Now we introduce a suitable algorithm to handle in a realistic and efficient way the nonlinear integro-differential equations of fractional derivative order (1) defined in the form:

$$\begin{aligned} &D^{(\alpha)}\,y\,(x\,) + f\,(x\,)\,y\,(x\,) \\ &+ \int_a^b w\,(x, t\,)\,y^{(q)}\,(t\,)\,y^{(m)}\,(t\,)\,d\,t = g\,(x\,), \end{aligned} \qquad (21a)$$

with the initial conditions:

$$\frac{d^k\,y\,(0)}{d\,x^k} = h_k\,,\; k = 0, 1, 2, \dots, n-1;\; h_k \in R\,, \qquad (21b)$$

In view of the homotopy technique, we can construct the following homotopy:

$$\begin{aligned} &D^{(\alpha)}\,y\,(x\,) + f\,(x\,)\,y\,(x\,) - g\,(x\,) \\ &= p\left[-\int_a^b w\,(x, t\,)\,y^{(q)}\,(t\,)\,y^{(m)}\,(t\,)\,d\,t\right], \end{aligned} \qquad (22a)$$

or

$$\begin{aligned} &D^{(\alpha)}\,y\,(x\,) - g\,(x\,) = p\,[-f\,(x\,)\,y\,(x\,) \\ &- \int_a^b w\,(x\,,\,t\,)\,y^{(q)}\,(t\,)\,y^{(m)}\,(t\,)\,d\,t\,, \end{aligned} \qquad (22b)$$

where $p \in [0, 1]$. The homotopy parameter $p$ always change from zero to unity.
In case $p = 0$ Equation (22) becomes the linearized equation:

$D^{(\alpha)} y(x) = g(x) - f(x) y(x)$, or

$$D^{(\alpha)} y(x) = g(x), \tag{23}$$

and when $p = 1$ equation (22) turns out to be the original nonlinear integro-differential Equation (21). The basic assumption is that the solution if Equation (22) can be written as a power series in $p$:

$$y = y_0 + p y_1 + p^2 y_2 + \dots . \tag{24}$$

Finally, we approximate the solution $y(x)$ by:

$$y(x) = \sum_{n=0}^{\infty} y_n , \tag{25a}$$

where the $n$-term approximate solution is:

$$y(x) = y_0 + y_1 + y_2 + \dots + y_{n-1} . \tag{25b}$$

The main advantage of the new modification of the two methods, as we will see in the next section, is that they can be applicable simply to a wide class of linear and nonlinear integro-differential equations with fractional derivative order.

## 3. Applications.

In this section we present some examples to illustrate the power of the given methods with the considered algorithms.

**Example 3.1.** Consider the following example with $y^{(q)}(t) = 1$, $m = 0$, $n = 1$ and $0 < \alpha \le 1$:

$$D_x^{\alpha} y(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x}{4} + \int_0^1 x\, t\, y(t)\, dt , \tag{26a}$$

with the initial condition:

$$y(0) = 0 . \tag{26b}$$

In view of the NIM, from (14a), we obtain:

$$y_0(x) = x^2 - \frac{x^{1+\alpha}}{4\gamma(2+\alpha)} .$$

Therefore, from (13) the integro-differentia Equation (26) is equivalent to the integral equation:

$$y(x) = x^2 - \frac{x^{1+\alpha}}{4\Gamma(2+\alpha)} + I_x^{\alpha}\left[\int_0^1 x\, t\, y(t)\, dt\right].$$

Let $N(y) = I_x^{\alpha}\left[\int_0^1 x\, t\, y(t) dt\right]$. Therefore, from (15) we can obtain easily the following first few components of the solution for Equation (26):

$$y_0(x) = x^2 - \frac{x^{1+\alpha}}{4\Gamma(2+\alpha)},$$

$$y_1(x) = \frac{(3+\alpha)\Gamma(2+\alpha) - 1}{4(3+\alpha)\Gamma(2+\alpha)^2} x^{1+\alpha},$$

$$y_2(x) = \frac{(3+\alpha)\Gamma(2+\alpha) - 1}{4(3+\alpha)^2\Gamma(2+\alpha)^3} x^{1+\alpha},$$

$$\vdots$$

$$y_5(x) = \frac{(3+\alpha)\Gamma(2+\alpha) - 1}{4(3+\alpha)^5\Gamma(2+\alpha)^6} x^{1+\alpha},$$

and so on, in the same manner the rest of components can be obtained.
The 6-term approximate solution for Equation (26) is:

$$\begin{aligned} y(x) &= \sum_{i=0}^{5} y_i \\ &= x^2 - \frac{x^{1+\alpha}}{4\Gamma(2+\alpha)} + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)\Gamma(2+\alpha)^2} x^{1+\alpha} \\ &\quad + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^2\Gamma(2+\alpha)^3} x^{1+\alpha} \\ &\quad + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^3\Gamma(2+\alpha)^4} x^{1+\alpha} \\ &\quad + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^4\Gamma(2+\alpha)^5} x^{1+\alpha} \\ &\quad + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^5\Gamma(2+\alpha)^6} x^{1+\alpha} . \end{aligned} \tag{27}$$

The same results can be obtained by using Equation (9) instead to Equation (15).
In view of the HPM, the homotopy for Equation (26) by equation (22), can be constructed as:

$$D_x^{\alpha} y(x) = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{x}{4} = p\left[\int_0^1 t\, y(t)\, dt\right]$$
. (28)

Substituting (24) and the initial condition (26b) into (28) and equating terms with identical powers of *p*; we obtain the following set of fractional integro-differential equations:

$$p^0 : D_x^{\alpha} y_0 = \frac{2x^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{x}{4};\ y_0(0) = 0,$$

$$p^1 : D_x^{\alpha} y_1 = x\int_0^1 t\, y_0(t)\, dt;\ y_1(0) = 0,$$

$$p^2 : D_x^{\alpha} y_2 = x\int_0^1 t\, y_1(t)\, dt;\ y_2(0) = 0,$$

$$p^3 : D_x^{\alpha} y_3 = x\int_0^1 t\, y_2(t)\, dt;\ y_3(0) = 0,$$

$$\vdots$$

Consequently, the first few components of the homotopy perturbation solution for Equation (26) are derived as follows:

$$y_0(x) = x^2 - \frac{x^{1+\alpha}}{4\Gamma(2+\alpha)},$$

$$y_1(x) = \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)\Gamma(2+\alpha)^2}\, x^{1+\alpha},$$

$$y_2(x) = \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^2\Gamma(2+\alpha)^3}\, x^{1+\alpha},$$

$$\vdots$$

$$y_5(x) = \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^5\Gamma(2+\alpha)^6}\, x^{1+\alpha},$$

and so on, in the same manner the rest of components can be obtained. The 6-term approximate solution for Equation (26) is:

$$y(x) = \sum_{i=0}^{5} y_i = x^2 - \frac{x^{1+\alpha}}{4\Gamma(2+\alpha)} + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)\Gamma(2+\alpha)^2}\, x^{1+\alpha} + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^2\Gamma(2+\alpha)^3}\, x^{1+\alpha} + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^3\Gamma(2+\alpha)^4}\, x^{1+\alpha} + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^4\Gamma(2+\alpha)^5}\, x^{1+\alpha} + \frac{(3+\alpha)\Gamma(2+\alpha)-1}{4(3+\alpha)^5\Gamma(2+\alpha)^6}\, x^{1+\alpha},$$
(29)

which is the same result obtained by the NIM in (27).

**Table 1.**

| $x$ | $\alpha = 0.25$ | $\alpha = 0.50$ | $\alpha = 0.75$ | $\alpha = 1.0$ | $y_{Exact}$ |
|---|---|---|---|---|---|
| **0.2** | **0.039956** | **0.039992** | **0.039999** | **0.0399998** | **0.04** |
| **0.4** | **0.159896** | **0.159978** | **0.159969** | **0.1899990** | **0.16** |
| **0.6** | **0.359828** | **0.359960** | **0.359992** | **0.3599980** | **0.36** |
| **0.8** | **0.639753** | **0.639938** | **0.349987** | **0.6399980** | **0.64** |
| **1.0** | **0.999674** | **0.999914** | **0.99981** | **0.9999960** | **1.00** |

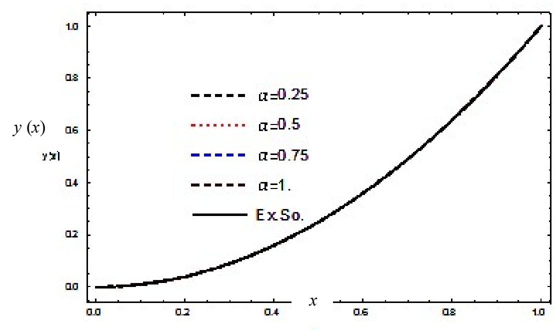


**Figure 1.**

Table 1 and Figure 1 showing the 6-term approximate solution for Equation (26) obtained for different values of $\alpha$ by the two methods. When $\alpha = 1$, the obtained approximate solutions are in high agreement with those obtained from the exact solution $y(x) = x^2$ for Equation (26). Of course the accuracy can be improved by computing more of terms than the computed sex-term.

**Example 3.2.** Consider the following example with $y^{(q)}(t) = 1$, $m = 1$, $n = 2$ and $1 < \alpha \le 2$.

$$D_x^{\alpha} y(x) = D_x^{\alpha} \cos x + x + \int_0^{\pi/2} x\, t\, y'(t)\, dt, \tag{30a}$$

with the initial conditions:

$$y(0) = 1, \qquad y'(0) = 0. \tag{30b}$$

By the NIM, as the above example, we obtain:

$$y_0(x) = \cos x + \frac{x^{1+\alpha}}{\Gamma(2+\alpha)}.$$

Therefore, the integro-differential equation (30) is equivalent to the integral equation:

$$y(x) = \cos x + \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} + I_x^{\alpha}\left[\int_0^{\pi/2} x\, t\, y'(t)\, dt\right].$$

Let $N(y) = I_x^{\alpha}\left[\int_0^{\pi/2} x\, t\, y'(t)\, dt\right]$. Therefore, from (15) we can obtain easily the following first few components of the solution for Equation (30).

$$y_0(x) = \cos x + \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_1(x) = \left(\frac{\pi}{2}\right)^{2+\alpha} \frac{x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)} - \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$p^3 : D_x^{\alpha} y_3 = x \int_0^{\pi/2} t\, y_2'(t)\, dt,$$

$$y_3(0) = 0, \qquad y_3'(0) = 0$$

$$\vdots$$

$$y_2(x) = \left(\frac{\pi}{2}\right)^{2(2+\alpha)} \frac{(1+\alpha) x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^2} - \left(\frac{\pi}{2}\right)^{2+\alpha} \frac{x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)}$$

$$\vdots$$

$$y_5(x) = \left(\frac{\pi}{2}\right)^{5(2+\alpha)} \frac{(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5} - \left(\frac{\pi}{2}\right)^{4(2+\alpha)} \frac{(1+\alpha)^3 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^4},$$

and so on, in the same manner the rest of components can be obtained. The 6-term approximate solution for Equation (30) is:

$$y(x) = \sum_{i=0}^{5} y_i = \cos x + \left(\frac{\pi}{2}\right)^{5(2+\alpha)} \frac{(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5}. \tag{31}$$

The same results can be obtained by using equation (9).

By the HPM, the homotopy for Equation (30) obtained from Equation (22) can be constructed as:

$$D_x^{\alpha} y(x) - D_x^{\alpha} \cos x - x = p\left[x \int_0^{\pi/2} t\, y'(t)\, dt\right]. \tag{32}$$

Substituting (24) and the initial condition (30b) into (32) and equating the terms with identical powers of *p*, we obtain the following set of fractional integro-differential equations:

$$p^0 : D_x^{\alpha} y_0 = D_x^{\alpha} \cos x + x,$$

$$y_0(0) = 1, \; y_0'(0) = 0,$$

$$p^1 : D_x^{\alpha} y_1 = x \int_0^{\pi/2} t\, y_0'(t)\, dt,$$

$$y_1(0) = 0, \; y_1'(0) = 0,$$

$$p^2 : D_x^{\alpha} y_2 = x \int_0^{\pi/2} t\, y_1'(t)\, dt,$$

$$y_2(0) = 0, \; y_1'(0) = 0,$$

$$\vdots$$

Consequently, the first few components of the homotopy perturbation solution for Equation (30) are derived as follows:

$$y_0(x) = \cos x + \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_1(x) = \left(\frac{\pi}{2}\right)^{2+\alpha} \frac{x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)} - \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_2(x) = \left(\frac{\pi}{2}\right)^{2(2+\alpha)} \frac{(1+\alpha) x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^2} - \left(\frac{\pi}{2}\right)^{2+\alpha} \frac{x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)},$$

$\vdots$

$$y_5(x)=\left(\frac{\pi}{2}\right)^{5(2+\alpha)}\frac{(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5}$$
$$-\left(\frac{\pi}{2}\right)^{4(2+\alpha)}\frac{(1+\alpha)^3 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^4},$$

and so on, in the same manner the rest of components can be obtained. The 6-term approximate solution for Equation (30) is:

$$y(x)=\sum_{i=0}^{5} y_i = \cos x +$$
$$\left(\frac{\pi}{2}\right)^{5(2+\alpha)}\frac{(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5}, \qquad (33)$$

which is the same result obtained by the NIM in (31).

**Table 2.**

| $x$ | $\alpha=1.25$ | $\alpha=1.50$ | $\alpha=1.75$ | $\alpha=2.0$ | $y_{\text{Exact}}$ |
|---|---|---|---|---|---|
| **0** | **0.1** | **0.1** | **0.1** | **0.1** | **0.1** |
| **5** | **0.999883** | **0.997036** | **0.996359** | **0.996223** | **0.9961950** |
| **10** | **1.002350** | **0.989558** | **0.985916** | **0.985034** | **0.9848078** |
| **15** | **1.009620** | **0.979015** | **0.969305** | **0.966689** | **0.9659260** |
| **20** | **1.023160** | **0.966563** | **0.947148** | **0.941502** | **0.9396930** |
| **25** | **1.044200** | **0.953248** | **0.920079** | **0.909842** | **0.9063078** |

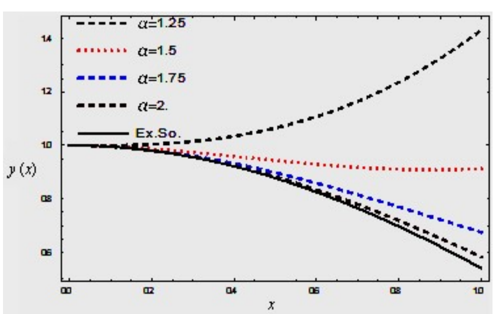


**Figure 2.**

Table 2 shows the 6-term approximate solution for Equation (30) obtained for different values of $\alpha$ by the two methods. When $\alpha=2$, the obtained approximate solutions are in high agreement with those obtained from the exact solution $y(x)=\cos x$ for Equation (30). Of Course the accuracy can be improved by computing more of terms than the computed six-term.

**Example 3.3.** Consider the following example with $y^{(q)}(t)=1$, $m=0$, $n=0$ and $2<\alpha\le 3$:

$$D_x^{\alpha} y(x)=e^x - x + \int_0^1 x\,t\,y(t)\,dt, \qquad (34a)$$

with the initial conditions

$$y(0)=y'(0)=y''(0)=1. \qquad (34b)$$

In this example, we obtain:

$$y_0(x)-e^x - \frac{x^{1+\alpha}}{\Gamma(2+\alpha)}.$$

Therefore, the integro-differential equation (34) is equivalent to the integral equation:

$$y(x)=e^x - \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} + I_x^{\alpha}\left[\int_0^1 x\,t\,y(t)\,dt\right].$$

Let $N(y)=I_x^{\alpha}\left[\int_0^1 x\,t\,y(t)\,dt\right]$. Therefore, we can obtain the following approximate solutions:

$$y_0(x)=e^x - \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_1(x)=-\frac{x^{1+\alpha}}{(3+\alpha)\Gamma(2+\alpha)^2}+\frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_2(x)=-\frac{x^{1+\alpha}}{(3+\alpha)^2\Gamma(2+\alpha)^3}$$
$$+\frac{x^{1+\alpha}}{(3+\alpha)\Gamma(2+\alpha)^2},$$

$\vdots$

$$y_5(x)=-\frac{x^{1+\alpha}}{(3+\alpha)^5\Gamma(2+\alpha)^6}$$
$$+\frac{x^{1+\alpha}}{(3+\alpha)^4\Gamma(2+\alpha)^5},$$

and so on. The 6-term approximate solution for Equation (34) is:

$$y(x)=\sum_{i=0}^{5} y_i$$
$$=e^x - \frac{x^{1+\alpha}}{(3+\alpha)^5\Gamma(2+\alpha)^6}. \qquad (35)$$

The same results can be obtained by using equation (9).

As the above Examples, the homotopy for Equation (34) becomes:

$$D_x^{\alpha} y(x) - e^{x} + x = p\left[ x\int_0^1 t\, y(t)\, dt \right]. \tag{36}$$

Also, we can obtain the following set of fractional integro-differential equations:

$$p^0 : D_x^{\alpha} y_0 = e^{x} - x,$$
$$y_0(0) = y_1'(0) = y_1''(0) = 1,$$
$$p^1 : D_x^{\alpha} y_1 = x\int_0^1 t\, y_0(t)\, dt,$$
$$y_0(0) = y_1'(0) = y_1''(0) = 0,$$
$$p^2 : D_x^{\alpha} y_2 = x\int_0^1 t\, y_1(t)\, dt,$$
$$y_2(0) = y_2'(0) = y_2''(0) = 0,$$
$$p^3 : D_x^{\alpha} y_3 = x\int_0^1 t\, y_2(t)\, dt,$$
$$y_3(0) = y_3'(0) = y_3''(0) = 0,$$
$$\vdots$$

solving the above set of equations, we obtain the following first few components of the homotopy perturbation solution for Equation (34):

$$y_0(x) = e^{x} - \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$
$$y_1(x) = -\frac{x^{1+\alpha}}{(3+\alpha)\Gamma(2+\alpha)^2} + \frac{x^{1+\alpha}}{\Gamma(2+\alpha)},$$
$$y_2(x) = -\frac{x^{1+\alpha}}{(3+\alpha)^2\,\Gamma(2+\alpha)^3} + \frac{x^{1+\alpha}}{\Gamma(3+\alpha)\Gamma(2+\alpha)^2},$$
$$\vdots$$
$$y_5(x) = -\frac{x^{1+\alpha}}{(3+\alpha)^5\,\Gamma(2+\alpha)^6} + \frac{x^{1+\alpha}}{\Gamma(3+\alpha)^4\,\Gamma(2+\alpha)^5}.$$

In the same manner the rest of components can be obtained. the 6-term approximate solution for Equation (34) is:

$$y(x) = \sum_{i=0}^{5} y_i = e^{x} - \frac{x^{1+\alpha}}{(3+\alpha)^5\,\Gamma(2+\alpha)^6}, \tag{37}$$

which is the same result obtained by the NIM in (35).

**Table 3.**

| $x$ | $\alpha = 2.25$ | $\alpha = 2.50$ | $\alpha = 2.75$ | $\alpha = 3.0$ | $y_{\text{Exact}}$ |
|---|---|---|---|---|---|
| 0.2 | 0.122140 | 0.122140 | 0.122140 | 0.122140 | 1.221403 |
| 0.4 | 1.491820 | 1.491820 | 1.491820 | 1.491820 | 1.497825 |
| 0.6 | 1.822120 | 1.822120 | 1.822120 | 1.822120 | 1.822119 |
| 0.8 | 2.225540 | 2.225540 | 2.225540 | 2.225540 | 2.225541 |
| 1.0 | 2.718280 | 2.718280 | 2.718280 | 2.718280 | 2.718282 |

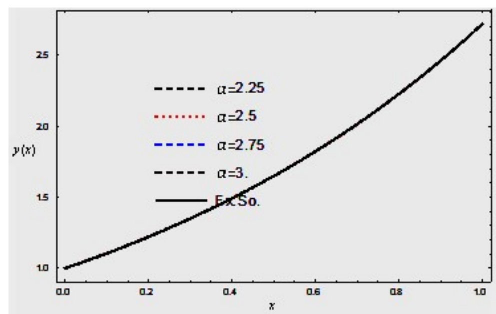


**Figure 3.**

From Table 3 and Figure 3, it is clear that the obtained 6-term approximate solutions, for different values of $\alpha$, by the two methods are in high agreement with those obtained from the exact solution $y(x) = e^{x}$ for Equation (34).

**Example 3.4.** Consider the following integro-differential equation:

$$D_x^{\alpha} y(x) = D_x^{\alpha}(x e^{x}) - x(e-1) + \int_0^1 x t\, y'(t)\, dt, \quad 1<\alpha\le 2, \tag{38a}$$

with the initial conditions:

$$y(0) = 0, \qquad y'(0) = 1. \tag{38b}$$

Therefore, the integro-differential equation (38) is equivalent to the integral equation:

$$y_0(x) = x\,e^x - \frac{(e-1)x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_1(x) = -\frac{(e-1)x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)} + \frac{(e-1)x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_2(x) = -\frac{(e-1)(1+\alpha)x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^2} + \frac{(e-1)x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)},$$

$$\vdots$$

$$y_5(x) = -\frac{(e-1)(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5} + \frac{(e-1)(1+\alpha)^3 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^4},$$

as so on. The 6-term approximate solution for Equation (38) is:

$$y(x) = \sum_{i=0}^{5} y_i = x\,e^x - \frac{(e-1)(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5}. \tag{39}$$

The same results can be obtained by using Equation (9).

Also, as above, the homotopy for Equation (38) is:

$$D_x^\alpha y(x) - D_x^\alpha (x\,e^x) + x(e-1) = p\left[ x \int_0^1 t\, y'(t)\, dt \right]. \tag{40}$$

In the same manner, we can obtain the following set of equations:

$$p^0 : D_x^\alpha y_0 = D_x^\alpha (x\,e^x) - x(e-1),$$
$$y_0(0) = y_0'(0) = 1,$$

$$p^1 : D_x^\alpha y_1 = x \int_0^1 t\, y_0'(t)\, dt,$$
$$y_1(0) = y_1'(0) = 0,$$

$$p^2 : D_x^\alpha y_2 = x \int_0^1 t\, y_1'(t)\, dt,$$
$$y_2(0) = y_2'(0) = 0,$$

$$p^3 : D_x^\alpha y_3 = x \int_0^1 t\, y_2'(t)\, dt,$$
$$y_3(0) = y_3'(0) = 0,$$

$$\vdots$$

and so on. Solving the above set of equations, we obtain the following first few components of the homotopy perturbation solution for Equation (38):

$$y_0(x) = x\,e^x - \frac{(e-1)x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_1(x) = -\frac{(e-1)x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)} + \frac{(e-1)x^{1+\alpha}}{\Gamma(2+\alpha)},$$

$$y_2(x) = -\frac{(e-1)(1+\alpha)x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^2} + \frac{(e-1)x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)},$$

$$\vdots$$

$$y_5(x) = -\frac{(e-1)(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5} + \frac{(e-1)(1+\alpha)^3 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^4}.$$

The 6-term approximate solution for Equation (38) is:

$$y(x) = \sum_{i=0}^{5} y_i = x\,e^x - \frac{(e-1)(1+\alpha)^4 x^{1+\alpha}}{\Gamma(1+\alpha)\Gamma(3+\alpha)^5}, \tag{41}$$

which is the same result obtained in (39) by the NIM.

**Table 4.**

| $x$ | $\alpha = 1.25$ | $\alpha = 1.50$ | $\alpha = 1.75$ | $\alpha = 2.0$ | $y_{Exact}$ |
|---|---|---|---|---|---|
| **0.2** | **0.244252** | **0.244276** | **0.244280** | **0.244280** | **0.244281** |
| **0.4** | **0.596603** | **0.596726** | **0.596726** | **0.596729** | **0.596730** |
| **0.6** | **1.092960** | **1.093260** | **1.093260** | **1.093270** | **1.093271** |
| **0.8** | **1.779830** | **1.780410** | **1.780410** | **1.780430** | **1.780433** |
| **1.0** | **2.717290** | **2.718230** | **2.718230** | **2.718270** | **2.718282** |

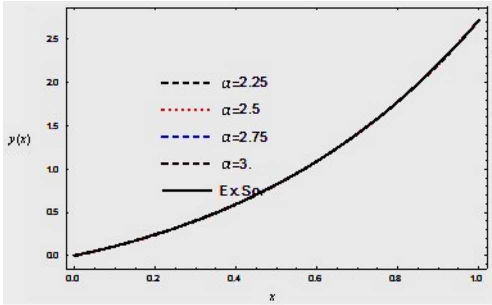


**Figure 4.**

From Table 4 and Figure 4, it is clear that the obtained 6-term approximate solutions, for different values of $\alpha$, by the two methods are in high agreement with those obtained from the exact solution $y(x)=x\,e^{x}$ for Equation (38).

**Example 3.5.** Consider the following nonlinear integro-differential equation with: $q=0$, $m=1$, $n=2$ and $1<\alpha\leq 2$.

$$D_x^{\alpha}\, y(x)=\frac{6x^{3-\alpha}}{\Gamma(4-\alpha)}-\frac{3x}{7}+\int_0^1 x\,t\,y(t)\,y'(t)\,dt, \quad (42a)$$

with the initial conditions:

$$y(0)=0, \qquad y'(0)=0. \quad (42b)$$

In this example, we obtain:

$$y_0(x)=x^3-\frac{3x^{1+\alpha}}{7\Gamma(2+\alpha)},$$

and the nonlinear integro-differential equation (42) is equivalent to the nonlinear integral equation:

$$y(x)=x^3\,\frac{3x^{1+\alpha}}{7\Gamma(2+\alpha)}+I_x^{\alpha}\left[\int_0^1 x\,t\,y(t)\,y'(t)\,dt\right].$$

Let $N(y)=\left[\int_0^1 x\,t\,y(t)\,y'\,dt\right]$. Therefore, from (9) we can obtain the following approximate solutions for Equation (42):

$$y_0(x)=x^3-\frac{3x^{1+\alpha}}{7\Gamma(2+\alpha)},$$

$$y_1(x)=[3x^{1+\alpha}[3(5+\alpha)+7(3+2\alpha)\cdot \Gamma(1+\alpha)(-4-\alpha+(5+\alpha)\Gamma(2+\alpha))]] /[49(15+13\alpha+2\alpha^2)\Gamma(1+\alpha)\,\Gamma(2+\alpha)^2,$$

$$y_2(x)=3x^{1+\alpha}\left[\frac{27(5+\alpha)^2}{3+2\alpha}-126(20+9\alpha+\alpha^2)\Gamma(1+\alpha)+588(4+\alpha)\cdot(3+2\alpha)\Gamma(1+\alpha)^2+147\alpha(4+\alpha)\cdot(3+2\alpha)\Gamma(1+\alpha)^2+147\alpha(1+\alpha)\cdot(4+\alpha)(3+2\alpha)\Gamma(1+\alpha)^3-3773\alpha\cdot(1+\alpha)(4+\alpha)\Gamma(1+\alpha)^4-686\alpha^5\cdot(4+\alpha)(3+2\alpha)\Gamma(\alpha)^3\,\Gamma(2+\alpha)+735(4+\alpha)+735(4+\alpha)(3+2\alpha)\Gamma(1+\alpha)^2\,\Gamma(2+\alpha)-4116(4+\alpha)(3+2\alpha)\Gamma(1+\alpha)^3\,\Gamma(2+\alpha)+343(5+\alpha)^2(3+2\alpha)^2\,\Gamma(1+\alpha)^3\,\Gamma(2+\alpha)^3\right]/[2401915+13\alpha+2\alpha^2)^2\,\Gamma(1+\alpha)^3\,\Gamma(2+\alpha)^4],$$

$$\vdots$$

and son on. By the HPM, the homotopy for Equation (42) obtained from Equation (22) takes the form:

$$D_x^{\alpha}\, y(x)-\frac{6x^{3-\alpha}}{\Gamma(4-\alpha)}+\frac{3x}{7}=p\left[x\int_0^1 t\,y(t)\,y'(t)\,dt\right]. \quad (43)$$

Substituting (24) and the initial conditions (42b) into (43) and equating the terms with identical powers of *p*, we obtain the following set of linear integro differential equations of fractional derivative order:

$$p^0: D_x^{\alpha}\, y_0=\frac{6x^{3-\alpha}}{\Gamma(4-\alpha)}-\frac{3x}{7},$$

$$y_0(0)=y_0'(0)=0,$$

$$p^1: D_x^{\alpha}\, y_1=x\int_0^1 t\,y_0(t)\,y_0'(t)\,dt,$$

$$y_1(0)=y_1'(0)=0,$$

$$p^2 : D_x^{\alpha} y_2 = x \int_0^1 t\,(y_0(t)\,y_1'(t) + y_1(t)\,y_0'(t))\,dt\,,$$
$$y_2(0) = y_2'(0) = 0\,,$$
$$p^3 : D_x^{\alpha} y_3 = x \int_0^1 t\,(y_0(t)\,y_2'(t) + y_1(t)\,y_1'(t) + y_2(t)\,y_0'(t))\,dt$$
$$y_3(0) = y_3'(0) = 0\,,$$
$$\vdots$$

and so on. Solving the above set of equations, we obtain the following first few components of the homotopy perturbation solution for Equation (42):

$$y_0(x) = x^3 - \frac{3x^{1+\alpha}}{7\Gamma(2+\alpha)}\,,$$

$$y_1(x) = [3x^{1+\alpha}[3(5+\alpha) + 7(3+2\alpha)\cdot \Gamma(1+\alpha)(-4-\alpha+(5+\alpha)\Gamma(2+\alpha))]]$$
$$49(15+13\alpha+2\alpha^2)\Gamma(1+\alpha)\Gamma(2+\alpha)^2\,],$$

$$y_2(x) = 3x^{1+\alpha}[4(1+\alpha)9[3(5+\alpha) +7(3+2\alpha)\Gamma(1+\alpha)(-4-\alpha+(5+\alpha)\cdot \Gamma(2+\alpha))][-5-\alpha+7(3+2\alpha)\Gamma(2+\alpha)) +3(-3(15+13\alpha+2\alpha^2)\alpha[3(5+\alpha) +7(3+2\alpha)\Gamma(1+\alpha)(-4-\alpha+5(5+\alpha) \Gamma(2+\alpha))]+4(6+17\alpha+11\alpha^2)\Gamma(2+\alpha) (1+\alpha)[3(5+\alpha)+7(3+2\alpha)\Gamma(1+\alpha)\cdot \Gamma(1+\alpha)(-4-\alpha(5+\alpha)\Gamma(2+\alpha))]])]$$
$$/[1372(1+\alpha)(5+\alpha)^2(3+2\alpha)^2\Gamma(1+\alpha) \Gamma(2+\alpha)^4\,],$$
$$\vdots$$

and son on. The 4-term approximate solution for equation (42), obtained by the two methods, for different values of $\alpha$ are listed in Table 5. It is clear that the approximate solutions are in high agreement with those obtained from the exact solution $y(x) = x^3$ for Equation (42).

**Table 5.**

| $X$ | Method | $\alpha = 1.50$ | $\alpha = 1.75$ | $\alpha = 2.0$ | $y_{Exact}$ |
|---|---|---|---|---|---|
| 0.2 | NIM | $7.96502^{-3}$ | $7.99220^{-3}$ | $7.99840^{-3}$ | $8.00^{-3}$ |
| | HPM | $7.96500^{-3}$ | $7.99210^{-3}$ | $7.99839^{-3}$ | |
| 0.4 | NIM | $6.38021^{-2}$ | $6.39475^{-2}$ | $6.39872^{-2}$ | $6.40^{-2}$ |
| | HPM | $6.38020^{-2}$ | $6.39474^{-2}$ | $6.39871^{-2}$ | |
| 0.6 | NIM | $2.15455^{-1}$ | $2.15840^{-1}$ | $2.15957^{-1}$ | $2.16^{-1}$ |
| | HPM | $2.15453^{-1}$ | $2.15838^{-1}$ | $2.15956^{-1}$ | |
| 0.8 | NIM | $5.10881^{-1}$ | $5.11647^{-1}$ | $5.11898^{-1}$ | $5.12^{-1}$ |
| | HPM | $5.10880^{-1}$ | $5.11646^{-1}$ | $5.11897^{-1}$ | |
| 1.0 | NIM | $9.98044^{-1}$ | $9.99348^{-1}$ | $9.99800^{-1}$ | 1.00 |
| | HPM | $9.98043^{-1}$ | $9.99348^{-1}$ | $9.99800^{-1}$ | |

**Example 3.6.** Finally, consider the following nonlinear integro-differential equation with: $q = 0$, $m = 0$, $n = 2$ and $1 < \alpha \le 2$;

$$D_x^{\alpha} y(x) = e^x - \frac{(e^2+1)x}{4} + \int_0^1 x\,t\,y^2(t)\,dt\,, \tag{44a}$$

with the initial conditions:

$$y(0) = 1, \qquad y'(0) = 1. \tag{44b}$$

As above, we obtain:

$$y_0(x = e^x - \frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)}\,,$$

also, the nonlinear integro-differential equation (44) is equivalent to the nonlinear integral equation:

$$y(x) = e^x - \frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)} + I_x^{\alpha}\left[\int_0^1 x\,t\,y^2(t)\,dt\right].$$

Let $N(y) = I_x^{\alpha}\left[\int_0^1 x\,t\,y^2(t)\,dt\right]$. Therefore, from (9) we can obtain the following approximate solutions for Equation (44):

$$y_0(x) = e^x - \frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)},$$

$$y_1(x) = \frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)} - \frac{(e^2+1)x^{1+\alpha}}{2(3+\alpha)\Gamma(2+\alpha)^2} + \frac{(e^2+1)^2 x^{1+\alpha}}{16(4+2\alpha)\Gamma(2+\alpha)^3},$$

$$y_2(x) = -\frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)} + \frac{(e^2+1)^2 x^{1+\alpha}}{2(3+\alpha)\Gamma(2+\alpha)^2} - \frac{(e^2+1)^2 x^{1+\alpha}}{16(4+2\alpha)\Gamma(2+\alpha)^3} + \frac{(e^2+1)x^{1+\alpha}}{1024\Gamma(2+\alpha)} \cdot \left[256 + \frac{(e^2+1)^3}{2(2+\alpha)^3\Gamma(2+\alpha)^6} - \frac{16(e^2+1)^2}{2(2+\alpha)^2(3+\alpha)\Gamma(2+\alpha)^5} + \frac{128(e^2+1)}{(2+\alpha)(3+\alpha)\Gamma(2+\alpha)^4} - \frac{64(-1)^{-\alpha}(e^2+1)^2(\Gamma(3+\alpha)-\Gamma(3+\alpha,-1))}{(2+\alpha)\Gamma(2+\alpha)^3} + \frac{1024(-1)^{-\alpha}(e^2+1)^2(\Gamma(3+\alpha)-\Gamma(3+\alpha,-1))}{(3+\alpha)\Gamma(2+\alpha)^3}\right],$$

$\vdots$

and so on.

By the HPM, the homotopy, for Equation (44) takes the form:

$$D_x^\alpha y(x) - e^x + \frac{(e^2+1)x}{4} = p\left[x\int_0^1 t\, y^2(t)\,dt\right]. \qquad (45)$$

As above, we can obtain the following set of linear integro-differential equations of fractional derivative order:

$$p^0 : D_x^\alpha y_0 = e^x - \frac{(e^2+1)x}{4},$$
$$y_0(0) = y_0'(0) = 0,$$

$$p^1 : D_x^\alpha y_1 = x\int_0^1 t\, y_0^2(t) y_0'(t)\,dt,$$
$$y_1(0) = y_1'(0) = 0,$$

$$p^2 : D_x^\alpha y_2 = x\int_0^1 2t\, y_0(t) y_1'(t)\,dt,$$
$$y_2(0) = y_2'(0) = 0,$$

$$p^3 : D_x^\alpha y_3 = x\int_0^1 t\,(2y_0(t)y_2(t) + y_1^2(t))\,dt,$$
$$y_3(0) = y_3'(0) = 0,$$

$\vdots$

and so on. Solving the above set of equations, we obtain the following first few components of the homotopy perturbation solution for Equation (44).

$$y_0(x) = e^x - \frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)},$$

$$y_1(x) = \frac{(e^2+1)x^{1+\alpha}}{4\Gamma(2+\alpha)} - \frac{(e^2+1)x^{1+\alpha}}{2(3+\alpha)\Gamma(2+\alpha)^2} + \frac{(e^2+1)^2 x^{1+\alpha}}{16(4+2\alpha)\Gamma(2+\alpha)^3},$$

$$y_2(x) = [-(1+e^2)x^{1+\alpha}[(1+e^2)(3+\alpha) - 16\Gamma(3+\alpha) + 8(6+5\alpha+\alpha^2)\Gamma(2+\alpha)^2]] / [64(6+5\alpha+\alpha^2)\Gamma(2+\alpha)^5],$$

$\vdots$

and so on.

**Table 6.**

| $x$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|
| $y_{\text{NIM}}$ | 1.22139 | 1.49175 | 1.82185 | 2.22491 | 2.71705 |
| $y_{\text{HPM}}$ | 1.22135 | 1.49171 | 1.82177 | 2.22480 | 2.71692 |
| $y_{\text{Exact}}$ | 1.22140 | 1.49182 | 1.82212 | 2.22556 | 2.71828 |

In Table 6, the 4-term approximate solution, obtained by the two methods, when $\alpha = 2$ with the corresponding exact solution $y(x) = e^x$ for equation (44). It is clear that the two solutions are in high agreement.

## 4. Conclusion.

In this work, the NIM and HPM with a reliable algorithms employed to solve linear and nonlinear integro-differential equations with fractional derivative order. The modified algorithms make the steps of solution are simple. The obtained results by the two methods, for different values of $\alpha$; are in high agreement with those obtained from the corresponding exact solutions. This exhibit that these methods needs much less computations and they have a very fast convergency when applied to solve this type of equations.

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**الملخص العربي :**

في هذا البحث قمنا باستخدام الطريقة التكرارية الجديدة (NIH) وطريقة الإطراب المتماثل (HPH) لحل المعادلات التفاضلية ـ التكاملية الخطية منها وغير الخطية ذات التفاضل الكسوري وذلك باستخدام تعريف Caputo للتفاضل الكسري وهذه الطرق تستخدم للحصول على الحلول التحليلية والتقريبية حيث يكون الحل في صورة سلسلة متقاربة بمكونات يمكن حسابها بسهولة

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