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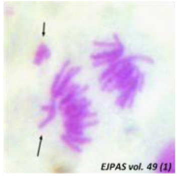

Picard Fixed Point Iteration Combined with Integrating Factor Approach

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ABSTRACT

We consider the Picard's iteration method as a technique for solving initial value problems of the first and second order linear differential equations. The basic idea is the use of integrating factors to collect some of the terms in a perfect differential term, and then use the decomposition techniques. For the second order differential equations, we transform the equation to a system of two first order equations and in addition we use the Gauss Seidel technique. The algorithm of the proposed method is discussed. Comparisons with the classical Picard method and modified Picard have illustrated the rapid convergence of the proposed method. Numerical examples have illustrated that the technique obtains the theoretical fixed point quicker than that obtained with other techniques including the modified Picard.

Introduction

Recently there has been interest in symbolic computations in treating initial and boundary value problems[1-3] and the references cited there. Many methods: Fixed point iteration method, Adomian decomposition method and Variational iteration method are good examples for methods in which symbolic computation takes place[3-5]. Also, there exist some codes for the power series and the method of Frobenius's. Among all polynomial approximations which can satisfy a differential equation on a finite number of points, there is only one polynomial which approach the solution between those points, and that is the truncated Maclaurin series, Rudmin[6]. The fixed point theorem is the main topic through which many mathematical methods are developed. For a long period it was considered as a method to prove existence not as method of solution. Picard iteration was assumed impractical as a solution method. The advent of computer algebra systems has removed this impracticality, Mathews[2] and Pruett et al[7]. Furthermore, attempts to design practical solution methods based upon Picard iteration have yielded not only successful new algorithms but also have raised intriguing theoretical issues to cover problems of boundary value type and problems in partial differential equations. Nevanlinna[8], considered first order linear systems with constant coefficients and he discussed the possibility of accelerating the convergence of the process using ideas common in accelerating iterations for linear algebraic systems of equations. Hyvönen[9], considered polynomial acceleration of the Picard-Lindelöf iteration for first order linear systems with constant coefficients in the sense that he can take linear combination of previous iterates. Youssef and Alarabawy[3], showed that the use of Gauss Seidel iteration with Picard technique even with equations with variable coefficients or simple nonlinearities improved considerably the results of the classical Picard iteration. From the functional analysis point of view, the contraction mapping principal, is considered as the main tool to introduce a rigorous proves

Definition: Let E be a complete metric space with distance ρ, let A be a mapping of E into itself. Then A is said to be a contraction mapping if there exists a constant α $(0 \leq \alpha < 1)$ such that the inequality

$$\rho(Ax, Ay) \leq \alpha\, \rho(x, y), \qquad (1)$$

holds for every pair of points $x, y \in E$.

If $A : E \to E$ is a contraction mapping, then the operator equation $x = A(x)$ has a unique solution and this solution is equal to the limit of the approximation

$$x_{n+1} = A(x_n), \qquad n = 0, 1, 2, \cdots \qquad (2)$$

Yildiz and Simsek[10], considered the semi-linear operator equation

$$x = A_0(x) + A_1(x) \qquad (3)$$

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in the E- Banach spaces, where $A_0(x)$ is a linear bounded operator defined in E such that $(I-A_0)^{-1}$ is a bounded operator, $A_1(x)$ is a nonlinear operator defined in E and I is the identity operator. The successive approximations corresponding to equation (3) is

$$x_{n+1}=A_0(x_{n+1})+A_1(x_n), \qquad n=0,1,2,\cdots \quad (4)$$

for arbitrary $x_0\in E$. The approximation (4) can be written as

$$x_{n+1}=(I-A_0)^{-1}A_1(x_n), \qquad n=0,1,2,\cdots \quad (5)$$

In this work we investigate the effect of reformulating the equations through differential identities (grouping terms, the rules of derivatives and integrating factor tools) before the application of the Picard iteration method. Also we consider the effect of this treatment on the rate of convergence of the Picard iteration. We restrict this work to problems up to second order initial value problems with smooth real valued data which admits only exponential behavior, which take the form:

$$y'+p(t)\,y=g(t,y) \qquad ; \qquad y(t_0)=y_0 \quad (6)$$

Or

$$y''+\alpha y'+\beta y=f(t)\,; \quad \alpha^2-4\beta\geq 0 \quad (7)$$

with the initial conditions $y(t_0)=\alpha_1$, $y'(t_0)=\alpha_2$, and the right hand side f(t) preserves the exponential behavior of the solution but with different attitudes.

It is well known that, the second order equation (7) can be written as a system of first order equations by taking

$$y_1(t):=y(t) \; and \; y_2(t):=y'(t), \quad (8)$$

Thus, we obtain

$$y_1'=y_2\,, \quad y_1(t_0)=\alpha_1 \quad (9)$$
$$y_2'=f(t)-\alpha\,y_2-\beta\,y_1, \qquad y_2(t_0)=\alpha_2$$

The approach extends straight forwards for higher order equations.

In this section, we briefly review Picard iteration method. Emile Picard's in 1891, begin by reformulating the initial value problem:

$$y'=f(t,y) \qquad ; \qquad y(t_0)=y_0\,, \quad (10)$$

as an equivalent integral equation

$$y(t)=y_0+\int_{t_0}^{t} f(t,y(t))\,dt \qquad , \quad (11)$$

whose solution y(t) can be obtained as the limit of a sequence of functions $y_n(t)$ generated by the recurrence formula[11-12],

$$y_n(t)=y_0+\int_{t_0}^{t} f(t,y_{n-1}(t))\,dt \quad ,n=1,2,3,\cdots \quad (12)$$

It is well known that if the right-hand side of (10), $f(t,y)$, satisfies Lipschitz condition with respect to y $|f(t,y)-f(t,y^*)|\leq L\,|y-y^*|\,, L=constant,$ in some closed rectangle $R=\{|t-t_0|\leq a, |y-y_0|\leq b$, then, irrespective of the choice of the initial function, the successive approximation $y_n(t)$ converge on some interval $[t_0,t_0+h]$ to the solution of the problem (10). Also, if $f(t,y)$ is continuous in the rectangle R, then the error of the approximate solution $y_n(t)$ is estimated by:

$$\varepsilon_n=|\,y(t)-y_n(t)|\leq ML^n\frac{(t-t_0)^{n+1}}{(n+1)!} \quad , \quad M=\max_{\{t,y\}\in R}|\,f(t,y)\,|$$

in the interval $[t_0,t_0+h]$, h is determined from the condition $h=\min(a,\frac{b}{M})$. Banach's fixed point theorem implies that the solution y(t) is the limit of the sequence $\{y_n\}_{n\geq 1}$ Boyce and Diprima[11].

The initial value problem (IVP) for a system of 2 first order ordinary differential equations (ODE's) can be written in the form:

$$\frac{dy_i}{dt}=f_i(t,y_1,y_2) \;\; ; \;\; y_i(t_0)=\alpha_i\,;\; i=1,2\,. \quad (13)$$

Accordingly, the classical Picard's method for the IVP (13) is obtained by the replacement of every equation in (13) by the corresponding integral form, as follows:

$$y_{i,n}=y_i(t_0)+\int_{t_0}^{t} f_i(t,y_{1,n-1},y_{2,n-1})dt\,;i=1,2,\; n=1,2,\cdots \;, \quad (14)$$

the corresponding modified Picard's method combined with Gauss Seidel method takes the form:

$$y_{1,n}=y_1(t_0)+\int_{t_0}^{t} f_1(t,y_{1,n-1},y_{2,n-1})dt, \quad (15)$$
$$y_{2,n}=y_2(t_0)+\int_{t_0}^{t} f_2(t,y_{1,n},y_{2,n-1})dt\,.$$

Material and methods

The objective of this work is the use of the integrating factor approach for the first order equations to accelerate the convergence of Picard iteration method. As well as decompose the system corresponding to the linear second order initial value problems into two parts and use the integrating factor for one part and use the Gauss seidel approach described in[3]

Case (I): Consider the equation

$$y'+p(t)\,y=g(t,y)\,;\; y(t_0)=y_0 \quad (16)$$

Using integrating factor, equation (16) can be written in the form:

$$\frac{d}{dt}\left(e^{\int p(t)dt}y\right)=e^{\int p(t)dt}\,g(t,y) \quad (17)$$

Applying Picard approach we have

$$y_{n+1}=y_o\,K\,e^{-\int p(t)dt}+e^{-\int p(t)dt}\int_{t_0}^{t} e^{\int p(x)dx}\,g(x,y_n)dx \quad (18)$$
$$K=\left[e^{\int p(t)dt}\right]_{t=t_0}, \quad n=0,1,2,\cdots$$

Case (II): The second order linear differential equation

$$y''+\alpha y'+\beta y=f(t)\,, \alpha^2-4\beta\geq 0 \quad (19)$$

with the initial conditions $y(a)=\alpha_1$, $y'(a)=\alpha_2$, which can be written as

$$y_1'=y_2\,, \quad y_1(a)=\alpha_1 \quad (20)$$
$$y_2'=f(t)\;-\alpha\,y_2-\beta\,y_1, \; y_2(a)=\alpha_2$$

Treating the first equation as in (3) and the second

$$\frac{d}{dt}\left(e^{\int \alpha dt}y_2\right)=e^{\int \alpha dt}\left(f(t)\;-\beta\,y_1\right), \; y_2(a)=\alpha_2\,.$$

Accordingly,

$$y_{1,n+1} = y_{1,0} + \int_a^t y_{2,n} \;, \qquad (21)$$

$$y_{2,n+1} = e^{-\int \alpha\, dt} K\, y_2(a) + e^{-\int \alpha\, dt} \int_a^t e^{\int \alpha\, dx} \left(f(x) - \beta\, y_{1,n+1}\right) dx \,,$$

$$K = e^{\alpha\, t_0}, \quad n = 0, 1, 2, \cdots .$$

Results and Illustrative Examples:

In this section, we apply the above technique to different examples of second order differential equations with exponential behavior in the right hand side of the equation or implicitly in the complementary function.

Example (1): Consider the initial value problem of second order differential equation

$$y'' + 4y' + 4y = e^{-2x}, \; 0 \le x \le 1, \; y(0) = 1 \,, y'(0) = -1 \qquad (22)$$

The nonhomogeneous term is a part of the complementary function of the differential equation, the exact solution is:

$$y = \left(1 + x + \frac{x^2}{2}\right) e^{-2x}, \qquad (23)$$

The corresponding system is:

$$y_1' = y_2, \qquad y_1(0) = 1, \qquad (24)$$

$$y_2' = e^{-2x} - 4y_2 - 4y_1, \quad y_2(0) = -1.$$

Accordingly, the corresponding Picard method is

$$y_{1,n} = 1 + \int_0^x y_{2,n-1}\, dt \,, \qquad (25)$$

$$y_{2,n} = -1 + \int_0^x \left(e^{-2t} - 4\, y_{2,n-1} - 4\, y_{1,n-1}\right) dt \,, \quad n = 1,2,3, \ldots ,$$

the corresponding Picard method with Gauss Seidel is

$$y_{1,n} = 1 + \int_0^x y_{2,n-1}\, dt \qquad (26)$$

$$y_{2,n} = -1 + \int_0^x \left(e^{-2t} - 4\, y_{2,n-1} - 4\, y_{1,n}\right) dt \,, \quad n = 1,2,3, \ldots ,$$

the corresponding Picard with integrating factor is

$$y_{1,n} = 1 + \int_0^x y_{2,n-1}\, dt \qquad (27)$$

$$y_{2,n} = -e^{-4x} + \int_0^x e^{4(t-x)} \left(e^{-2t} - 4\, y_{1,n-1}\right) dt \,, \quad n = 1,2,3, \ldots$$

and the corresponding Picard modified by Gauss Seidel with integrating factor is

$$y_{1,n} = 1 + \int_0^x y_{2,n-1}\, dt \;, \qquad (28)$$

$$y_{2,n} = -e^{-4x} + \int_0^x e^{4(t-x)} \left(e^{-2t} - 4\, y_{1,n}\right) dt \,, \quad n = 1,2,3, \ldots .$$

The results of the Picard method as given by formula (25), the Picard method modified with Gauss Seidel as given by formula (26), the Picard method modified with integrating factor as given by formula (27) and the Picard method modified with integrating factor with Gauss Seidel as given by formula (28), are summarized along the interval [0, 1] in the following tables:

Table 1: The numerical results with only seven iterations.

x_i	*Exact* y	*Picard* $y_{1,7}$	*Picard-Gauss S.* $y_{1,7}$	*Picard with I.F* $y_{1,7}$	*Picard- G.S. with I.F* $y_{1,7}$
0.1	0.904697	0.904697	0.904697	0.904697	0.904697
0.2	0.81779	0.81779	0.81779	0.81779	0.81779
0.3	0.738152	0.738152	0.738144	0.738152	0.738152
0.4	0.665007	0.665012	0.664927	0.665007	0.665007
0.5	0.597804	0.597835	0.597307	0.597807	0.597804
0.6	0.536126	0.53626	0.533887	0.536137	0.536126
0.7	0.479631	0.48009	0.471588	0.479665	0.479631
0.8	0.428021	0.429353	0.403523	0.428108	0.428021
0.9	0.381014	0.384422	0.315237	0.381212	0.381014
1.0	0.338338	0.346225	0.178454	0.338746	0.338338

Table 2: The numerical results with eleven seven iterations.

x_i	*Exact* y	*Picard* $y_{1,11}$	*Picard-Gauss S.* $y_{1,11}$	*Picard with I.F* $y_{1,11}$	*Picard- G.S. with I.F* $y_{1,11}$
0.1	0.904697	0.904697	0.904697	0.904697	0.904697
0.2	0.81779	0.81779	0.81779	0.81779	0.81779
0.3	0.738152	0.738152	0.738152	0.738152	0.738152
0.4	0.665007	0.665007	0.665007	0.665007	0.665007
0.5	0.597804	0.597804	0.597803	0.597804	0.597804
0.6	0.536126	0.536126	0.536119	0.536126	0.536126
0.7	0.479631	0.479631	0.479585	0.479631	0.479631
0.8	0.428021	0.428022	0.427776	0.428021	0.428021
0.9	0.381014	0.381019	0.379948	0.381014	0.381014
1.0	0.338338	0.338357	0.334336	0.338339	0.338338

From tables (1, 2) we find that the use of integrating factor with Gauss Seidel approach gives results similar to the exact solution after only seven steps in comparison with at least eleven steps when using Gauss Seidel without integrating factor moreover the results of the modified Picard method with Gauss Seidel improved by the integrating factor does not change during the calculations from the seventh's iteration until the elevens iteration, which means that we have obtained the fixed point.

Example (2): Consider the initial value problem

$$y'' + y' - 2y = 2x + e^x(x^2 - 1),\ \ 0 \le x \le 1,\ \ y(0) = 0, y'(0) = 1. \quad (29)$$

The nonhomogeneous term contains a part of the complementary function of the differential equation multiplied by a second degree polynomial plus another simple function, the exact solution is:

$$y = \frac{88}{81}e^x - \frac{95}{162}e^{-2x} - x - \frac{1}{2} + x\,e^x\left(\frac{x^2}{9} - \frac{x}{9} - \frac{7}{27}\right) \quad (30)$$

The corresponding system takes the form:

$$y_1' = y_2, \qquad y_1(0) = 0, \quad (31)$$

$$y_2' = 2x + e^x(x^2 - 1) - y_2 + 2y_1, \quad y_2(0) = 1,$$

the corresponding Picard modified by Gauss Seidel with integrating factor is

$$y_{1,n} = \int_0^x y_{2,n-1}\,dt\,,\ \ y_{1,0}(0) = 0,\ \ n = 1,2,\ldots \quad (32)$$

$$y_{2,n} = e^{-x} + \int_0^x e^{(t-x)}\left(2\,t + e^t(t^2 - 1) + 2\,y_{1,n}\right)dt\,,$$

$$y_{2,0}(0) = 1.$$

Table 3: The numerical results with only six iterations.

x_i	*Exact* y	*Picard* $y_{1,6}$	*Picard-Gauss S.* $y_{1,6}$	*Picard with I.F* $y_{1,6}$	*Picard- G.S. with I.F* $y_{1,6}$
0.1	0.0908019	0.0908019	0.0908019	0.0908019	0.0908019
0.2	0.166192	0.166192	0.166192	0.166192	0.166192
0.3	0.230241	0.230239	0.230241	0.230241	0.230241
0.4	0.286632	0.286613	0.286632	0.28663	0.286632
0.5	0.33885	0.338763	0.338848	0.33884	0.33885
0.6	0.390365	0.390065	0.390361	0.390334	0.390365
0.7	0.444821	0.443962	0.444812	0.444735	0.444821
0.8	0.50623	0.504101	0.506211	0.506022	0.50623
0.9	0.579183	0.574459	0.57915	0.578734	0.579183
1.0	0.669092	0.659482	0.669038	0.668202	0.669092

Table 4. The numerical results with minimum number of iterations which gives the exact solution for each method.

x_i	*Exact* y	*Picard* $y_{1,13}$	*Picard-Gauss S* $y_{1,9}$	*Picard with I.F* $y_{1,10}$	*Picard- G.S. with I.F* $y_{1,6}$
0.1	0.0908019	0.0908019	0.0908019	0.0908019	0.0908019
0.2	0.166192	0.166192	0.166192	0.166192	0.166192
0.3	0.230241	0.230241	0.230241	0.230241	0.230241
0.4	0.286632	0.286632	0.286632	0.286632	0.286632
0.5	0.33885	0.33885	0.33885	0.33885	0.33885
0.6	0.390365	0.390365	0.390365	0.390365	0.390365
0.7	0.444821	0.444821	0.444821	0.444821	0.444821
0.8	0.50623	0.50623	0.50623	0.50623	0.50623
0.9	0.579183	0.579183	0.579183	0.579183	0.579183
1.0	0.669092	0.669092	0.669092	0.669092	0.669092

From table (3) one can see that the use of integrating factor with Gauss Seidel approach gives results similar to the exact solution after only six iterations. Table (4) illustrates the number of different iterations required to obtain the same results with the different methods.

Example (3): Consider the initial value problem

$$y'' - 2y' + y = 4 + xe^x,\ \ 0 \le x \le 1,\ \ y(0) = 1\ y'(0) = 1. \quad (33)$$

The nonhomogeneous term contains part of the complementary function of the differential equation multiplied by a first degree polynomial plus another simple term. The exact solution is:

$$y = (\,4x - 3)e^x + \frac{x^3}{6}e^x + 4. \quad (34)$$

The system of the IVP takes the form

$$y_1' = y_2, \qquad y_1(0) = 0, \quad (35)$$

$$y_2' = 2x + e^x(x^2 - 1) - y_2 + 2y_1, \quad y_2(0) = 1\,.$$

the corresponding Picard modified by Gauss Seidel with integrating factor is

$$y_{1,n} = \int_0^x y_{2,n-1}\,dt\,,\ \ y_{1,0}(0) = 1, \quad n = 1,2,\ldots \quad (36)$$

$$y_{2,n} = e^{-x} + \int_0^x e^{2(t-x)}\left(4 + te^t - \,y_{1,n}\right)dt\,, \qquad y_{2,0}(0) = 1.$$

From table (5) we find that the integrating factor approach with Gauss Seidel gives results similar to the exact solution after only six iterations. Table (6) illustrates the number of different iterations required to obtain the same results with the other methods.

In Table (5-6) we introduce the comparison between the Picard's solution and the approximate solution by using the proposed technique.

Table 5. The numerical results with only six iterations.

x_i	*Exact* y	*Picard* $y_{1,6}$	*Picard-Gauss S.* $y_{1,6}$	*Picard with I.F* $y_{1,6}$	*Picard- G.S. with I.F* $y_{1,6}$
0.1	1.12674	1.12674	1.12674	1.12674	1.12674
0.2	1.31454	1.31454	1.31454	1.31454	1.31454
0.3	1.57633	1.57633	1.57625	1.57632	1.57633
0.4	1.92736	1.92735	1.9269	1.92732	1.92736
0.5	2.38563	2.38558	2.38388	2.38546	2.38563
0.6	2.97232	2.97217	2.96711	2.97177	2.97232
0.7	3.71237	3.7119	3.69918	3.71084	3.71237
0.8	4.63502	4.63381	4.60556	4.63129	4.63502
0.9	5.7746	5.77179	5.71468	5.76629	5.7746
1.0	7.17133	7.16533	7.05815	7.15411	7.17133

Table 6. The numerical results with minimum number of iterations which gives the exact solution for each method.

x_i	*Exact* y	*Picard* $y_{1,10}$	*Picard-Gauss S* $y_{1,13}$	*Picard with I.F* $y_{1,9}$	*Picard-Green G.S.* $y_{1,6}$
0.1	1.12674	1.12674	1.12674	1.12674	1.12674
0.2	1.31454	1.31454	1.31454	1.31454	1.31454
0.3	1.57633	1.57633	1.57633	1.57633	1.57633
0.4	1.92736	1.92736	1.92736	1.92736	1.92736
0.5	2.38563	2.38563	2.38563	2.38563	2.38563
0.6	2.97232	2.97232	2.97232	2.97232	2.97232
0.7	3.71237	3.71237	3.71237	3.71237	3.71237
0.8	4.63502	4.63502	4.63502	4.63502	4.63502
0.9	5.7746	5.7746	5.7746	5.7746	5.7746
1.0	7.17133	7.17133	7.17133	7.17133	7.17133

Example (4): Consider the initial value problem

$$x_1' = 5x_1 - x_2,\ \ x_1(0) = 3, \qquad 0 \le t \le 1, \quad (37)$$
$$x_2' = 3x_1 + x_2, \quad x_2(0) = 2.$$

whose exact solution is

$$x_1 = \frac{-1}{2}e^{2t} + \frac{7}{2}e^{4t}, \qquad (38)$$
$$x_2 = \frac{-3}{2}e^{2t} + \frac{7}{2}e^{4t}.$$

The corresponding Picard iteration is given by

$$x_{1,n} = x_{1,0} + \int_0^t \left(5x_{1,n-1} - x_{2,n-1}\right) dt\,, \qquad (39)$$
$$x_{2,n} = x_{2,0} + \int_0^t \left(3x_{1,n-1} + x_{2,n-1}\right) dt.$$

The modified Picard iteration with Gauss-Seidel method is

$$x_{1,n} = x_{1,0} + \int_0^t \left(5x_{1,n-1} - x_{2,n-1}\right) dt\,, \qquad (40)$$
$$x_{2,n} = x_{2,0} + \int_0^t \left(3x_{1,n} + x_{2,n-1}\right) dt.$$

The corresponding Picard modified by Gauss Seidel with integrating factor is

$$x_{1,n} = x_{1,0}\, e^{5t} - \int_0^t e^{5(t-s)} \left(x_{2,n-1}(s)\right) ds\,, \qquad (41)$$
$$x_{2,n} = x_{2,0}\, e^{t} + 3\int_0^t e^{(s-t)} x_{1,n}(s)\, ds.$$

Table 7. The numerical results with only six iterations.

t_i	*Exact* x_1	*Picard* $x_{1,6}$	*Picard-Gauss S.* $x_{1,6}$	*Picard with I.F* $x_{1,6}$	*Picard- G.S. with I.F* $x_{1,6}$
0	3	3	3	3	3
0.1	4.61069	4.61068	4.61068	4.61069	4.61069
0.2	7.04348	7.04332	7.04307	7.04348	7.04348
0.3	10.7093	10.7064	10.7022	10.7094	10.7093
0.4	16.2228	16.1997	16.1688	16.2232	16.2228
0.5	24.5026	24.3854	24.2408	24.5049	24.5026
0.6	36.9211	36.4742	35.9663	36.9322	36.9211
0.7	55.5287	54.1249	52.6606	55.5732	55.5287
0.8	83.3873	79.5595	75.9048	83.5422	83.3873
0.9	125.069	115.691	107.523	125.553	125.069
1.0	187.399	166.267	149.533	188.791	187.399

A Comparison of the exact solution for x_2 with its successive approximations

Table 8. The numerical results with only six iterations.

t_i	*Exact* x_2	*Picard* $x_{2,6}$	*Picard-Gauss S.* $x_{2,6}$	*Picard with I.F* $x_{2,6}$	*Picard- G.S. with I.F* $x_{2,6}$
0	2	2	2	3	2
0.1	3.38928	3.38928	3.38928	3.38928	3.38928
0.2	5.55166	5.5515	5.55162	5.55166	5.55166
0.3	8.88723	8.88432	8.88623	8.88726	8.88723
0.4	13.9973	13.9742	13.9871	13.9976	13.9973
0.5	21.7843	21.6674	21.7228	21.7863	21.7843
0.6	33.6009	33.1549	33.3319	33.6106	33.6009
0.7	51.4735	50.0723	50.5323	51.5124	51.4735
0.8	78.4343	74.6131	75.6351	78.5699	78.4343
0.9	119.019	109.657	111.659	119.445	119.019
1.0	180.01	158.911	162.435	181.238	180.01

Table 9. The numerical results with different number of iterations for each method.

t_i	*Exact* x_1	*Picard* $x_{1,12}$	*Picard-Gauss S.* $x_{1,12}$	*Picard with I.F* $x_{1,11}$	*Picard- G.S.with I.F* $x_{1,12}$
0	3	3	3	3	3
0.1	4.61069	4.61069	4.61069	4.61069	4.61069
0.2	7.04348	7.04348	7.04348	7.04348	7.04348
0.3	10.7093	10.7093	10.7093	10.7093	10.7093
0.4	16.2228	16.2228	16.2228	16.2228	16.2228
0.5	24.5026	24.5026	24.5025	24.5026	24.5026
0.6	36.9211	36.921	36.9206	36.9211	36.9211
0.7	55.5287	55.5282	55.5255	55.5287	55.5287
0.8	83.3873	83.3847	83.3696	83.3873	83.3873
0.9	125.069	125.056	124.989	125.069	125.069
1.0	187.399	187.347	187.092	187.399	187.399

Table 10. The numerical results with different number of iterations for each method.

t_i	Exact x_2	Picard $x_{2,11}$	Picard-Gauss S. $x_{2,12}$	Picard with I.F $x_{2,12}$	Picard- G.S. with I.F $x_{2,12}$
0	2	2	2	2	2
0.1	3.38928	3.38928	3.38928	3.38928	3.38928
0.2	5.55166	5.55166	5.55166	5.55166	5.55166
0.3	8.88723	8.88723	8.88723	8.88723	8.88723
0.4	13.9973	13.9973	13.9973	13.9973	13.9973
0.5	21.7843	21.7843	21.7843	21.7843	21.7843
0.6	33.6009	33.6009	33.6009	33.6009	33.6009
0.7	51.4735	51.473	51.4728	51.4735	51.4735
0.8	78.4343	78.4316	78.4304	78.4343	78.4343
0.9	119.019	119.007	119.	119.019	119.019
1.0	180.01	179.958	179.925	180.01	180.01

Discussion

The fundamental objective of this work is to find some multipliers that can be used to accelerate the convergence of the Picard iteration method. We find that the ideas of integrating factors can be used to collect some terms in a single perfect differential term. We used the ideas introduced by Yildiz[10] to decompose the equation, and define integrating factor to the linear part of the decomposed equation. Also, we considered the Gauss Seidel treatment introduced in, Youssef[3].

The goal has been achieved successfully. The new modified Picard iteration method is relatively straightforward to apply at least with the assistance of a powerful computer algebra packages and in simple cases it gives exact solutions and in most cases it gives a series that converges rapidly to the unique solution. The method presented here in addition to its deeply mathematical roots is easier straightforward in comparison with the other mentioned techniques and it gives the same results as in Picard's method and Taylor's method with smaller number of iterations as shown from the tables. Moreover, the calculated results illustrate that, we obtain the theoretical fixed point, there is no change in the values with more iterations.

The accuracy of the new modified Picard iteration method has been confirmed by comparison with the exact solution as shown in the tables.

The convergence of the method has been confirmed by comparison with the Picard method, our treatment gives the exact solution or at least the solution of the Picard iteration modified by the Gauss Seidel[3].

In comparison with the power series method, which requires the right hand side of the equation to be analytic, whereas for using the method of successive approximations the analyticity of the right-hand side is not obligatory? Therefore, the method of successive approximations is generally speaking, more widely used: it also used when the expansion of the solution of a differential equation in a power series is impossible[3]. But this method, unfortunately, has its own shortcoming, which consists in that it calls for the necessity to compute more and more cumbersome integrals.

This approach is promising and will help in treating boundary value problems and other applications in partial differential equations. In a next subsequent work we will try to use this approach to problems with variable coefficients, nonlinear types and also we try to find some general forms for the multipliers even for standard classes of differential equations.

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