ANALYTIC SOLUTION OF TWO-DIMENSIONAL TRANSIENT HEAT CONDUCTION

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حل تحليلى لإنتقال الحراره الغير مستقر ثنائى الأبعاد

يتم فى هذا البحث دراسة إنتقال الحراره خلال سطح زعنفه ثنائية الابعاد ، تتبادل الحرارة مع الوسط المحيط عبر سطحيها الاعلى والاسفل، عند تعرض جذرها إلى تغير مفاجئ فى درجة الحرارة مع الاخذ فى الإعتبار التغير الزمنى فى درجات الحرارة.. وقد تم التوصل إلى حل تحليلى خالص بدون أى فروض أو تبسيط رياضى للمعادلات المتحكمة فى توزيع درجات الحراره خلال الوسط.. والحل المستنتج يعطى علاقة مباشرة بين درجة الحرارة وكل من الإحداثى الطولى والإحداثى المتعامد والزمن بحيث يمكن إيجاد قيمة درجة الحرارة عند أى نقطة داخل الوسط عند أى قيمة للزمن بدون الحاجة إلى معرفة قيم درجات الحراره عند أى نقطه اخرى أو زمن سابق.. والعلاقة المستنتجة فى هذاالبحث يمكن تطبيقها على أى وسط ثنائى الابعاد حيث تعتمد على النسبة بين إرتفاع الزعنفه و طولها.. وبمقارنة توزيع درجات الحراره لمجموعات من المتغيرات بنتائج الطرق التحليلية التقريبيه والطرق العددية ، تم التحقق من ان الحل المستنتج فى هذا البحث يعكس بوضوح تأثير المتغيرات الفيزيائية المختلفة ويعطى نتائج دقيقة فى زمن حسابى اقل.

ABSTRACT

Analysis of the transient heat conduction in a two-dimensional rectangular fin, which exchanges heat with th
surroundings by convection, is considered. Through the
of finite integral transform, a complete analytic s
is obtained. The solution is compared to other
from a numerical method and from an approximat
method. The analytic solution presented refl
of different physical parameters, is accur
rapidly.

INTRODUCTION

The use of extended surf
of heat exchange has been
Finned surfaces are wid
electric transformers
and other heat t
cooling electror
equipments. T
phenomenon in

* Assistant Pr
** Associate Pro

Efforts has been paid to analysis transient heat conduction in two-dimensional fins. However, most researchers tends to simplify this problem to the one-dimensional one on the assumption that conduction in the transverse direction is much smaller than that in the longitudinal direction. This assumption, however, is restricted to cases in which Biot number is small [1].

Relatively little efforts has been directed to obtain analytical solution to transient heat conduction in two-dimensional fins. Among others, Chu [2] studied the unsteady heat conduction in straight fin using the Laplace transform and separation of variables techniques and obtained a solution which converges very slowly at small times. Chu et al [3] studied the same problem using Laplace transformation and utilized Fourier series to obtain the inverse of Laplace transformation. Because the series converges very slow, hundreds of terms are retained in the solution in order to obtain numerical results accurate to three significant figures.

More recently, Yi-Hsu et al [4] used a regular perturbation technique and an averaging method to reduce the problem of 2-D rectangular fin to that of a 1-D problem. A linear operator method was then used to obtain the solution. The assumption used is valid for thin fins as the temperature distribution predicted using this method tends to deviate from the actual distribution when the thickness of the fin gets larger.

In this paper we present an analytical solution to the transient heat conduction in a two dimensional rectangular domain. No mathematical simplifications are attempted. The obtained solution accounts for the temperature variation in the transverse direction as well as the temperature variation in the longitudinal direction. The solution obtained from this method is valid for both small and large time and allows for parametric investigation of the different parameters involved.

PROBLEM FORMULATION

The transient heat
nduction in a two-dimen-
nal rectangular domain,
re (1), is considered.
surface at x=0 is
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ature, T_o, while the
at x=L is completely
d. The surface at y=0
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a heat transfer

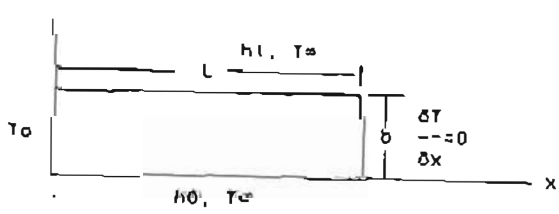

Fig.(1) Schematic of physical Problem

coefficient h_0 and the surface at y=b is convecting to the ambient with a coefficient h_1. The initial temperature of the entire domain is taken as T_∞. The material thermophysical properties are constants.

The temperature of the two-dimensional domain given above is described by the following differential equation, written in a non-dimensional form:

$$\theta_\tau(\eta,\xi,\tau)-\theta_{\eta\eta}(\eta,\xi,\tau)-\frac{1}{\epsilon^2}\theta_{\xi\xi}(\eta,\xi,\tau)=0, \qquad (1)$$

Subject to the following dimensionless boundary conditions

$$\theta(0,\xi,\tau) = 0 \qquad (2a)$$
$$\theta_\eta(1,\xi,\tau) = 0 \qquad (2b)$$
$$\theta_\xi(\eta,0,\tau)-B_0\theta(\eta,0,\tau)=-B_0 \qquad (2c)$$
$$\theta_\xi(\eta,1,\tau)+B_1\theta(\eta,1,\tau)=B_1 \qquad (2d)$$

and the initial condition

$$\theta(\eta,\xi,0) = 1, \qquad 0<\eta<1, \; 0<\xi<1. \qquad (2e)$$

Where, the subscripts η and $\eta\eta$ represents the first and second partial derivatives, respectively. The same with respect to ξ and $\xi\xi$.

In these equations, B_0 and B_1 are Biot numbers at the lower and upper surfaces, respectively, and are defined as

$$B_0=\frac{h_0 b}{k} \qquad \text{and} \qquad B_1=\frac{h_1 b}{k} \qquad (3a)$$

The other dimensionless parameters are defined as follows:

$$\eta=\frac{x}{L}, \qquad \xi=\frac{y}{L}, \qquad \tau=\frac{kt}{\rho c L^2} \qquad (3b)$$

$$\epsilon=\frac{b}{L}, \qquad \theta=\frac{T(x,y,t)-T_a}{T_\infty-T_a} \qquad (3c)$$

The problem considered is general as it analyzes the transient heat transfer in any two-dimensional rectangular domain subject to different kinds of boundary conditions. For instance, when the ratio ϵ is small the problem tends to the case of two-dimensional rectangular fin. In addition, the boundary conditions in the transverse direction are nonsymmetrical. However, the symmetric case can be obtained by setting B_0=B_1. By setting any of B_0 or B_1 equal zero, we obtain the case of complete insultaion at the respective boundary.

SOLUTION METHODOLOGY

The system described by eqs.(1) and (2) is now solved using the finite integral transform method (FIT). The solution procedure is initiated by considering the homogeneous problem associated with the given system. Then employing the generalized FIT technique [5], we recast the given equations into a system of first order ordinary differential equations in the transform variable which is easily solved. The obtained transformed variable is then inverted back into the dimensionless field variable (temperature). These steps are shown in the following sections.

Associated Problem

The homogeneous problem associated with eqs. (1) and (2) is taken as

$$\theta_\tau(\eta,\xi,\tau) - \theta_{\eta\eta}(\eta,\xi,\tau)-\frac{1}{\varepsilon^2}\theta_{\xi\xi}(\eta,\xi,\tau)=0, \tag{4}$$

and the auxiliary conditions are

$$\theta(0,\xi,\tau) = 0 \tag{5a}$$
$$\theta_\eta(1,\xi,\tau) = 0 \tag{5b}$$
$$\theta_\xi(\eta,0,\tau)-B_0\theta(\eta,0,\tau)= 0 \tag{5c}$$
$$\theta_\xi(\eta,1,\tau)+B_1\theta(\eta,1,\tau)= 0 \tag{5d}$$

and

$$\theta(\eta,\xi,0) = 1, \quad 0<\eta<1, \ 0<\xi<1. \tag{5e}$$

Employing the principle of separation of variables on the associated problem yields the following Sturm-Liuoville problems.

$$X_{\eta\eta} + \mu_n^2 X = 0 \tag{6a}$$
$$X(0) = 0 \tag{6b}$$
$$X_\eta(1) = 0 \tag{6c}$$

and

$$Y_{\xi\xi} + \nu_n^2 Y = 0 \tag{7a}$$
$$Y_\xi(0) - B_0 Y(0) = 0 \tag{7b}$$
$$Y_\xi(1) - B_1 Y(1) = 0 \tag{7c}$$

The eigenfunctions associated with the problem given by eq.(6) is

$$X_n(\eta) = \sin\mu_n\eta \tag{8a}$$

and the eigenvalues are the positive roots of the transcendental equation

$$\cos\mu_n = 0. \tag{8b}$$

The normalization integral for this problem is found to be

$$N(\mu_n) = \frac{1}{2}. \tag{8c}$$

The eigenfunctions, the transcendental equation for the eigenvalues and the normalized integral, respectively, associated with the problem given by eq.(7) are

$$Y_n(\xi) = \nu_n \cos(\nu_n \xi) + B_o \sin(\nu_n \xi) \tag{9a}$$

$$\tan(\nu_n) = \frac{\nu_n(B_o+B_1)}{\nu_n^2 - B_o B_1} \tag{9b}$$

$$N(\nu_n) = \frac{1}{2}\left[(\nu_n^2+B_o^2)(1+\frac{B_1}{\nu_n^2+B_1^2})+B_o\right] \tag{9c}$$

Transformation Pairs

Making use of the orthogonality properties of the eigenfunctions given in eq.(8a) and (9a), we can define the transformation pairs needed for the solution as follows.

The transformation pair for $\theta(\eta,\xi,\tau)$ with respect to the η variable is taken as:

Integral Transform:

$$\bar{\theta}(\mu_n,\xi,\tau) = \int_0^1 \theta(\eta',\xi,\tau) X_n(\eta') d\eta' \tag{10a}$$

Inversion Formula:

$$\theta(\eta,\xi,\tau) = \sum_{n=1}^{\infty} \frac{X_n(\eta)\bar{\theta}(\mu_n,\xi,\tau)}{N(\mu_n)} \tag{10b}$$

The transformation pair for the $\bar{\theta}(\eta_n,\xi,\tau)$ with respect to ξ variable is taken as

Integral Transform:

$$\Psi(\mu_n,\nu_m,\tau) = \int_0^1 \bar{\theta}(\mu_n,\xi',\tau) Y_m(\xi') d\xi' \tag{11a}$$

Inversion Formula:

$$\bar{\theta}(\mu_n,\xi,\tau) = \sum_{m=1}^{\infty} \frac{Y_m(\xi)\Psi(\mu_n,\nu_m,\tau)}{N(\nu_m)} \tag{11b}$$

Problem Transformation and solution

The solution of the problem given by eqs.(1)-(2) is obtained by the successive application of the one-dimensional integral transforms to remove from the equation one of the partial derivatives with respect to the space variable in each step. Thus, taking the integral transform of the problem first with respect to the η variable using the transform (10a) and then with respect to the ξ variable using the transform (11a), we obtain,

$$\frac{d\Psi}{d\tau}(\eta_n,\nu_m,\tau) + (\eta_n^2+\frac{\nu_m^2}{\epsilon^2})\Psi(\eta_n,\nu_m,\tau) = \frac{\Pi(\eta_n,\nu_m)}{\epsilon^2} \quad (12a)$$

subject to the transformed initial condition

$$\Psi_o(\eta_n,\nu_m,0) = \frac{1-\cos(\eta_n)}{\eta_n}\,[\sin\nu_m + \frac{B_o}{\nu_m}(1-\cos\,\nu_m)] \quad (12b)$$

where,

$$\Pi(\eta_n,\nu_m) = \frac{1-\cos(\eta_n)}{\eta_n}\,[B_1\nu_m\cos(\nu_m) + B_oB_1\sin\nu_m + B_o\nu_m] \quad (13)$$

Equation (12) is solved and successively inverted by the inversion formulas (10b) and (11b) to find the solution of the problem described by eqs.(1)-(2) as

$$\theta(\eta,\xi,\tau) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{X_n(\eta)Y_m(\xi)\Psi(\eta_n,\nu_m,\tau)}{N(\eta_n)N(\nu_m)} \quad (14)$$

All terms on the R.H.S. of eq.(14) are defined in the preceding sections. The summations in eq.(14) runs from 1 to ∞, however; a finite number of the eigen values needs to be retained in the solution.

RESULTS AND DISCUSSION

The transient temperature distribution in a two dimensional rectangular domain is obtained from the solution formula given in eq.(14). This formula posses several features: First, it satisfies the boundary conditions given by eqs.(5). For instance, eq.(5a) is satisfied through the substitution of eq.(8a) at $\eta=0$ into eq.(14). Likewise, eq.(5b) is satisfied through the utilization of eq.(6c) in the first partial derivative of eq.(14) with respect to η. In the same way, it can be shown that eq.(14) satisfies the conditions given by eqs.(5c) and (5d). Second, the steady state solution is obtainable for eq.(14) by noticing that Ψ in eq.(12a) is, in this case, independent of τ and therefore, θ depends only on η and ξ. Third, the solution as derived in this paper admits for the symmetric cooling (when $B_0=B_1$) as well as the nonsymmetrical one. It is also valid for the analysis of thin fins (when ϵ is small) and regular rectangular domains (large ϵ). Fourth, the temperature at any values of η, ξ, and/or τ can be obtained without knowledge of the temperature at any other location or time.

In summary, the solution obtained here posses the characteristics of analytic solutions, even though it is calculated over a summation of a finite number of eigenvalues. Five eigenvalues in each direction were enough to obtain accurate results at large values of time (τ>0.1). But up to ten eigenvalues were needed to achieve the same accuracy at early time. However, the time needed to compare the temperature distribution in the entire domain was very small (in order of few seconds). Therefore ten eigenvalues were retained in the solution to get the results shown in this work.

We now proceed in comparing results obtained in this work to those presented in the literature. A numerical solution to the 2-D symmetric convectively cooled fin [2] and an analytic solution to the 1-D approximation of the 2-D case [1] are used in the comparison. Fig.(2a) shows the temperature profile of the fin center when Biot number equal 0.1 at both the top and the bottom surfaces of the fin, for different values of time. The plots in the figure agrees well with results presented in [1], which is reproduced here in Fig.(2b). One observes that the curve at t=4 is a little above that of t=1 in Fig.(2a) whereas the two curves coincide in Fig.(2b), which represents results from a 1-D approximation of the 2-D problem.

The temperature distribution at the surface and center of the fin is shown in Fig.(3). Fig.(3a) displays the results obtained from eq.(14) and Fig.(3b) shows the results obtained from the 1-D approximation. Our results compares well with the results obtained from the numerical solution.

A further validation of our analytic solution is done by comparing the effect different Biot numbers on the temperature profiles at the fin center at given times, Fig(4). Fig.(4b) displays the results obtained in this work which agree with the results obtained in [1], reproduced in Fig.(4a), for a thin fin (ϵ=0.1). Figs.(4c)and (4d) show same effects at different values of time. The plots in Fig.(4) reflects the physical phenomena as expected. For small values of Biot number, the heat exchange due to convection is insignificant and thus the temperature inside the fin medium (being initially at the ambient temperature) cools off due to the sudden change in temperature at the fin root, (θ=0). For higher Biot numbers, the convection heat transfer plays a noticeable role in making up for the heat lost at the root, and thus maintains higher temperature in the medium. This is reflected by the curves shown in Figs(4b-d). This phenomenon gets more obvious as time proceeds and the steady state case gets established.

The development of temperature profile at the center and at the surface of the fin are shown in Figs.(5a) and (5b) for $\epsilon=0.5$ and $\epsilon=1.0$, respectively. as time progresses from $\tau=0.1$ to $\tau=1.0$, a significant deviation between the two curves occurs. For such cases, the 1-D approximations, which is based on averaging the temperature in the transverse direction [1], is expected to predict the temperature inaccuratly. Stated differently, the approximation may acceptably predict the temperature when ϵ and/or τ are relatively small and is restricted for small Biot numbers [6].

The temperature profile in the transverse direction is displayed in Fig.(6) at different locations along the η direction, namely, at $\eta=0.01$, 0.1, and 1. Fig.(7) displays the temperature history at the same locations. From these two figures, it is noticed that as the time increases the temperature in the fin is reaching the steady state distribution.

CONCLUSION

A complete analytical solution to the transient heat transfer through a 2-D rectangular domain was presented. The solution is valid for both small and large time. This analytic solution overcomes the inaccuracy problem associated with the approximate solutions. The form of the solution presented, being analytic, allows for parametric studies and thus gives insight of the effect of the parameters involved, which is obtained by numerical methods on the expense of computational time. Comparison of the results reveals that the solution method presented in this paper is efficient, accurate and computationally rapid.

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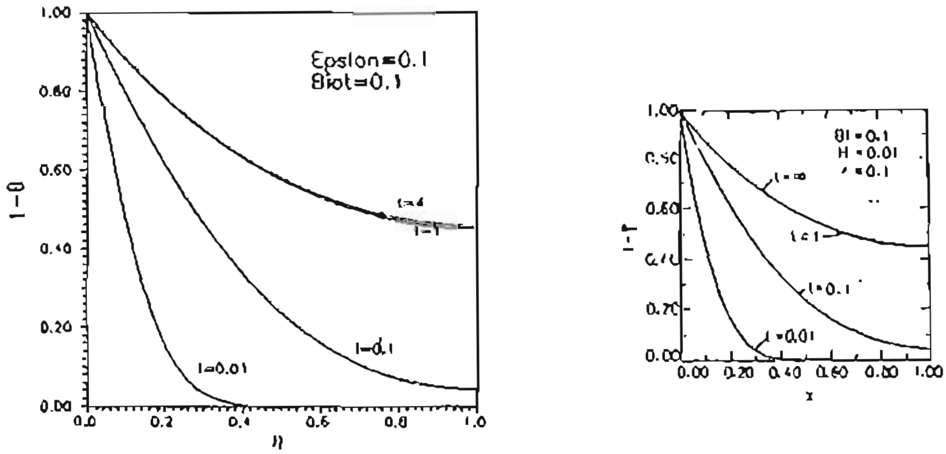
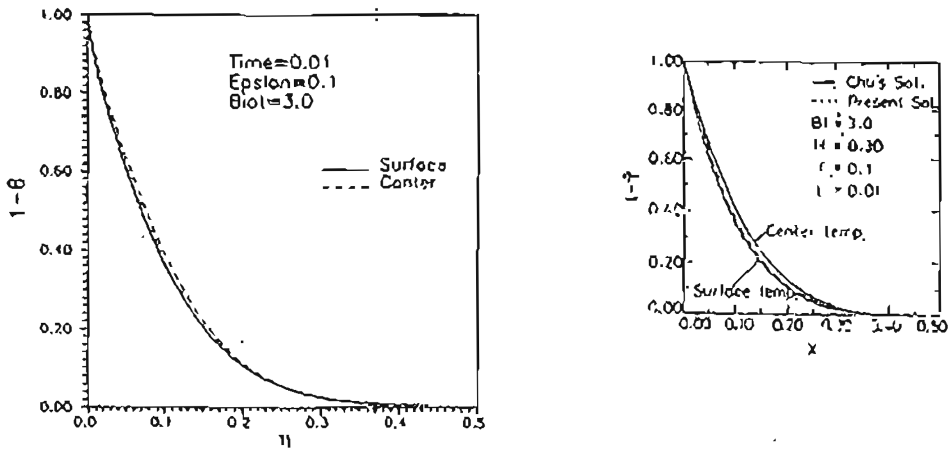


(a) Obtained from Eq.(14) (b) Reproduced from [1]

Fig (2) Temperature Profile of the Fin Center for Bi=0.1



(a) Obtained from Eq.(14) (b) Reproduced from [1]

Fig (3) Comparison of Surface and Center Temperature of the Fin for Bi=3.0

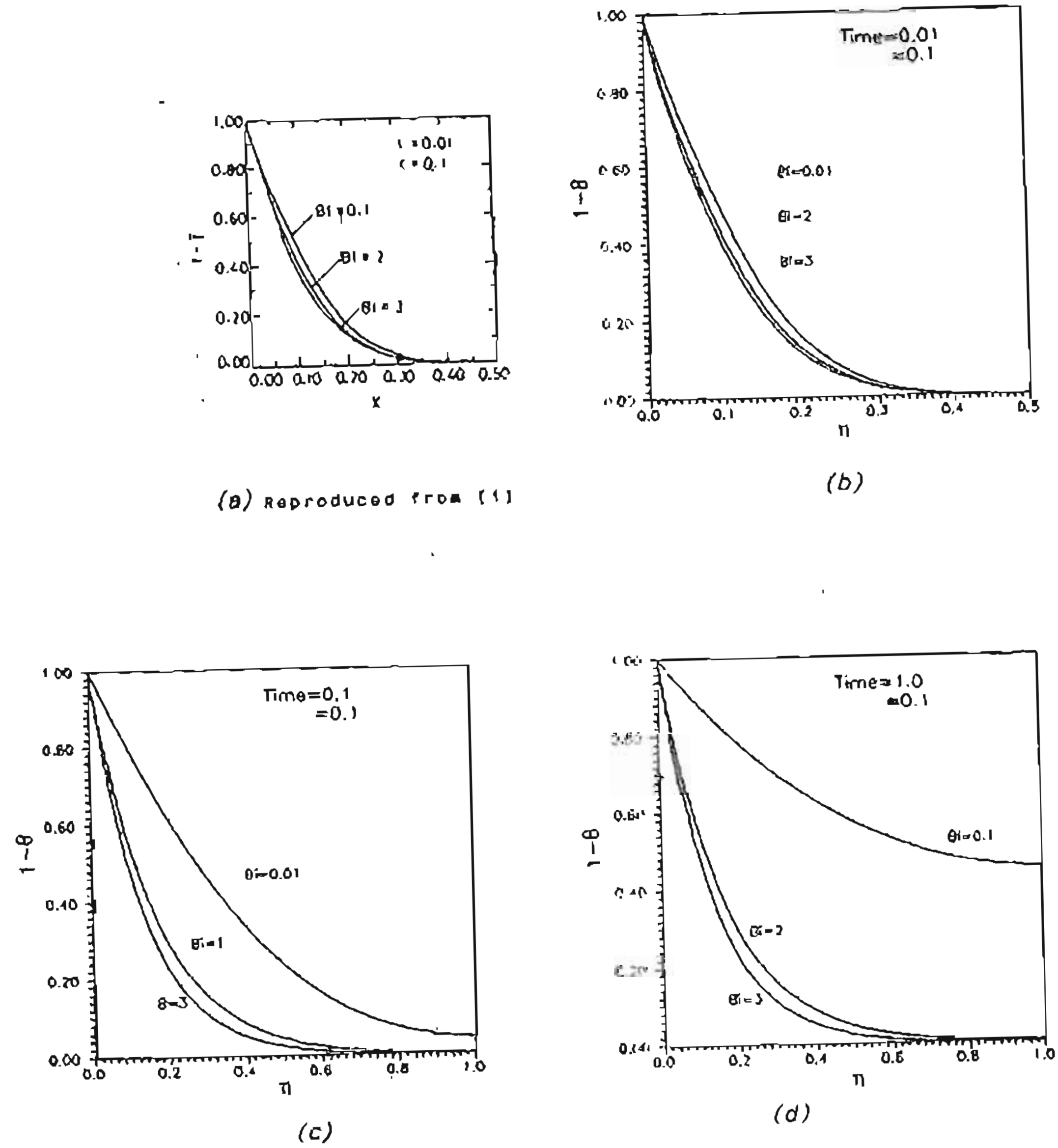


Fig (4) Effect of Biot Number on Temperature Profile at Different Times

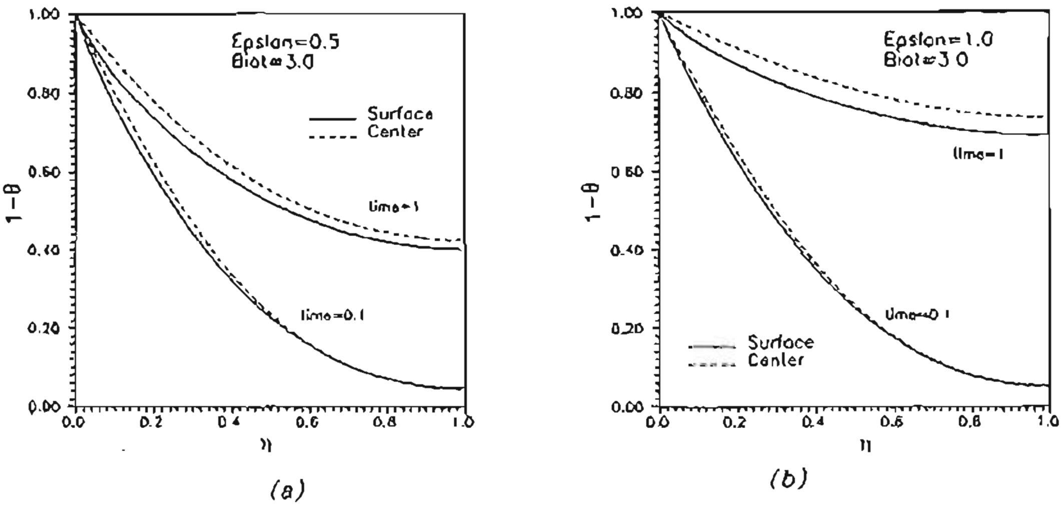


Fig (5) Effect of on the Surface and Center Temperature Profiles

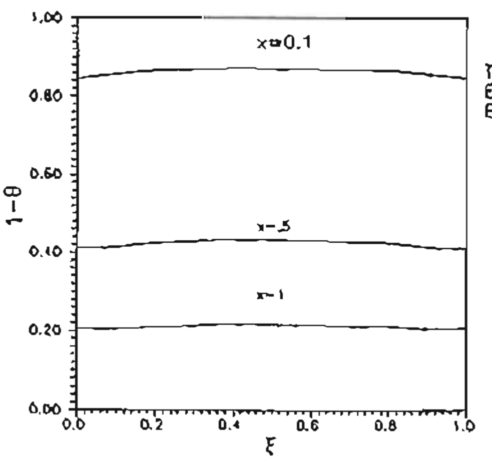


Fig (6) Temperature Variation in the Transverse Direction at Different Locations

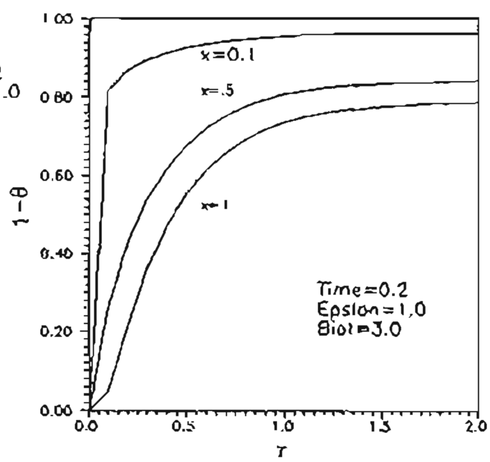


Fig (7) Temperature History of Some Locations on the Center of the Fin