Some Notes On the Concept of Seqed and Sloping Span Length in Ancient Egypt

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Abstract

  Seqed or the so-called Seked is an ancient Egyptian expression indicating the concept of the inclination, slope, slant or tendency in a particular direction that diverges or causes to diverge from the vertical or horizontal position, it also indicates the angle at which a straight line or level is inclined to another. In ancient Egypt, it points out the inclination of the triangular faces of a right pyramid. The slope length is calculated through the Pythagorean Theorem, which is completed by multiplying the square root of a plane figure with four equal straight sides and four right angles (the square) and by the sum of the squared width and the squared height. The Pythagorean theorem is calculated the sloping span length, where the slope length is studied using the vertical distance is the rise and the horizontal distance is the run, which is $\text{rise}^2 + \text{run}^2 = \text{slope length}^2$. Mathematics is the abstract science of an arithmetical value, expressed by a word, symbol, or figure, representing a particular quantity and used in counting and making calculations and for showing order in a series or for identification (Number), the amount or number of a material or immaterial thing not usually estimated by spatial measurement (Quantity), and the dimensions of height, depth, and width within which all things exist and move, or the position of two or more items at a distance from one another (Space). Mathematics can be studied in its own right as pure mathematics, or as it is applied to other disciplines such as physics and engineering or the so-called applied mathematics. The ancient Egyptian system based on the length measure called the royal cubit, which is subdivided into seven palms, where every palm is also sub-divided into four digits or numbers. Therefore, the inclination of measured slopes is expressed as the number of horizontal palms and digits relative to the rise of every royal cubit.

Keywords: Seqed, Sloping Span Length, Mathematics, Royal Cubit, Ancient Egypt

Introduction

Science is an issue of discussing information by means of identifying the struggle between fact and belief. The fact is a matter known or proved to be true, whilst the belief or opinion may be as a view or judgment established about a thing that is unspecified or unknown, which is not necessarily based on fact or knowledge. The fact in a scientific context is generally accepted reality, but still open to scientific inquiry, unlike an absolute truth that is not a part of science. The hypotheses and theories are generally based on objective inferences, unlike opinions that are generally based on subjective influences. Therefore, we can say "the facts versus the opinions or beliefs". In determining the factual length of an ancient Egyptian cubit or the true value of a *Seqed*, as discussed in the Rhind and Moscow Mathematical Papyruses, one will notice confusing differences. Varying opinions have been advanced, e.g., by Carter & Gardiner, Iversen, Hayes, Budge, Noblecourt, Lorenzen, Müller, Gardiner, Stricker, Gay, Legon, Gay & Shute, Naguib, Roik, Shaffer, and the Encyclopaedia Britannica. Possibly, though, also ancient Egyptian sources might have offered

different values. The question now is how to measure a *Seqed* (*sḳd*) in ancient Egypt accurately? And is the value of the *Seqed* or the angle θ correct applying the rules of modern trigonometry? The question arises as to whether the problems of the *Seqed* are actually mirrored in the ancient Egyptian architecture methods and construction techniques?

How is it possible that one can actually measure the *Seqed* (*sḳd*) in some pyramids whilst in others a different or adjusted method was used to determine the pyrimid's sloping angle? Although scientists differ on how to measure and determine the true value of a *Seqed*, we can assume that the ancient Egyptians depended on a natural method of measuring dimensions such as the cubit, which was used as a measure of length, approximately equal to the length of a forearm. Traditionally, this was measured from the bent elbow to the tips of the fingers, measuring approximately ≈44 cm (≈18 inches), and a long cubit of about≈52 cm (≈21 inches). Noteworthy that these numbers given are not in accordance with fact in comparison with modern units, where 18"= 45.72 cm, and 21"= 53.34 cm. The second method was using the span, i.e. the width of the palm of the hand, with the human fingers used as digits of measuring the width. Four digits equalled the sign of one palm and seven palms equalled one cubit. In present-day trigonometry, the cotangent requires the same units for both the horizontal run and vertical rise. Ancient sources, however, such as the Rhind Papyrus (dating from the First Intermediate Period, 16 cent. B.C.) use palms for the run and cubits for the rise, resulting in characteristically different, mathematics. In ancient Egyptian mathematics there were seven palms in a cubit, in addition to the *Seqed*, which was seven times the cotangent. The Egyptian *Seqed*/*Seked* is the ratio of the run to the rise of a slope of a cotangent. The Rhind Papyrus which is an ancient Egyptian mathematical document mentions the *Seqed* repeatedly (e.g. 56, 57, 58, 59, 59 b and 60), in connection with many problems or issues. The ancient Egyptians knew about the concept of a slope which is equal in value to the cotangent of a tangent angle or along a tangent point. The tangent is a line, curve, or surface meeting another line, curve or surface at a common point and sharing a common tangent line or tangent plane in that point. The abbreviation 'tan' is the trigonometric function of an acute angle in a right triangle that is the ratio of the tan θ, which is equal in value to the ratio of the length of the side opposite the angle to the length of the side adjacent to the angle. Ancient Egyptians used the tangent angle by identifying it as a *Seqed*, which is the ratio of two lengths of the sides of a right-angled triangle. The ancient Egyptians used the equivalent of similar triangles to measure dimensions and lengths of slopes defining the *Seqed* as the ratio of the run to the rise, which is the reciprocal of the recent definition of the slope. For example, the *Seqed* of a pyramid is considered as the number of palms in the horizontal corresponding to a rise of one cubit or seven palms. Thus, if the *Seqed* is 51/4 and the base is 140 cubits, the height becomes 931/3 cubits that is mentioned in the Rhind Papyrus (issue No. 57). The ancient Egyptian calculations indicate that the *Seqed* is equal in value to one Royal cubit, which equals seven palms or twenty-eight fingers. Mathematically, the *Seqed* can be expressed by the following equation; *Seqed* = 1 Royal cubit = 7 Palms = 28 Digits/Fingers. The ancient Egyptians have not measure angles by a direct way, but used the theory of a slope which is equal to the tangent of an angle; their angle concerning measure was the *Seqed*, which is defined as the rate of two lengths of the sides of a right triangle taken in a certain method, where the *Seqed* is the length from a human elbow to the

end of the middle finger, the approximate width of the human hand is equal a palm, and the estimated width of a finger is equal in value to a digit. The ratio of the two shorter sides of the triangle is measured in cubits for the vertical side (H) and in palms along of the horizontal side (W), wherefore the $Seqed = \frac{\text{Width by Palms}}{\text{Height by Cubits}} = 7\tan(\theta)$.

Here θ is the angle between the vertical and the plummet line. The 7 tan (θ) is due to the fact that one cubit equal in value to seven palms. For example, the secant of the angle θ equal in value to the ratio of the length of the face of the Great Pyramid of Giza (the slope height), inclined at an angle θ to the surface of the earth, to half-length of the side of the square base.

Different and conflicting *Seqed* values

The Egyptian *Seqed*/*Seked* is the ratio of the run to the rise of a slope of a cotangent, broadly based on the ancient Egyptian measurements of the Royal cubit, the palm and the digit [1]. But there are some differences which have caused confusion when we want to measure the length of a cubit or to determine the accurate value of a *Seqed*. It should be mentioned that the confusion and the difficulties explaining the *Seqed* problems arise in part from the technical words employed. EISENLOHR, the first editor of the Rhind Papyrus, mentions the different interpretations by scientists as early as 1877. He was well aware of the difficulties related to the issues of a *Seqed*. EISENLOHR concluded that the term *Seqed* referred to the ratio of two lines. Also, he concluded that the *Seqed* should refer to either the ratio of the half-base to the apothem or to the ratio of the half-diagonal to the sharp edge formed by the meeting of two flat or curved surfaces. In contemporary terminology, these ratios correspond to cos(α) and cos(β) respectively [2, 3]. Moreover, PETRIE (1877) argued that the cubit derived from the original value of the typical average, which may be slightly increased by continual copying over the ages [4]. CARTER & GARDINER (1917) emphasized that a cubit measured 523 millimetres; this is according to information about the tomb of Ramesses IV mentioned in the Turin plan of a royal tomb [5]. Later, it was studied again by WEEKS within the frame of the Berkeley Theban Mapping Project/BTMP. WEEKS compared the measurements of the papyrus and those taken by the BTMP, and then used CARTER's suggested value of the cubit of 1 Cubit = 0.5231 meters [6]. IVERSEN (1955) believed that Egyptian metrology is based on measures of 4 small cubits and that two canons were used successively. The unit of the first canon was the fist and, when squared to form a grid, it automatically divided the height of the standing male figure into 18 squares. According to the second canon the height of the figure was divided into 21 squares. He assumed the reason for the change to be found in metrology: about the time of the 26th Dynasty the small cubit was replaced for all practical purposes by the royal cubit [7]. HAYES (1957) referred to this fact by studying a 22 cm high diorite statuette of Sennemut (now kept in the Metropolitan Museum of Art, accession No. 48.149.7) which is a reproduction at a scale of 1:7 (i.e. one palm to the cubit) of Cairo CG 579 showing Sennemut presenting a sistrum to Mut. The long autobiographical text which occupies the back pilaster of Cairo CG 579 finds no place at all on its small replica [8]. BUDGE (1960) clarified that a cubit is 0.525 meters [9, 1]. NOBLECOURT (1965) mentioned that in King Tutankhamun's tomb, four models of the famous Egyptian unit of length were found and that they measured one foot and seven and a half inches [10, 1].

LORENZEN (1970) indicated that there was a system labelled the single-inch system consisting of a 19 squares grid each square being 5 inches. The top edge of a square inscribed in the circle is approximately mid-way between 5 x 19 fists and 4 x 24 fists. This is why the first canon pattern is 19 squares high. Noteworthy that the basic measure or the prevalent module during the 18th-19th dynasties was the so-called 'Scale A1' or the double-inch system which equals 5-inch per fist(the hand-unit of scale A). The fathom, the royal cubit, and the foot remained unchanged as 3/4, ¼ and 1/8 of the diameter but became subdivided into 16 double-fists = 32 half double-fists = 128 half double-inches (written 128"). This leads to the equation: one fathom =3 royal cubits = 4 *pechys* = the length between the roots of the thumbs when the arms are outstretched [11]. MÜLLER (1970) discussed the royal and the smaller (or normal) cubit, and explained how the proportions of the human body became 'translated' into a canon. He then argues that the Old Kingdom system of axially crossing lines became in the Middle Kingdom replaced by a grid adjusting not only the width of the figures but also the lay-out of entire walls. The system remained unaltered during the New Kingdom and afterwards, but in the Saite Period one passed from the royal to the smaller cubit meaning that the height of the standing figure was henceforth 28 instead of 24 handbreadths. This resulted in an 'unnatural' length of the feet and slimness of the bodies [12]. GARDINER (1973) noted that the value of a cubit totalled 20.6 inches or 523 millimetres [13, 1]. STRICKER (1976) believed that the Egyptian royal cubit measured seven palms, and the small cubit six palms. He states, for instance, that the length of one cubit of the newly-born children in the Pap. Westcar alludes to innocence, the cubit being the latter's measure also in other instances [14].

GILLINGS (1982) suggested that the *Seqed* was the ancient Egyptian unit of measure for the inclination of the triangular faces of a typical pyramid. The measurement method was based on the Royal cubit of seven palms subdivided into four digits. The inclination of slopes was thus expressed as a value of horizontal palms and digits for every Royal cubit rise. This inclination may be a type of the contemporary measure belonging to the so- called Gradient which is in connection with the contemporary concept of cotangent and tangent [15]. Gay (1985) noted against IVERSEN's explanation for the change in the canon of proportions during the 26th Dynasty that a change in the grid system preceded any changes in the metrical system. In order to conform to the 6 palms of a small cubit the forearm was divided into 6 squares instead of 5. This allowed to generate a new grid system for standing figures from the old one using increments of one fifths of a square. These subdivisions, although realized in a few models from the Middle Kingdom onwards, were rarely used with such accuracy, however; the new grid system, therefore, caused only minimal changes in the principal ratios [16]. GAY & SHUTE (1985) concluded that in ancient Egyptian buildings with inclined walls (such as pyramids and pylons) the inclination was contrived to give a horizontal displacement measured in palms, half-palms or quarter-palms for a drop of one royal cubit of seven palms. The inclinations of the edges between adjacent walls were not relevant to the design. Evidence is presented suggesting that in two-dimensional art the obliquity of slanting lines may have been determined by a similar system, but with a drop of six units instead of seven. After the introduction of the squared grid-obliquity was probably controlled by reference to grid intersections. So, there is no valid cause for assuming that the irrational values π/ pi and ϕ / phi were involved, except coincidentally in the inclination angle of the

pyramids or in the slope of slanting lines [17]. As is well known that ratio of the circumference to the diameter,'π/pi'(≈3.1416) and the proportion of 'golden section', ϕ / phi (≈1.6180).
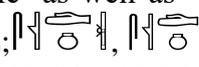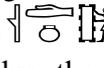
LEGON (1988) believed that the precision with which the Great Pyramid of the King Khufu had been built suggests that the builders had wanted to express particular dimensions as accurately as possible. Only limited or partially accurate estimations of the use of the Royal cubit in this pyramid have hitherto been put forward. A number of correlations between the various dimensions have thus gone unnoticed, he assumed, with the result that the intentions of the architect have not been properly understood. Far from being the outcome of a series of changes of plan the arrangement of the passages in the Great Pyramid reveals a logical and highly-integrated design according to his argument [18]; whilst GAY & SHUTE (1990) explained and defended their view on the 14 to 11 proportion. They maintain that the size and the shape of a pyramid were predetermined by the lengths of the sides of its square base and by the slope of its triangular faces. The unit of measure was usually the *Seqed* relating to lateral displacement in palms for a vertical drop of seven palms or one royal cubit. The *Seqed* of the Great Pyramid, e.g., is believed to be 51/2, which converts into an inclination of 51(51'). The other interpretation of pyramidal form lays particular emphasis on the height being more hypothetical since the height, unlike the slope, cannot be checked directly at all building stages. The approximation of the values 'π/ pi and ϕ / phi' are discussed and this last value or number derived from the golden ratio. The square root of 'phi' relates to the height of the Great Pyramid as a result of the use of a *Seqed* of 51/2. But it does not follow from this that the ancient Egyptians knew of this mathematical ratio or that they had any concept of phi and its square root [19].

NAGUIB (1991) believed that the rod (nbi) was a measure of length where longer linear units were needed believing it to be a full arm of 70 cm length. He sees a relation between royal cubit and rod: 1 rod = 1 1/3 royal cubit and 1 royal cubit = 3/4 rod [20]. By way of example he cites the rod and its divisions having been used for the principal architectural features of the tomb of Hem-Min (M43) at the cemetery of El-Hawawish at Akhmim (end of the 5th/-beginning of the 6th Dynasty) [20, 21, 22, 23]. ARNOLD (1991) considered the possibility that the Egyptian cubit was longer than a typical forearm. It seems to have consisted of 7 palms of 4 digits, which equals 28 units of altogether 52.3 to 52.4 cm in length [24]. The earliest certified gauge is the royal cubit of about 523 to 525 mm (= 20.6 to 20.64 inches). In ancient Egyptian architecture the royal cubit is known from the Old Kingdom onwards [24, 25, 26]; whilst ROIK (1993) considered the royal cubit as the basic measure of length in ancient Egypt, and set forth her views about the existence of a yet undetected ancient Egyptian measuring system basing her reasoning on measurements found in the tomb of Tawosre. She explored the evidence for this system, which she sees based on the existence of the nbi (Greek naubion?), a measure of length provisionally estimated to be about 65 cm. Also, she concluded that the nbi system was used as basis of the grid furnishing examples of measurements of the seated and standing human body [27]. LEGON (1996) examined the evidence for the use of the small and royal cubits in Egyptian monuments, concluding that the Egyptian artists' canon of proportions was based on the royal cubit and not upon the small cubit as hitherto claimed. The

'canonical height' of the standing figure is thus found to have been three royal cubits. The length of the cubit is shown to have been divided in practice into simple fractions, as well as into the more customary units of palms and fingers **[28]**.

According to SHAFFER (1996) the cubit is a measure of length from the elbow to the end of the middle finger; (18 inches or 45.72 centimeters) **[29, 1]**. The ENCYCLOPAEDIA BRITANNICA/MERRIAM-WEBSTER (2016) describes the cubit as an ancient unit of length based on the length of the forearm from the elbow to the tip of the middle finger and usually equal to about 18 inches or 46 centimeters **[30, 1]**. The value of the *Seqed* in ancient Egyptian architecture varies therefore. The slopes of the faces of the Great Pyramid of Giza measure a *Seqed* of 5 1/2, or 5 palms and 2 digits, which amounts to a slope of 51.84° **[31]**. EDWARDS (1979) considered this to have been the 'normal' or most typical slope choice for pyramids **[32]**. A lot of the smaller pyramids in Egypt show other slope angles. Although looking like the Great Pyramid of Giza, the pyramid at Meidum is thought to have a slope of nearly 51.842° or 51° 50' 35" amounting to a *Seqed* of 5 1⁄2 palms **[33]**. The similarity of the slope at Meidum and at Giza is believed to be deliberate wishing to make certain that the circuit of the base of the pyramids was exactly comparable to the length of a circle, this if the pyramid's height were used as a radius **[34]**. Noteworthy that these relations of areas and of circular ratio are so systematic those were included in the builder's design **[35]**.

The writing and semantics of *Seqed* in the ancient Egyptian language

The linguistic structure of the *Seqed* in the ancient Egyptian language reflects their alternative forms. Therefore, there are some written forms which indicate the meaning of 'slope angle', 'tangent angle' as well as 'to measure the cotangent point'. These written forms were as follows; [hieroglyphs], [hieroglyphs] **sḳd/sqd**, which also means 'slope of pyramid'[(1)] **[36, 37, 38, 39, 40, 41, 42]**. Mathematically, it means, 'degree or extent per cubit in height, altitude or amount, which is half the baseline divided by the Pyramid height, expressed in handbreadths a unit of measure angles' **[41, 43, 44]**. It has been noticed that different determinatives or ideograms were attached to the previous word, such as [hieroglyph], [hieroglyph] **[13]**. Gardiner's sign-list classifies [hieroglyph] (Aa 28) as a representation of the ideogram used in the verb [hieroglyphs], [hieroglyphs] **ḳd/qd**, which means to 'build'. The instrument represented by the ideogram was used by bricklayers, the sign [hieroglyph] sometimes stands alone such as in the scene of the funerary buildings of the King Saḥu-Ra [(2)] **[13, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]**. Another suggestion is a striker used in measuring corn[(3)] **[55, 56]**. Another form of the first ideogram is [hieroglyph], which was used in the Old Kingdom and continued until the New

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[(1)] Wb. IV, 309, 20.

[(2)] Borchardt, S'aḥu-Re, Pl.54; Sethe, Pyr.514d, 576c, 642c-643a, 1129a-b, 1597c; De Buck, CT I, 174e; Sethe, Urk. I, 4, 10; 95, 9; 121, 13; Steindorff, Urk. VII, 56, 9; Kitchen, KRI. II, 311, 7; Helck, Nfr.tj, KÄT, Vol. 2, 58; Helck, Merikare, KÄT, Vol.7, 47, 9-10.

[(3)] Quibell, The tomb of Hesy, 26, Pl.17; Griffith, Siût and Dér Rifêh, No. I, 236.

Kingdom [1] **[45, 57, 13, 33, 49, 58, 59, 60]**. In Hieratic ⎰, is usually transcribed as ⎰[2] **[61]**. Furthermore, there were other related words, previously mentioned such as; **sḳd/sqd – sḳdy/sqdy** meaning 'line of measure', **ḳd/qd** meaning 'to extend, length, ratio, dimension', **ḳddw/qddw** meaning 'characters of people, who measure the extent or the length', **ḳdwt/qdwt** which means 'sketch', or the action of indicating the exact position or location of a border or a borderline[3] **[40]**. Also **sḳd/sqd** meaning 'length, ratio, dimension' [4] **[39, 46]**. This is beside the written form of the word **sḳdw/sqdw** meaning 'builders', perhaps used for 'those who commissioned the work of measuring slopes'[5] **[37, 39]**. In addition **ḳdt/qdt** means 'kite (= weight of one tenth of a Deben or 9.1 grams'[6] **[36, 40, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71]**. In Coptic the variant forms of the writing were kite, kit[7] **[72, 73, 74]**. In the Sahidic Dialect the written forms were kwte, kwt, kot [8] **[75, 76, 77]**. The original meaning 'pot' led via the mythological picture of god Khnum forming man on a potter's wheel to the meaning of **ḳd/qd** 'personality, character' as can be found in the expression 'the day of the reckoning of the characters/ **hrw ḥsb ḳdwt**' referring to judgement of the deceased in the Osirian 'Hall of Judgement'[9] **[78, 79]**.

Seqed and Sloping Span Length in Ancient Egypt

GILLINGS (1982) defined the *Seqed* of a pyramid as follows: 'The *Seqed* of a right pyramid is the inclination of any one of the four triangular faces to the horizontal plane of its base, and is measured as so many horizontal units per one vertical unit rise. It is thus a measure equivalent to our modern cotangent of the angle of slope. In general, the *Seqed* of a pyramid is a kind of fraction, given as so many palms horizontally for each cubit of vertically, where 7 palm equal one cubit. The Egyptian word '*Seqed*' is thus related to our modern word 'gradient' **[15]**. The method for the measuring of a sloping span length in ancient Egypt becomes clear looking at religious and funeral structures. A very impressive structure is the great pyramid of King Khufu on the Giza plateau **[80, 81, 82, 83, 84, 85, 86, 87, 88, 89]**. PETRIE (1883) believed that for an accurate determination of the value of the usual Egyptian cubit the King's Chamber in the Great Pyramid (called 'Khufu's Horizon') was certainly the most significant source as it is the most accurately built best preserved and the most precisely measured **[31]**, (Figs.1, 2 , 3, 4, 5).

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[1] Borchardt, S'aḥu-Re, Pl.54;Davies, Ptahhetep and Akhethetep, Vol. I, 13, No.271; Sethe, Urk.I,16,4; Wreszinski, P. Ebers, 19, 13; Davies, Puyemrê, 20 (Dyn.18).

[2] Möller, Hieratische Paläographie, Vol. 1, Nos. 457, 472.

[3] Dickson, Dictionary, 195,304.

[4] Sethe, Pyr. 361.

[5] Hannig, Groβes Hand Wörterbuch, 837.

[6] Dickson, Dictionary, 195, 304; Wb. V, 79, 15-80; Wilson, A Ptolemaic Lexikon, 1072.

[7] Crum, A Coptic dictionary, 123-124; Černý, Coptic Etymological Dictionary, 65.

[8] Plumley, Sahidic Dialect, 12; Westendorf, Koptisches Handwörterbuch, 71; Till, Koptische Grammatik, 324.

[9] Grieshammer, Das Jenseitsgericht, 49-50, 107; Seeber, Untersuchungen zur Darstellung des Totengerichts, 74.
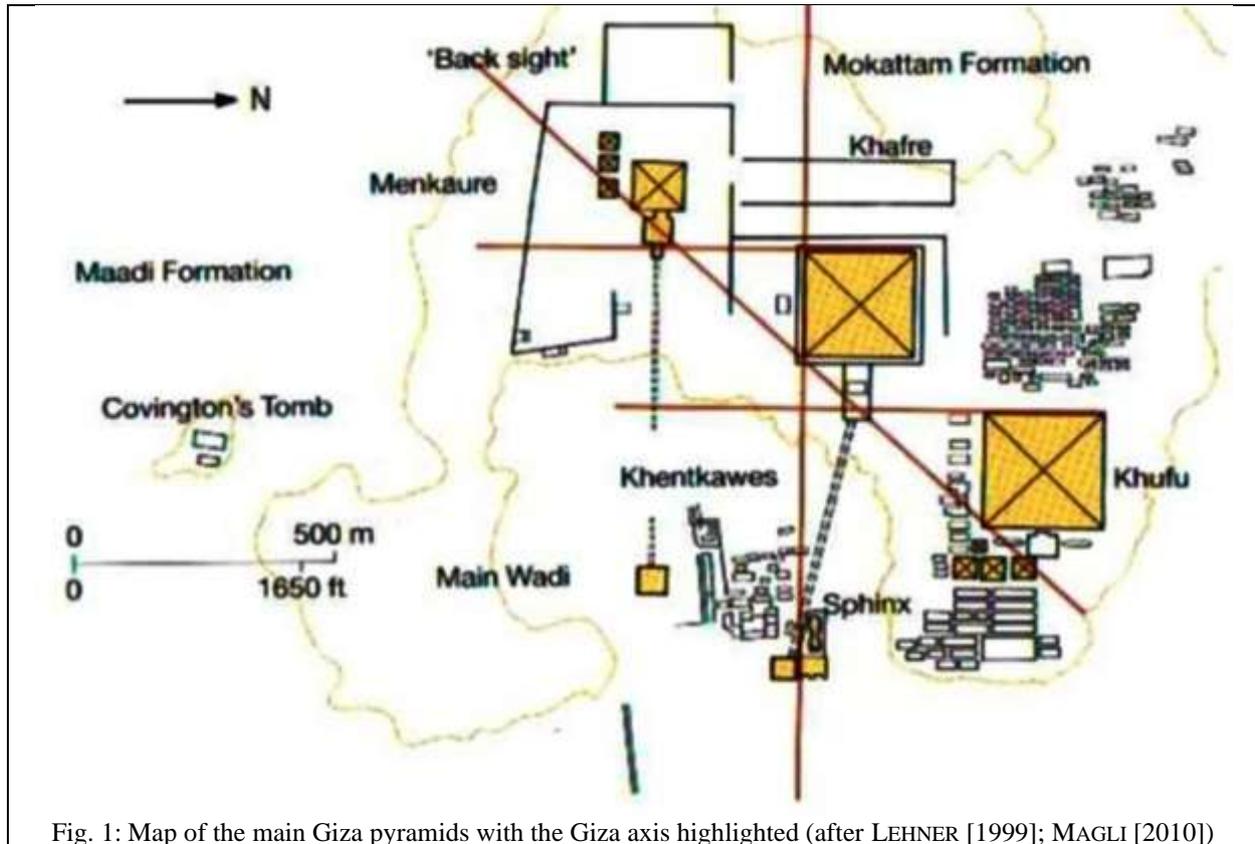
Fig. 1: Map of the main Giza pyramids with the Giza axis highlighted (after LEHNER [1999]; MAGLI [2010])

The most important aspect of pyramid construction was the precise mathematical calculation and the determination of the sloping span length of a structure, which was kept carefully close to the line of the horizon.
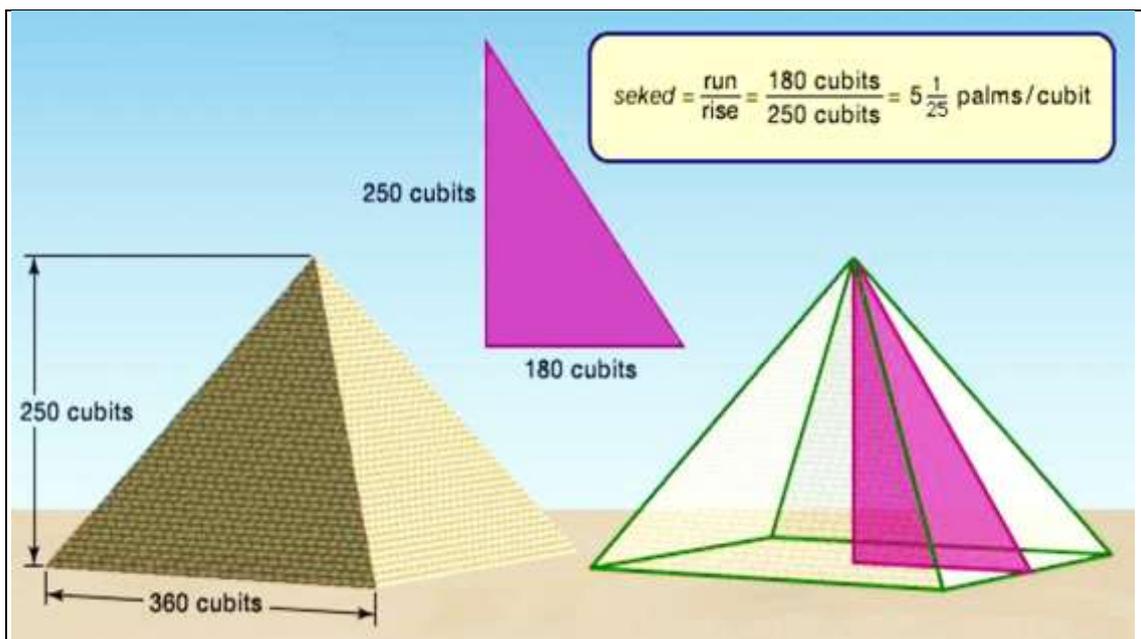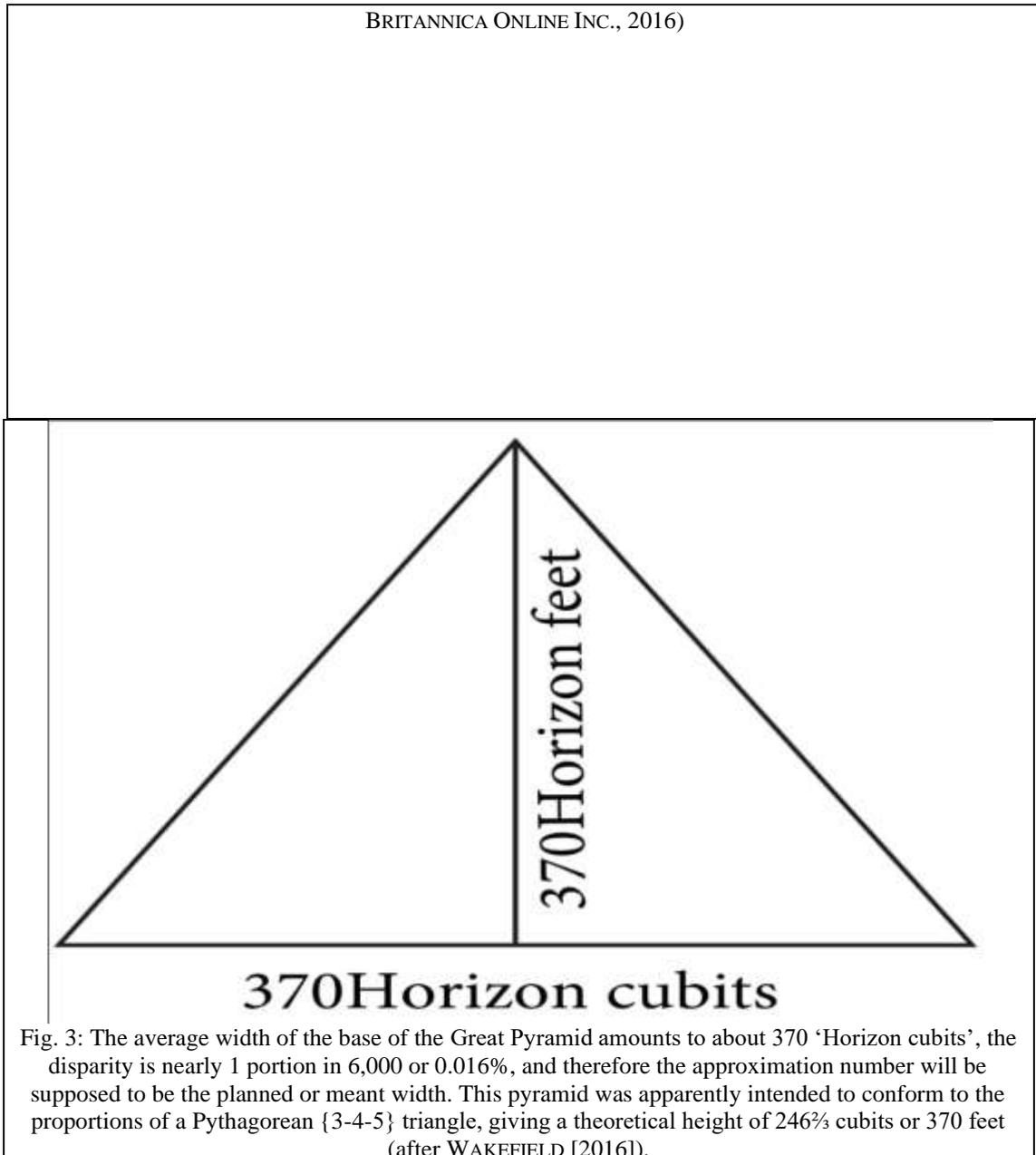

$$seked = \frac{run}{rise} = \frac{180 \text{ cubits}}{250 \text{ cubits}} = 5\frac{1}{25} \text{ palms/cubit}$$

Fig. 2: The *Seqed* of a pyramid is considered to be the number of palms in the horizontal corresponding to a rise of one cubit or seven palms (after MERRIAM-WEBSTER'S INTERNATIONAL ENCYCLOPÆDIA
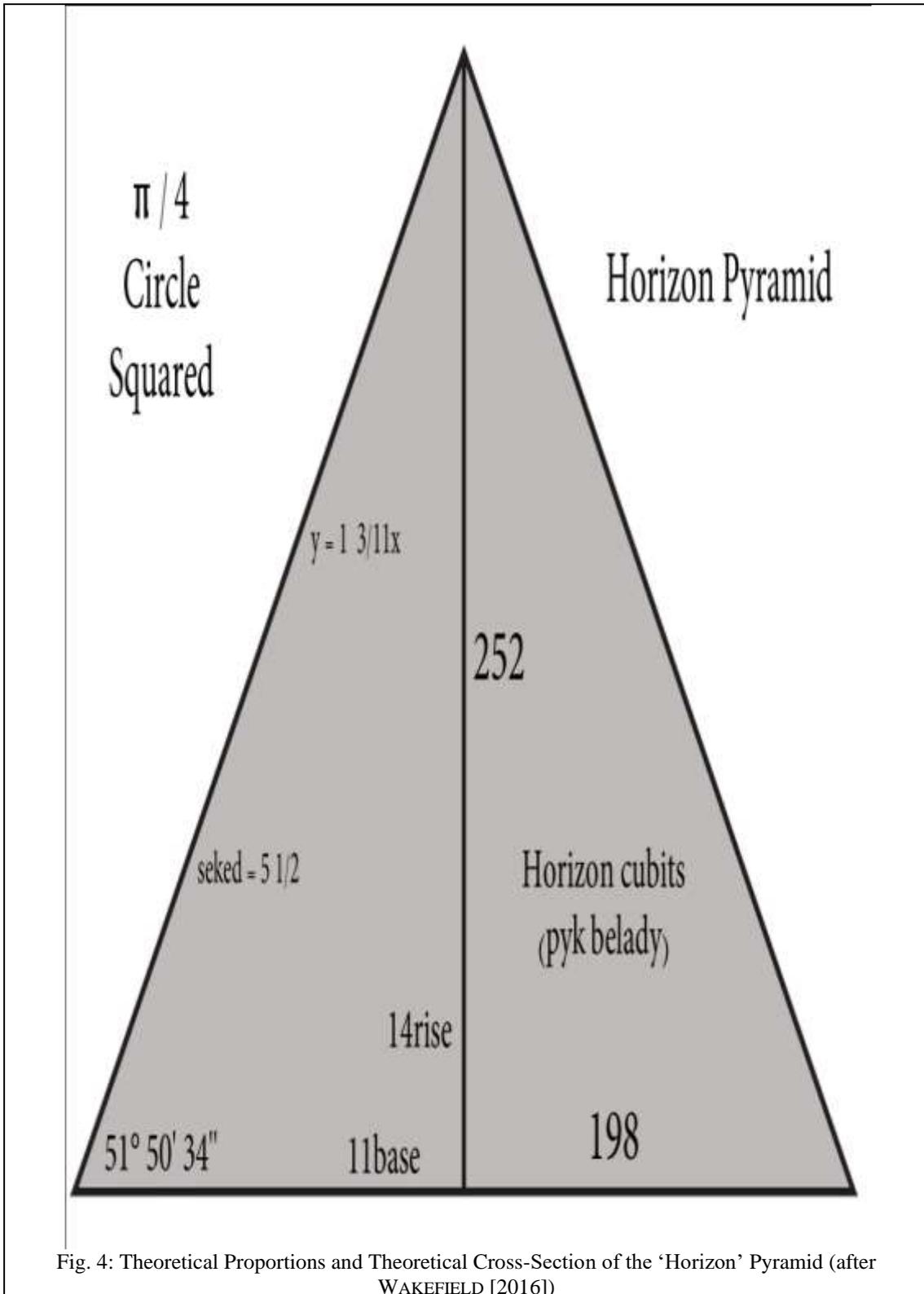
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Fig. 3: The average width of the base of the Great Pyramid amounts to about 370 'Horizon cubits', the disparity is nearly 1 portion in 6,000 or 0.016%, and therefore the approximation number will be supposed to be the planned or meant width. This pyramid was apparently intended to conform to the proportions of a Pythagorean {3-4-5} triangle, giving a theoretical height of 246⅔ cubits or 370 feet (after WAKEFIELD [2016]).

Fig. 4: Theoretical Proportions and Theoretical Cross-Section of the 'Horizon' Pyramid (after WAKEFIELD [2016])

The Giza necropolis with The-6 square grid of the master plan.

a. Khufu's Great Pyramid

b. Khafre's Pyramid

c. Queen's Pyramids

d. Cult Pyramids

e. Mortuary temples

f. Valley temples

g. Causeways. The-6 square grid

h. The Sphinx and temple

i. Royal Cemetery

j. Mastabas

k. The trial passage
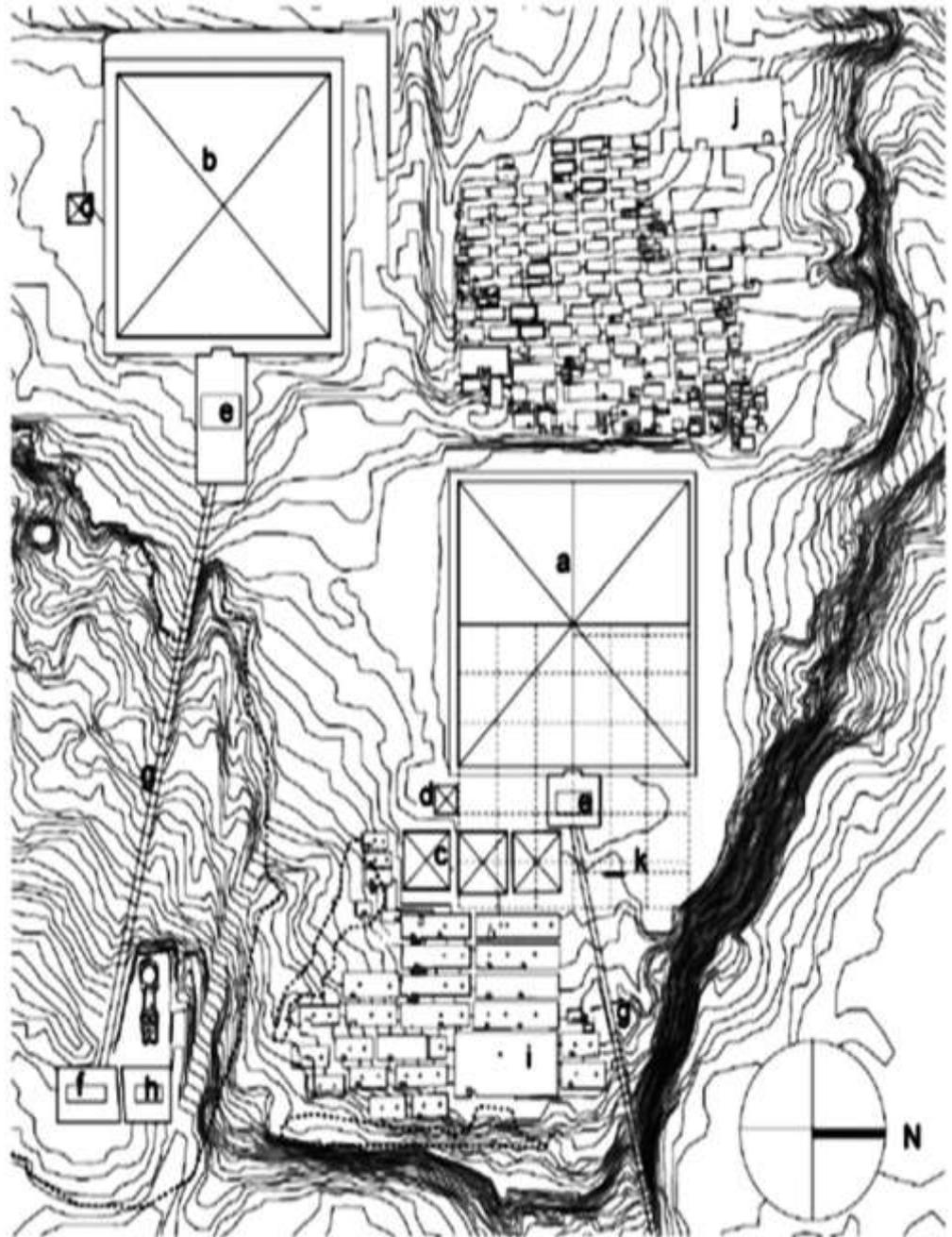
Fig.5: Diagram illustrating the building processes on the Giza plateau (after BRYN [2010])

The ancient Egyptians knew about the concept of a slope which is equal in value to the cotangent of a tangent angle or along a tangent point. The tangent is a line, curve, or surface meeting another line, a curve, or surface at a common point and sharing a common tangent line or tangent plane in that point. The abbreviation 'tan' is the trigonometric function of an acute angle in a right triangle that is the ratio of the tan θ, which is equal in value to the length of the side opposite the angle to the length of the side adjacent to the angle (Fig. 6).

Fig.6: Diagram illustrating the calculated *Seqed* that is the measure of an angle θ. In current trigonometry *Seqed* (or θ) means the cotangent of the angle between the base of the pyramid and one of its faces. Thus, if the tangent of an angle is a/b then the cotangent of that angle is b/a. It is noteworthy that the right triangle (ABS), where (AB) is the height and (BS) is half the length of the base (after NICKEL [2009])

The preceding diagram explains the concept of the right triangle (ABS), where AB is the height and BS is half the length of the base. The Rhind Papyrus is clearly dealing with this type of triangle because the solution involves calculation with half the value of the base; that is to say $\frac{1}{2}$ of 360 = 180 the reason being that the tangent value in this triangle is $\frac{AB}{BS}$ and the complementary rate or the cotangent angle is $\frac{BS}{AB}$. The two angles are complementary if the total of them is meant to be 90°. It is noteworthy that the two acute triangles of a true triangle always emphasize the values of each other and it should be symmetrical or integral. It means that the tan θ equals cot (90° – θ), where 90° – θ equals ∠BAS, and cot θ equals tan (90° – θ). Additionally, tan θ equals $\frac{AB}{BS}$ and cot θ equals $\frac{BS}{AB}$ in a right triangle. For this reason there is a mutual benefit or a reciprocity relationship between the tangent rates and cotangent values **[93, 94]** (Fig.7)
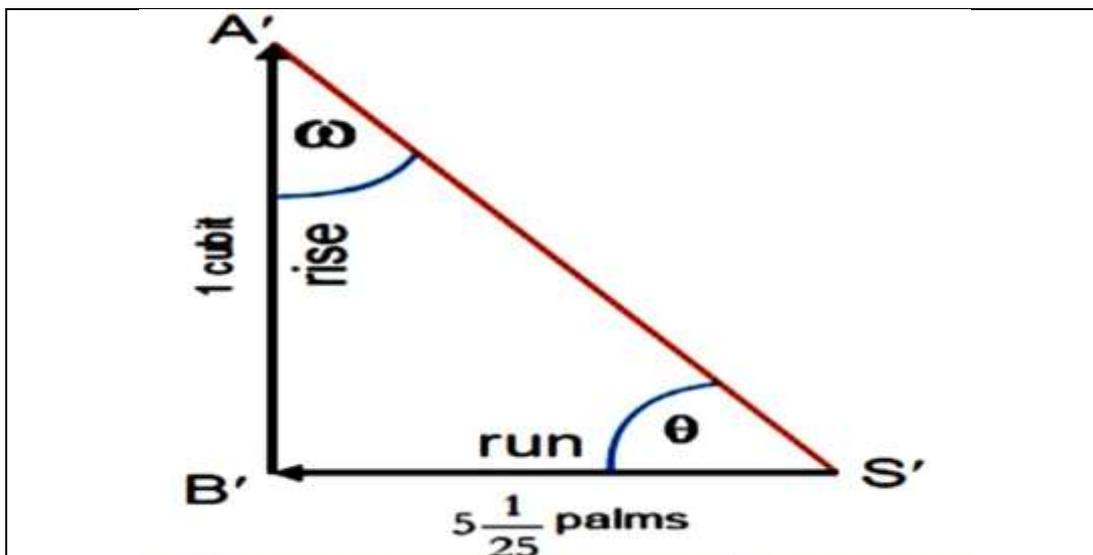

Fig.7: Most modern scientific calculators offer no function for finding the inverse cotangent using

the complement of the cotangent, i.e. the tangent instead. We set ω = 90° – θ. Hence, ω = tan-1 (0.72) ⇒ ω = 35.75° (the complement of the angle we are looking for). Since ω = 90° – θ, then θ = 90° – ω. Hence, θ = 90° – 35.75°= 54.25°. This value corresponds closely to the actual inclination angles of the pyramids of Egypt (after NICKEL [2009])

The ancient Egyptians used the tangent angle and identified it as a *Seqed*, which is the taken ratio of two lengths of the sides of a right-angled triangle. The ancient Egyptians used equivalent of similar triangles to measure dimensions and the length of slopes and they defined the *Seqed* as the ratio of the run to the rise, which is the reciprocal of the above definition of the slope. For example, the *Seqed* of a pyramid is considered to be the number of palms in the horizontal corresponding to a rise of one cubit or seven palms. Thus, if the *Seqed* is 51/4 and the base is 140 cubits, the height becomes 931/3 cubits which is mentioned in issue No. 57 of the Rhind Papyrus **[30, 42, 95, 96, 97, 98, 99, 100, 101, 123]**. Issue No. 56 concerns the following: 'if a pyramid is 250 cubits height and the side of its base 360 cubits length, what is its *Seqed*?' The solution will become clear considering the following: take half of 360, which results in180; then multiply250 by a value of the cubit so as to arrive at 180; this value is $\frac{1}{\langle 21/|51/|50\rangle}$ of a cubit. A cubit is 7 palms. Multiply 7 by$\frac{1}{\langle 21/|51/|50\rangle}$. The *Seqed* is thus$5\frac{1}{25}$ palms, which are 3 + 1/2 + 1/3 1 + 1/15 + $\frac{1}{10}$ + 1/25 **[2]** (Fig.8, a, b, c).

Fig. 8, a: Problem No. 56 of the Rhind Papyrus, written in Hieratic by the scribe Ahmose (c. 1550 B.C). (after IMHAUSEN [2016];
http://www.britishmuseum.org/system_pages/beta_collection_introduction/beta_collection_object_details/beta_collection_image_gallery.aspx?assetId=366139001&objectId=110036&partId=1#more-views
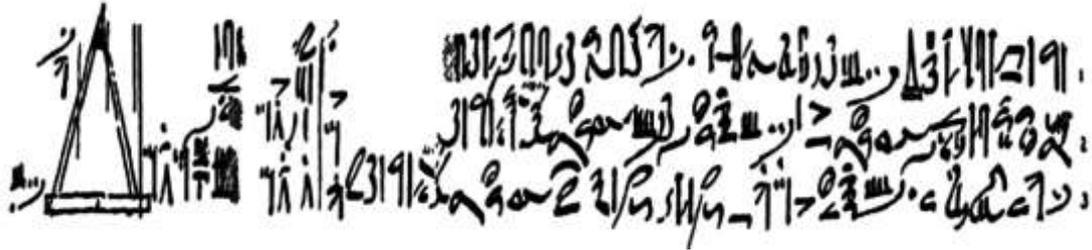http://www.britishmuseum.org/)

Fig.8, b: Facsimile of problem No. 56 of the Rhind Papyrus
(after http://www.britishmuseum.org/system_pages/beta_collection_introduction/beta
_collection_object_details.aspx?assetId=766120001&objectId=117389&partId=1
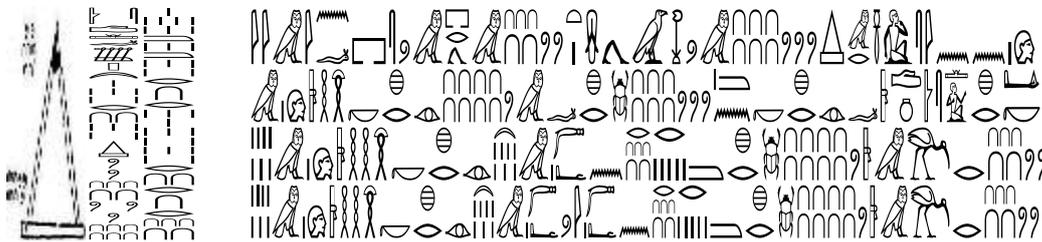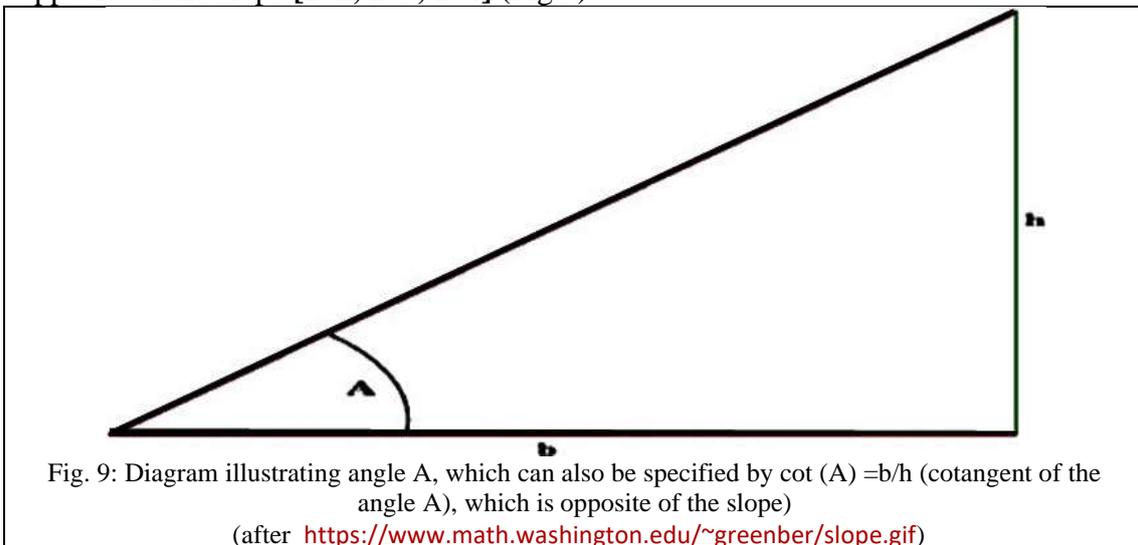http://www.britishmuseum.org/)

Fig. 8, c: Hieroglyphic transcription of problem No. 56.

The text accompanying issue No. 59a reads as follows: 'If the side of a pyramid is 12 cubits and the height are 8 cubits what is its *Seqed* ?' The solution will become clear considering the following; Take **½** of the side. The result: 6. Determine the fraction of the height 8 which gives 6? The result: **¾**. Then 1 cubit = 7 palms; 1 palm = 4 fingers. Take **¾** of 7 palms. The result: 5 palms and 1 finger. This is the *Seqed* **[2, 3]**. There is a noteworthy belief that metrological standards in ancient times matched the same principles that are used in the present; therefore, the value of a *Seqed* would have possessed exactly the same absolute value in whatever the context. It would appear that in monumental architecture units were consistent only within the framework of one particular building project. A certain set of standards would be created for each pyramid, e.g., and ritually dedicated specifically for that pyramid **[31, 91]**. The average width of the base of the Great Pyramid is about 370 'Horizon' cubits ('pyk baladi'. Etymologically, the term 'pyk belady/baladi' is related to the Greek 'pekhus and pygon', both also meaning a cubit, with that being the number of cubits to the Greek stade (about 600 feet [180 metres]). Hence 'pyk belady' can be taken to mean 'national cubit [of Egypt]'), therefore the difference amounting to just 1 part per 6,000 or 0.016% . For our purpose we assume the approximate figure corresponding to the planned or desired width. Mathematical ratios existing in this pyramid seems to be close to the proportions or values of a Pythagorean (3-4-5) triangle resulting in a theoretical height of $246^{2}/_{3}$ cubits or 370 feet. In fact, it is well known that the angle θ for the Pythagorean triangle is 53.13°, which is more than 1° greater than the angle for the Khufu's Pyramid. The *Seqed* of a pyramid is calculated by finding **x** in terms of **y** then multiplying the coefficient of the determination of **y** by **7**. Or the other way round: it is **7** times the cotangent of the pyramid's dihedral angle. In order to determine the slope of an angle the latter should

be considered as being formed by a horizontal line and a slanted line as illustrated below. Such an angle can be defined by just giving its slope. In the illustration the slope of angle A is the ratio h/b. In geometry this mathematical relation is also referred to as the tangent of the angle A and denoted by Tan (A). The angle A can also be specified by cot (A) =b/h, which is called the cotangent of the angle A and is opposite of the slope **[102, 103, 104]** (Fig.9).

Fig. 9: Diagram illustrating angle A, which can also be specified by cot (A) =b/h (cotangent of the angle A), which is opposite of the slope)
(after https://www.math.washington.edu/~greenber/slope.gif)

The ancient Egyptian mode of calculation shows the *Seqed* to equal the value of one Royal cubit of seven palms and twenty-eight fingers. Mathematically, the *Seqed* can be expressed by the following equation:$\text{Seked} = \frac{\text{Width by Palms}}{\text{Height by Cubits}} = 7\tan(\theta)$ whereby 1 *Seqed* equals 1 Royal cubit = 7 Palms = 28 Digits/Fingers (Fig.10).

Fig.10: Diagram illustrating the equation of *Seqed*
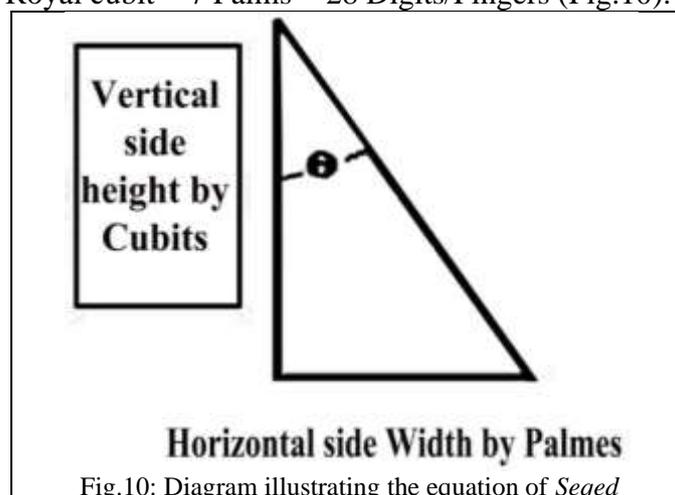
The tan (θ) is the value of the angle between the vertical side of height and the horizontal side of width. Then 7 tan (θ) is a result of the issue that 1 Royal cubit equals 7 palms = 28 fingers (or digits). The ancient Egyptians used similar triangles to measure dimensions and lengths of slopes, and they defined the *Seqed* as the ratio of the run to the rise. Pyramid slopes were fixed in the designing stage of the building process via the tangent angle. Using the simplest and lowest possible numbers and

fractions resulted in an approximation differing just 1% of the actual measured slopes of the pyramids. It is noteworthy that later sources such as the Rhind Papyrus from the First Intermediate Period, are dealing with issues like methods of measuring the tangent angle or *Seqed*. The Rhind Papyrus discusses the tangent angles as obviously the best way of dealing with physical quantities **[42, 95, 96, 97, 98, 99, 100, 123]** (Fig.11, a, b).
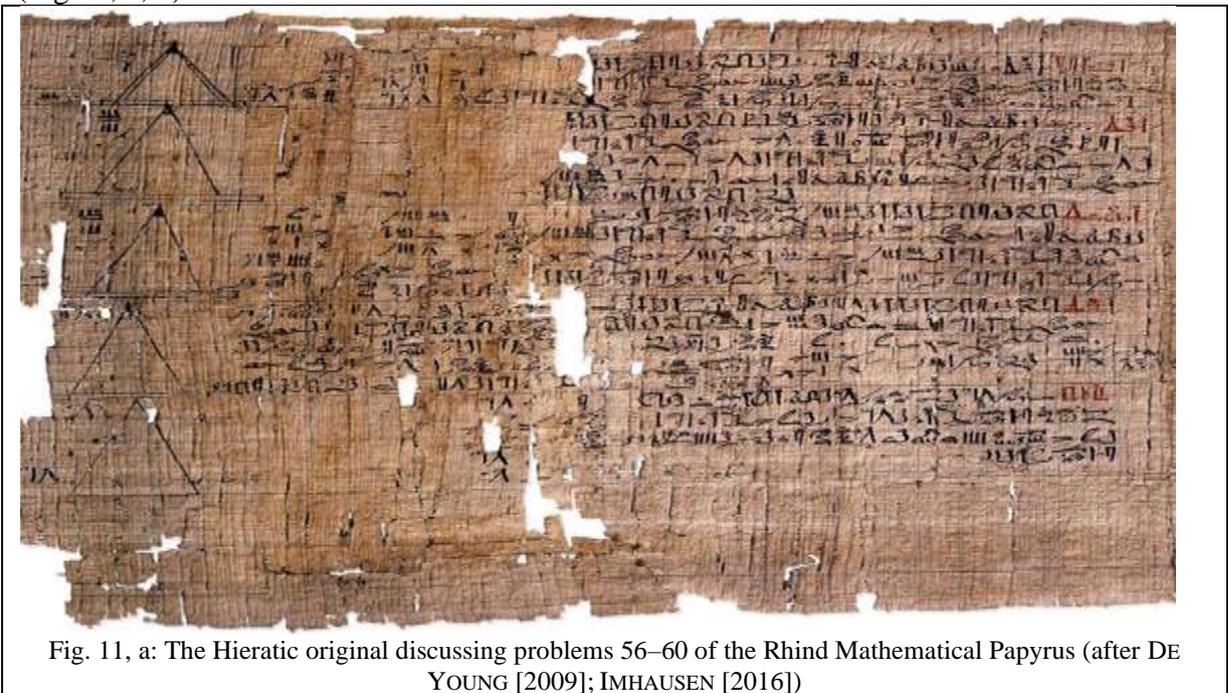

Fig. 11, a: The Hieratic original discussing problems 56–60 of the Rhind Mathematical Papyrus (after DE YOUNG [2009]; IMHAUSEN [2016])
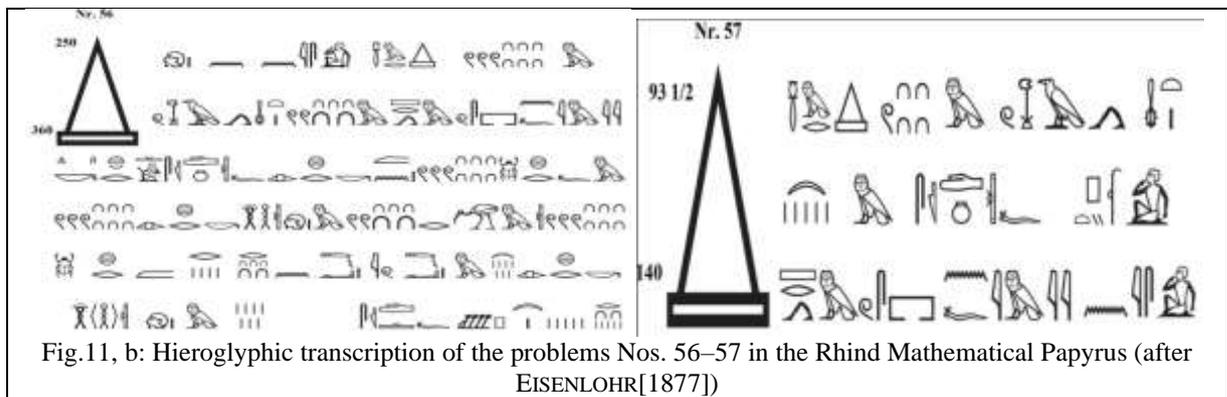

Fig.11, b: Hieroglyphic transcription of the problems Nos. 56–57 in the Rhind Mathematical Papyrus (after EISENLOHR [1877])

Another mathematical papyrus discussing the tangent angle is the Moscow Mathematical Papyrus dated to the Middle Kingdom. It is also known as the GOLENISHCHEV Mathematical Papyrus after its first owner, Egyptologist VLADIMIR GOLENISHCHEV who bought the papyrus in 1892 or 1893 in Luxor. It later entered the collection of the Pushkin State Museum of Fine Arts in Moscow where it is kept today. This document too deals with the tangent angle called *Seqed* **[99, 106]** defined as the number of palms and digits/fingers spanned horizontally for each cubit rise–or as ratio of the base (numerator) in cubits of a right triangle of one cubit in height. The

Old Kingdom pyramid builders used the *Seqed* successfully but during the Middle Kingdom errors may have occurred using this measuring method[1] **[80, 107]** (Fig.12).
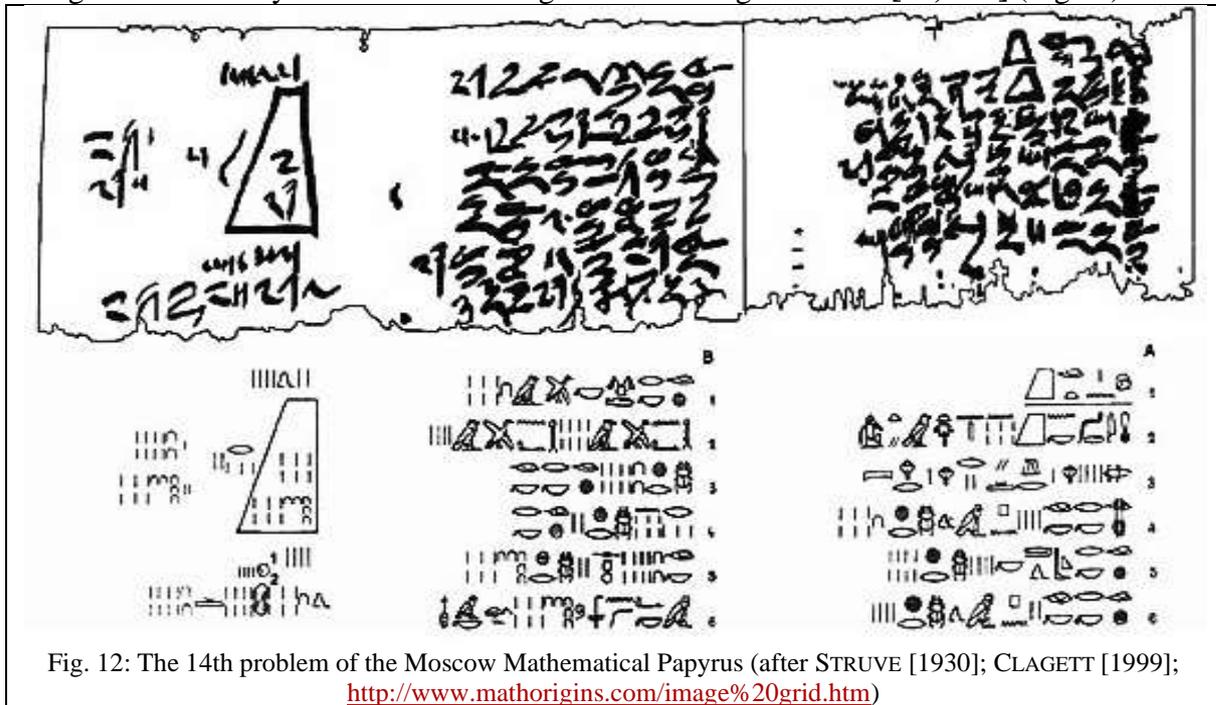

Fig. 12: The 14th problem of the Moscow Mathematical Papyrus (after STRUVE [1930]; CLAGETT [1999]; http://www.mathorigins.com/image%20grid.htm)

Slope angles of pyramids of the Fourth and Fifth Dynasties were noticeably subject to some experimenting. Their sloping angles range between 6/5 and 4/3. In connection with this observation it has been suggested that angles were aligned to the sun, i.e., that the 4/3 triangle arose from measuring the zenith of the sun at culmination during the winter solstice, this measure being close to the complementary angle **[84, 107, 108, 109, 110, 111, 112, 124, 125]**. It should be indicated that the height of the sun above the horizon at noon in the winter solstice $h_{min} = (90° – φ) – ε$, which is about 2500 years B.C.(during the Old Kingdom), where the inclination of the earth's axis to the equator – ε – was slightly less than 24°. The ratio of $h_{min} ≈ 36°$, so the slope-angle should be Θ≈54.0°, which gives a *Seqed* 5.08 (5 1/11). The *Seqed* of the Khufu's Pyramid is 5 ½, or θ≈51.8°. The difference of 2° is large enough to remain unnoticed. So the Pythagorean triangle probably originated from the surveying practice, in which the Egyptians were very sophisticated **[124]**. The occurrence of the phenomenon known as precession of the earth's axis of rotation, leading to a change in the direction of the axis in space. But the very precession has little in common with the value of the angle ε. Astronomers know that the Moon stabilizes the Earth's rotation in such a way that the variation of ε occurs in a very narrow range - only 2 degrees, with a period of about 41,000 years**[125]**. In more detail the earth rotates about an imaginary line called the axis of rotation which passes through the North and South Poles of the planet, in addition there are relationships between the zenith of the sun during the winter solstice with the axial precession and the orientation process of the astronomical bodies and Earth's rotational axis, therefore the orbit represents a symmetrical open curve (hyperbola) formed by the intersection of a plane with two

[1] Magli, Architecture, Astronomy and Sacred Landscape, 100-106, A3- A6.

identical cones on opposite sides of the same vertex. A hyperbola is concave toward North in the winter, concave toward South in the summer and a straight line at the Equinoxes. In connection with this the azimuth seems to be as the angle formed between specific directions from true north and a sight line of the observer to a specific point measured in the same scale as the provisional orthogonal direction to zenith as the highest point of the azimuth **[113, 114, 115,125]** (Fig. 13, a, b; 14, a, b).
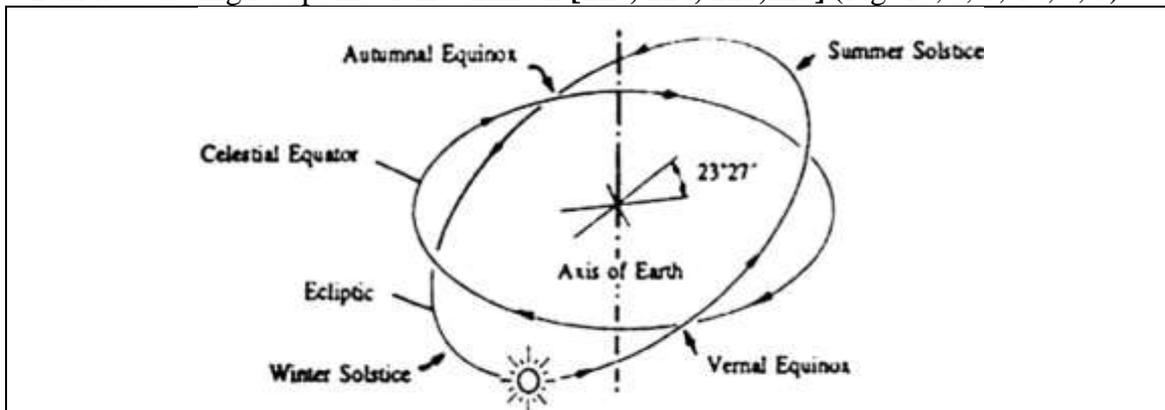

Fig.13, a: Diagram showing the axial precession and the orientation process of the astronomical bodies and Earth's rotational axis (after NEUGEBAUER [1980]; ISLER [1989]).
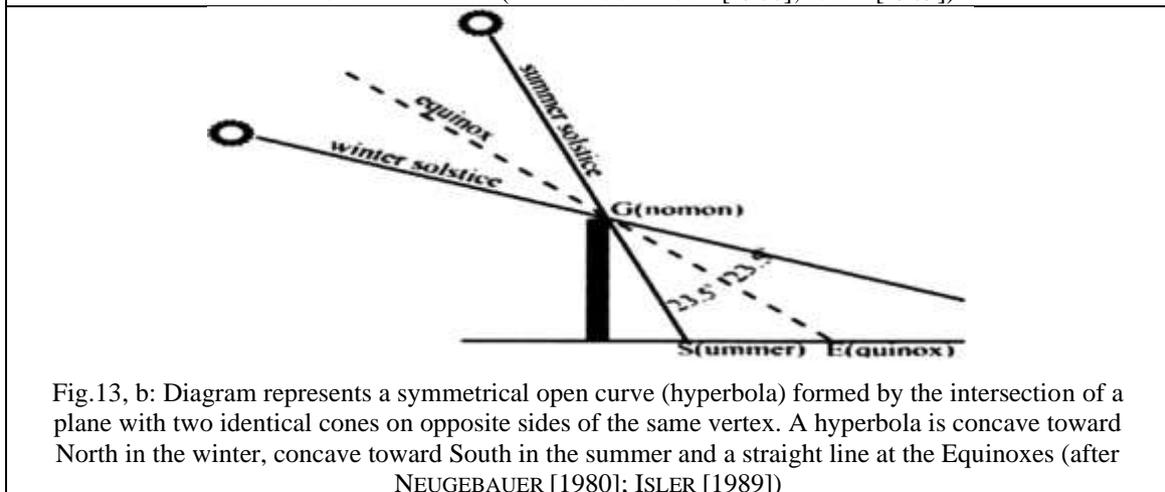

Fig.13, b: Diagram represents a symmetrical open curve (hyperbola) formed by the intersection of a plane with two identical cones on opposite sides of the same vertex. A hyperbola is concave toward North in the winter, concave toward South in the summer and a straight line at the Equinoxes (after NEUGEBAUER [1980]; ISLER [1989])
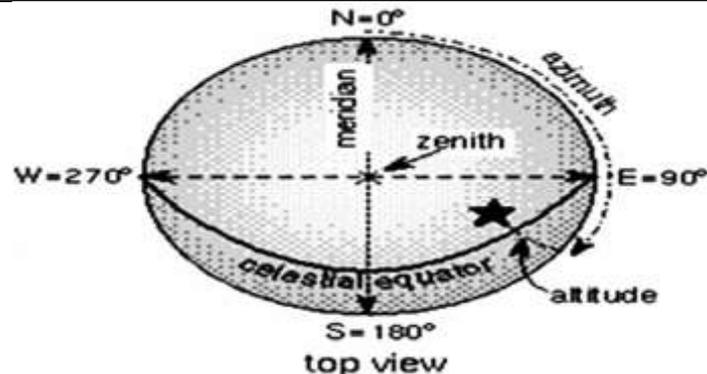
Fig. 14, a: Diagram represents the azimuth that is the angle formed between specific directions from true north and a sight line of the observer to a specific point measured in the same scale as the provisional orthogonal direction to zenith as the highest point of the azimuth. The true north on the stereographic diagram is the positive Y axis 'straight up' and is marked with N (after COUPRIE [2011]; MAGLI [2013]).
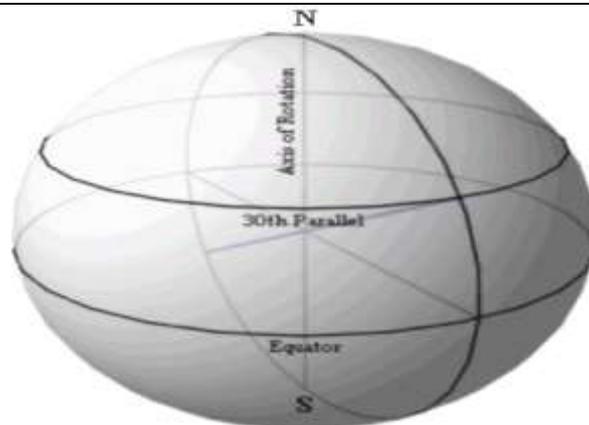

Fig. 14, b: Diagram showing the process of the Earth's rotation. The earth rotates about an imaginary line called the axis of rotation which passes through the North and South Poles of the planet (after WAKEFIELD [2016]).

Noteworthy that (Fig.13, a) illustrates the momentary position of the earth's orbit (ecliptic) to the celestial equator with the indicated equinoxes and solstices. In addition to the precession of the axis of rotation (with a period of about 25800 years), the rotation of the apsid line of the earth's orbit with a period of about 21,000 years, so the result is a change in the duration of the seasons in a year. Whilst (Fig.13, b) represents the diagram of the extreme altitudes of the sun at the solstices (with the figure being cut off from the side of the winter solstice), this is by Fixing the shadow points of the equinox and the solstices, and measuring the obliquity of the ecliptic with the help of a gnomon. It is believed that the alignment and appropriate relative location of the so-called air-shafts of the King's Chamber in the Great Pyramid was determined by observing a celestial process now known as precession. PETRIE's survey in 1881 mentions that these shafts sloped southward in order to align with the position of the stars known as 'Delta Orionis' and northward in order to align with the stars known as 'Alpha Draconis' around 2600 B.C [119]. In 1990 PETRIE'S data were used to specify that the southern shaft of the Queen's Chamber sloped towards the constellation known as 'Sirius' around 2750 B.C [120]. In 1993, the German engineer RUDOLF GANTENBRINK was able to obtain more precise measurements for the angles of these shafts confirming these hypothetical astral orientations and dates (c. 2450

B.C. +/- 25 years) **[121]**. In 1995 also the Scottish astronomer MARY BRUCK concurred with these findings but assumed a +/- 60 years margin of error **[122]**.

Results

1. In ancient Egyptian mathematics there were seven palms in a cubit, in addition to the *Seqed*, which was seven times the cotangent. The Egyptian *Seqed*/*Seqed* is the ratio of the run to the rise of a slope of a cotangent. The Rhind Papyrus which is an ancient Egyptian mathematical document mentions the *Seqed* repeatedly (e.g. 56, 57, 58, 59, 59 b and 60) in connection with many problems or issues.

2. The concept and significance of the *Seqed* in architecture becomes apparent when considering the inner slope or inclination of the triangular side of the pyramid. In modern trigonometric terms the *Seqed*-to-the-height ratio (in ancient palms) is the cotangent angle of the triangular surface.

3. The inclination of slopes was thus expressed as a value of horizontal palms and digits for every Royal cubit rise. The *Seqed* represents the run (or incline) which equals a vertical rise of 1 cubit, a word derived from the Latin 'Cubitum' for elbow (Greek πῆχυς [pechys]). Etymologically, the term 'pyk belady/baladi' is related to the Greek 'pekhus and pygon', both also meaning a cubit, with that being the number of cubits to the Greek stade (about 600 feet [180 metres]). Hence 'pyk belady' can be taken to mean 'national cubit [of Egypt]')

4. The value of the *Seqed* in ancient Egyptian architecture varies therefore. The slopes of the faces of the Great Pyramid of Giza measure a *Seqed* of 5 1/2, or 5 palms and 2 digits, which amounts to a slope of 51.84°. In current trigonometry *Seqed* (or θ) means the cotangent of the angle between the base of the pyramid and one of its faces. Thus, if the tangent of an angle is a/b then the cotangent of that angle is b/a. It is noteworthy that the right triangle (ABS), where (AB) is the height and (BS) is half the length of the base.

5. The average width of the base of the Great Pyramid amounts to about 370 'Horizon' cubits (*pechys baladi*), the disparity is nearly 1 portion per 6,000 or 0.016%, and for our purpose we assume the approximate figure corresponding to the planned or desired width. Mathematical ratios existing in this pyramid seems to be symmetrical with the proportions or values of a Pythagorean {3-4-5} triangle, giving a theoretical height of $246\,{}^{2}/_{3}$ cubits or 370 feet. But in fact, it is well known that the angle θ for the Pythagorean triangle is 53.13°, which is more than 1° greater than the angle for the Khufu's Pyramid. Furthermore the *Seqed* of a pyramid is calculated by finding **x** in terms of **y** then multiplying the coefficient of the determination of **y** by **7**. Or the other way round: it is **7** times the cotangent of the pyramid's dihedral angle.

Conclusion

The length of the historical cubit varied in ancient times. There were two main units: the first one measuring almost 18 inches and a second about 20 inches long. Also other variations occurred, some small-scale and some greater than usual. This may be due to very large deviations from the 'norm' at a given time. The concept and significance of the *Seqed* in architecture becomes apparent when considering the inner slope or inclination of the triangular side of the pyramid. There, the *Seqed* represents the run (or incline) which equals a vertical rise of 1 cubit, a word derived from the Latin 'Cubitum' for elbow (Greek πῆχυς [pechys]). In terms of modern geometry a right triangle must be designed to determine the *Seqed*, with triangle's vertical side representing the pyramids height and its horizontal side being half of the building's

base-line. Then one should draw a comparable triangle whose vertical side is equal to 1 cubit. The *Seqed* will then equal the length of the horizontal side of the second triangle. In modern trigonometric terms the *Seqed*-to-the-height ratio (in ancient palms) is the cotangent angle of the triangular surface. In the Rhind Papyrus it is said that the value of the cubit equalled 7 palms and that the value of the *Seqed* is expressed in palms and fingers (cf. specifically problem 59 a, but also problems 56, 57, 58, and 59 b). Problem 60 presents a different solution. Here the *Seqed* is considered as a fraction of a cubit. In solution of problem 59 a, the ratio between the value called and 7 palms equals 1 cubit representing exactly the ratio of the half-side to the height. Simply speaking this, then, is the concept behind the term *Seqed*. Also, this concept was suggested and considered in problems 56-60 of the Rhind Papyrus. The Egyptian *Seqed* is the rise/run ratio of the slope of a cotangent broadly based on the ancient Egyptian measurements values of the Royal cubit, the palm and the digit. There are, however, some modifications which have caused confusion concerning the length of a cubit or the accurate value of a *Seqed*. It should be mentioned that the confusion and the difficulties explaining the *Seqed* problems arise in part from the technical words employed. EISENLOHR, the first editor of the Rhind Papyrus, mentions the different interpetations by scientists as early as 1877. EISENLOHR concluded that the term *Seqed* referred to the ratio of two lines. Also, he concluded that the *Seqed* should refer to either the ratio of the half-base to the apothem or to the ratio of the half-diagonal to the sharp edge formed by the meeting of two flat or curved surfaces.

In ancient Egypt, there were seven palms in the cubit; in addition to the *Seqed* was seven times the cotangent. The Egyptian *Seqed* is the ratio of the run to the rise of a slope of the cotangent. The Rhind Papyrus which is an ancient Egyptian documentary source mentioned the *Seqed*, which is the base of many problems or issues such as; 56, 57, 58, 59 ,59 b and 60. The Rhind Papyrus contains 84 mathematical problems to be solved and there is a section assigned to the orientation of pyramids, where the term '*Seqed*' is used. The inclination of measured slopes was expressed as a value of horizontal palms and digits for every Royal cubit rise. This inclination a type of the contemporary measure, which related to 'Gradient', therefore it is a measure comparable to the slope angle of the current cotangent. The value of the *Seqed* in ancient Egyptian architecture is differentiated; the slopes of the faces of the Great Pyramid of Giza were a *Seqed* of 5 1/2, or 5 palms and 2 digits, which adjust to a slope of 51.84° from the horizontal, using the current 360 degree system. Technically, *Seqed* or θ is using the current trigonometry, the cotangent of the angle between the base of the pyramid and one of its faces. Thus, if the tangent of an angle is a/b then the cotangent of that angle is b/a. It is noteworthy that the right triangle (ABS), where (AB) is the height and (BS) is half the length of the base. Calculating the *Seqed* meant determining an angle θ. One could therefore say that also current trigonometry uses the *Seqed*. In ancient Egyptian as well as in present trigonometry the cotangent of an angle is the value between the base of a pyramid and one of its faces. Thus, if the tangent of an angle is a/b then the cotangent of that angle is b/a. The *Seqed* of a pyramid is considered to be the number of palms in the horizontal corresponding to a rise of one cubit or seven palms. The average width of the base of the Great Pyramid is about 370 'Horizon' cubits (*pechys baladi*), the disparity is nearly 1 portion in 6,000 or 0.016%, and therefore the approximation number will supposedly be the planned or the meant width. This pyramid was apparently intended to conform to the

proportions of a Pythagorean {3-4-5} triangle, giving a theoretical height of 246⅔ cubits or 370 feet. In fact, it is well known that the angle θ for the Pythagorean triangle is 53.13°, which is more than 1° greater than the angle for the Khufu's Pyramid.

References

[1] Waziry, A., (2016). Evidences on Harbingers of Mensuration Methodology in Ancient Egyptian Mathematics and Geometry, International Journal of Scientific & Engineering Research, Vol. 7, Issue 12,pp.1624-1638.

[2] Eisenlohr, A., (1877). Ein Mathematisches Handbuch der alten Aegypter (Papyrus Rhind des British Museum), 2 Vols. Hinrichs, Leipzig. (Digitized 30 Jan 2008 by the University of Michigan).

[3] Herz-Fischler, R., (2000). The Shape of the Great Pyramid, Wilfrid Laurier University Press, Canada.

[4] Petrie, W.M.F., (1877). Inductive Metrology, Saunders, London.

[5] Carter, H.& Gardiner ,A.H., (1917).The tomb of Ramesses IV and the Turin plan of a royal tomb, JEA, Vol. 4 , pp. 130-158.

[6] Weeks, K.R., (1979).The Berkeley Map of the Theban Necropolis: Report of the Second Season, 1979, University of California, Berkeley.

[7] Iversen, E., (1955). Canon and Proportions in Egyptian Art, Sidgwick & Jackson Publishers, London.

[8] Hayes, W., (1957). Varia from the Time of Hatshepsut, MDAIK, Vol. 15, pp.78-90.

[9] Budge, E.A.W., (1960). An Egyptian Hieroglyphic Dictionary, with an Index of English Words, King List and Geographical List with Indexes, List of Hieroglyphic Characters, Coptic and Semitic Alphabets, etc. Vol. I. A-Kha, Frederick Ungar Publishing Co., New York.

[10] Desroches-Noblecourt, Ch., (1965). Life and Death of a Pharaoh Tutankhamen, Doubleday & Company press, New York.

[11] Lorenzen, E., (1970). Along the line where columns are set. Book II continuing Technological Studies in Ancient Metrology, Translated by John R.B. Gosney, Nyt Nordisk Forlag, Arnold Busck, Copenhagen.

[12] Müller, H.W., (1973). Der Kanon in der ägyptischen Kunst, Der "Vermessene" Mensch. Anthropometrie in Kunst und Wissenschaft, München, pp. 9-31.

[13] Gardiner, A. H., (1973).Egyptian Grammar, 3rd. ed. Cambridge University Press, London.

[14] Stricker, B. H., (1976). De Maat der dingen, B.V. Noord-Hollandsche Uitgeversmaatschappij, Amsterdam /Mededelingen der Koninklijke Nederlandse Maatschappij van Wetenschappen, Afd. Letterkunde/ Amsterdam, Vol. 39.No.7,pp.1-36.

[15] Gillings, R., (1982). Mathematics in the Time of the Pharaohs, Dover Publications, USA.

[16] Robins, G., (1985). Standing Figures in the Late Grid System of the 26th Dynasty, SAK, Vol. 12, pp. 101-116.

[17] Gay, R., & Shute, C.D., (1985). Mathematical Bases of Ancient Egyptian Architecture and Graphic Art, Historia Mathematica, Vol. 12, pp.107-122.

[18] Legon, J.A.R., (1988). The Design of the Pyramid of Khufu, DE, Vol. 12, pp. 41-48.

[19] Gay, R. & Shute, C.D., (1990). Irrational Numbers and Pyramids, DE, Vol. 18, pp.43-53.

[20] Naguib, V.,(1991).The Rod (nbj) and Its Use in Egyptian Architecture, GM, Vol. 121, pp. 101-110.

[21] Kanawati, N., (1986). The Rock Tombs of El Hawawish, Vol. V: The Cemetery of Akhmim, Macquarie Ancient History Association Publisher, Sydney.

[22] Bard, K.A., & Shubert, S.B., (Eds.) (1999). Akhmim, Encyclopedia of the Archaeology of Ancient Egypt, Routlege/Taylor & Francis Group, London & New York, pp.124-127.

[23] Thompson, E.M., (2001). A Study of the Architecture of the Cemetery of El-Hawawish at Akhmim in Upper Egypt in the Old Kingdom, Thesis submitted for the Degree of Master of Arts within Macquarie University, Division of Humanities, School of Ancient History, Sydney.

[24] Arnold, D., (1991). Building in Egypt: Pharaonic Stone Masonry, Oxford University Press, Oxford.

[25] Lauer, J.P., (1930). Étude sur quelques monuments de la IIIe dynastie (Pyramide à degrés de Saqqarah), ASAE, Vol. 30, pp.137-146.

[26] Lauer, J.P., (1931).Étude sur quelques monuments de la IIIe dynastie, ASAE, Vol. 31, pp.49-64.

[27] Roik, E., (1993). Das Längenmasssystem im Alten Ägypten, Christian-Rosenkreuz-Verlag, Hamburg.

[28] Legon, J.A.R., (1996).The Cubit and the Egyptian Canon of Art, DE, Vol. 35, pp.61-76.

[29] Shaffer, B.S., (1996). Treasures of the Ancients, C.F.I. Press, U.S.A.

[30] Merriam-Webster, (2016). Merriam-Webster's International Encyclopædia Britannica since 1828-2016. Enc. Britannica Inc. http://corporate.britannica.com (at July 2016).

[31] Petrie, W.M.F., (1883).The Pyramids and Temples of Gizeh, Field & Tuer, The Leadenhall Press, London.

[32] Edwards, I.E.S., (1986). The Pyramids of Egypt: Revised Edition, Penguin Books, London.

[33] Petrie, W.M.F., (1892). Medium, David Nutt publisher, London.

[34] Lightbody, D., (2008). Egyptian Tomb Architecture: The Archaeological Facts of Pharaonic Circular Symbolism. British Archaeological Reports International Series, S1852, pp. 22–27.

[35] Petrie, W.M.F., (1940). Wisdom of the Egyptians, British School of Archaeology in Egypt and B. Quaritch Ltd., London.

[36] Erman, E., & Grapow, H., (1971). Wörterbuch der ägyptischen Sprache, 7 Vols. Akademie Verlag Publishing, Berlin.

[37] Hannig, R., (2005). Groβes Hand Wörterbuch, Ägyptische Deutsch, Die Sprache der Pharaonen Marburger Edition, Marburg – Germany.

[38] Borger, R., (2005). Ägyptische Wörterbuch II; Mittleres Reich und Zweite Zwischenzeit, der mich das Systematische Arbeitenlehrte, Marburg University Edition, Marburg – Germany.

[39] Faulkner, R. O., (1991) A concise dictionary of Middle Egyptian, Griffith Institute Ashmolean Museum Press, Oxford.

[40] Dickson, P., (2006), Dictionary of Middle Egyptian in Gardiner Classification Order, Creative Commons Press, USA.

[41] Peet, T.E., (1923). The Rhind Mathematical Papyrus British Museum 10057 and 10058: introduction, transcription, translation and commentary, Univ. Pr. of Liverpool; Hodder & Stoughton, London.

[42] Gunn, B. & Peet, T.E., (1929). Four geometrical problems from the Moscow mathematical papyrus, JEA, Vol. 15, pp.167-185.

[43] Robins, G., (1987). The Rhind Mathematical Papyrus: an ancient Egyptian text, London.

[44] Imhausen, A., (2003). Ägyptische Algorithmen: eine Untersuchung zu den mittelägyptischen mathematischen Aufgabentexten, Wiesbaden.

[45] Borchardt, L., (1913). Das Grabdenkmal des Königs S'aḫu-Re, Vol. II, Ausgrabungen der Deutschen Orient-Gesellschaft,Verlag Hinrichs, Leipzig.

[46] Sethe, K., (1908-1910). Die Altägyptischen Pyramidentexte, 2 Vols. J. C. Hinrichs'sche Buchhandlung, Leipzig.

[47] De Buck, A., (1935-1961). The Egyptian Coffin Texts, 7 Vols. University of Chicago Press, Chicago.

[48] Lesko, L., & Lesko, B., (Eds.). (2004). A dictionary of Late Egyptian, Vol.2, 2nd ed., Providence, RI : B.C. Scribe Publications, Providence – USA.

[49] Sethe, K., (1903). Urkunden des Alten Reichs, Vol.1, Leipzig.

[50] Steindorff, G., (1955-1961). Urkunden des äegyptischen Altertums, IV, Sethe, K., Urkunden der 18 Dynastie, 1-16, Berlin, 1914; Helck, W., Urkunden der 18 Dynastie, 17-22, Berlin .

[51] Kitchen, K. A., (1970). Ramesside inscriptions: historical and biographical, 6 Vols. B.H. Blackwell, Oxford.

[52] Helck, W., (2000). Die Prophezeiung des Nfr.tj, KÄT, Vol. 2, Wiesbaden.

[53] Helck, W., (1977). Die Lehre für König Merikare, KÄT, Vol.7, Wiesbaden.

[54] Goyon, G., (1936). Les travaux de Chou et les tribulations de Geb d'après le naos 2248 d'Ismaïlia, Kêmi, Vol. 6, pp. 1- 42.

[55] Quibell, J.E., (1911-1912). Excavations at Saqqara (1905-1914), 6 Vols. Impr. de l'Institut français d'archéologie orientale, Le Caire.

[56] Griffith, F. LI., (1889). The inscriptions of Siût and Dêr Rîfeh, Trübner Publisher, London.

[57] Davies, N. de G., (1900-1901). The Mastaba of Ptahhetep and Akhethetep at Saqqara, 2 Vols. The Archeological Survey of Egypt, Egyptian Exploration Fund, London.

[58] Wreszinski, W., (1913). Der Papyrus Ebers, J.C. Hinrichs'sche Buchhandlung, Leipzig.

[59] Davies, N. de G., (1922-1923). The Tomb of Puyemrê at Thebes, 2 Vols. Publications of the Metropolitan Museum of Art, New York.

[60] Baillet, J., (1906) Les noms de l'esclave en égyptien, Rec. Trav., Vol. 28, pp.113-131.

[61] Möller,G.,(1909-1936). Hieratische Paläographie, 4 Vols. J.C. Hinrichs'sche Buchhandlung, Leipzig.

[62] Vycichl, W., (1973). Zwei ägyptische Nomina der hamito-semitischen Nominalklasse katul. Berichtigung der bisherigen Lesung von Hm.t "Frau" und qd.t"Kite, Gewicht von 9,1 Gramm", ZÄS, Vol. 99, 2, pp. 135-139.

[63] Helck, W., & Otto, E., (1999). Maße und Gewichte, In: Kleines Lexikon der Ägyptologie, Harrassowitz, Wiersbaden, cols.178-180.

[64] Vleeming, S.P., (1980). Maße und Gewichte in den demotischen Texten, In: Helck, W., Westendorf, W., (Eds.), Lexikon der Ägyptologie, Vol. 3, Harrassowitz, Wiesbaden, cols. 1209-1214.

[65] Kamel, A., (2000). Maße und Gewichte, In: Wissenschaft im Alten Ägypten, Kemet, Vol. 4, Berlin, pp.38-40.

[66] Wilson, P., (1997). A Ptolemaic Lexikon. A Lexicographical Study of the Texts in the Temple of Edfu, Uitgeverij Peeters en Departement Oosterse Studies, Leuven.

[67] Devauchelle, D., (1986). 1D: deben or kite? Enchoria, Vol.14, pp.157-158.

[68] Eran, A., (1985). The Old-Egyptian Weight-Unit "Deben" - Its Reality, its Dispersion and its Late Echo, Acta Metrologiae Historicae, Vol. 14, pp.94-116.

[69] Raymond, W., (1924). La "kite" d'or de Byblos, Revue égyptologique, Vol. 2, pp. 21-37.

[70] Spiegelberg, W., (1923). [kd.t] als Bezeichnung der Kite, ZÄS, Vol. 58, p.154.

[71] Kammerzell, F., (1994). Zur Etymologie des ägyptischen Zahlworts "4", Lingua Aegyptia, Vol.4, pp.165-189.

[72] Crum, W. E., (1962). A Coptic Dictionary, Clarendon Press, Oxford.

[73] Černý, J., (1976). Coptic Etymological Dictionary, Cambridge University Press, Cambridge.

[74] Pierce, R.H., (1965). Notes on Obols and Agios in Demotic Papyri, JEA, Vol.51, pp. 155-159.

[75] Plumley, J.M., (1948). An introductory Coptic Grammar; Sahidic Dialect, Home and Van Thal, London.

[76] Westendorf, W., (1977). Koptisches Handwörterbuch. Bearbeitet auf Grund des Koptischen Handwörterbuchs Wilhelm Spiegelbergs, Carl Winter Universitätsverlag, Heidelberg.

[77] Till, W.C., (1955). Koptische Grammatik; Saïdischer Dialekt mit Bibliographie, Lesestücken und Wörterverzeichnissen, Otto Harrassowitz, Leipzig.

[78] Grieshammer, R., (1970). Das Jenseitsgericht in den Sargtexten, Otto Harrassowitz, Ägyptologische Abhandlungen, Herausgegeben von Wolfgang Helck und Eberhard Otto, Vol. 20, Wiesbaden, pp. 1-188.

[79] Seeber, Ch., (1976). Untersuchungen zur Darstellung des Totengerichts im Alten Ägypten, Deutscher Kunstverlag, MÄS/ Münchner Ägyptologische Studien herausgegeben von Hans Wolfgang Müller, Vol.35, München/Berlin, pp.1- 271.

[80] Rossi, C., (2003). Architecture and Mathematics in Ancient Egypt, Cambridge University Press, London

[81] Verner, M., (2002). The Pyramids; The Mystery, Culture, and Science of Egypt's Great Monuments Grove Press, New York.

[82] Lehner, M., (1999). The Complete pyramids, London, Thames and Hudson Press.

[83] Lehner, M., (1985). A contextual approach to the Giza pyramids, Archiv fur Orientforsch, Vol. 31, pp.136-158.

[84] Jeffreys, D., (1998). The Topography of Heliopolis and Memphis: some cognitive aspects, Beitrage zur Kulturgeschichte Ägyptens, Rainer Stadelmann gewidmet, Mainz, pp.63-71.

[85] Baines, J., & Malek, J., (1984). The Cultural Atlas of the World: Ancient Egypt, Oxford.

[86] Goyon, J.L., (1977). The secrets of the pyramids' builders, Pygmalion Press, Paris.

[87] Barta, M., (2005). Location of Old Kingdom pyramids in Egypt, Cambridge Arch. Journal, Vol. 15, pp.177-191.

[88] Stadelmann, R., (1991). Die ägyptischen Pyramiden, Vom Ziegelbau zum Weltwunder: P. von Zabern Press, Mainz.

[89] Spence, K., (2000). Ancient Egyptian Chronology and the Astronomical Orientation of Pyramids, Nature, Vol. 408, 16 November, pp. 320-324.

[90] Magli, G., (2010). Topography, astronomy and dynastic history in the alignments of the pyramid fields of the Old Kingdom, Mediterranean Archaeology and Archaeometry, Vol. 10, No.2, pp. 59-74.

[91] Wakefield, S.P., (2016). A Study of Ancient Unified Numerical and Metrological Systems, Mountains of the Moon, The Stone-Cored Pyramids of Aereia, Last update 13th July 2016, Chapter 3, Figs. 3-6.

[92] Bryn, O.J., (2010). Retracing Khufu's Great Pyramid; The "Diamond Matrix" and the number 7, Nordic Journal of Architectural Research, Vol. 22, No.1/2, pp.135-144, Figs.1-3.

[93] Nickel, J. D., (2009). The Proto-Trigonometry of the Pyramids, In: Biblical Christian World View, pp.1-2 www.biblicalchristianworldview.net (at 22/7/2016).

[94] Maor, E., (1998).Trigonometric Delights, Princeton University Press, Princeton.

[95] Faulkner, R. O., (1933). The Papyrus Bremner-Rhind, British Museum, No. 10188, Bibliotheca Aegyptiaca, Vol.3, pp. 1-10.

[96] Faulkner, R. O., (1937). The Bremner-Rhind Papyrus III, JEA, Vol. 23, pp.166-185.

[97] Spalinger, A., (1990). The Rhind Mathematical Papyrus as a Historical Document, SAK, Vol. 17, pp.295-337.

[98] Schott, S., (1952). Astronomie und Mathematik, Wissenschaftliche Literatur, In: Brunner, H., Kees, H., Morenz, S., & Spiegel, J., (Eds.), Handbuch der Orientalistik, 2.Vols. E. J. Brill, Leiden, pp.170-176.

[99] Struve, V., (1930). Mathematischer Papyrus des Staatlichen Museums der Schönen Künste in Moskau / hrsg. und kommentiert von W. W. Struve unter Benutzung einer hieroglyphischen Transkription von B. A. Turajeff, Berlin, J. Springer, XII, pp.1-197, figs.1-10 /Quellen und Studien zur Geschichte der Mathematik; Abteilung A, Quellen 1, pp. 120-123, Nos. 14,18.

[100] Toomer, G. J., (1971). Mathematics and Astronomy, The Legacy of Egypt, Second Ed., Edited by J.R. Harris, At the Clarendon Press, Oxford, pp. 27-54.

[101] Möller, G., (1913). Die Beiden Toten Papyrus Rhind des Museums zu Edinburg, Leipzig.

[102] https://www.math.washington.edu/~greenber/slope.gif (at 15/7/2016).

[103] https://www.math.washington.edu/~greenber/slope.html (at 15/7/2016).

[104] http://www.mathorigins.com/image%20grid.html (at 15/7/2016).

[105] De Young, G., (2009). Diagrams in ancient Egyptian geometry Survey and assessment, Historia Mathematica, Vol.36, pp.321–373, fig.2.

[106] Clagett, M., (1999). Ancient Egyptian Science: A Source Book. Vol. 3: Ancient Egyptian Mathematics, Memoirs of the American Philosophical Society 232, American Philosophical Society, Philadelphia.

[107] Magli, G., (2013). Architecture, Astronomy and Sacred Landscape in Ancient Egypt, Cambridge University Press, New York.

[108] Kaiser, W., (1956). Zu den Sonnenheiligtümern der 5 Dynastie, MDAIK, Vol.14, pp.104-116.

[109] Magdolen, D., (2000). On the Orientation of Old-Kingdom Royal Tombs. In: Bárta, M., & Krejcí, J., (Eds.), Abusir and Saqqara in the Year 2000, Prague: Academy of Sciences of the Czech, Oriental Institute, pp. 491-498.

[110] Stadelmann, R., (2000). Userkaf in Saqqara und Abusir, In: Saqqara in the Year 2000, Archiv Orientální Supplementa, Vol. 9, Prague, pp. 529-542.

[111] Goedicke, H., (2001). Abusir - Saqqara – Giza, In: Bárta, M., & Krejcí, J., (Eds.), Abusir and Saqqara in the Year 2000, Prague: Academy of Sciences of the Czech, Oriental Institute, pp.397-412.

[112] Nuzzolo, M., (2007). The Sun Temples of the Vth Dynasty: A Reassessment, SAK, Vol.36, pp. 217-247.

[113] Hohenkerk, C.Y., Yallop, B.D., Et al., (2005). Celestial Reference Systems, In: Seidelmann, P.K., (Ed.) Explanatory Supplement to the Astronomical Sausalito, University Science Books Press, USA, pp. 99-100.

[114] Mueller, I., (1969). Spherical and Practical Astronomy as applied to Geodesy, New York, pp.58-59.

[115] Neugebauer, O., (1950). The Alleged Babylonian Discovery of the Precession of the Equinoxes, Journal of the American Oriental Society, Vol. 70, No.1 , pp.1-2.

[116] Neugebauer, O., (1980). On the Orientation of the Pyramids, Centaurs, An International Journal of History of Science and its Cultural Aspects ,Vol.24, Issue.1, USA, Published Online at 26 July,2007,pp.1-3, Fig.2.

[117] Isler, M., (1989). An Ancient Method of Finding and Extending Direction, JARCE, Vol. 26, pp.191-200.

[118] Couprie, D. L., (2011). Heaven and Earth in Ancient Greek Cosmology from Thales to Heraclides Ponticus, New York, Figs. 2.13; 2.15.

[119] Trimble, V., (1964). Astronomical investigation concerning the so-called Air-shafts of Cheops's Pyramid, Mitteilungen des Instituts für Orientforschung, Vol.10, No. 2/3, pp.183-187.

[120] Bauval, R., (1990). The seeding of the star-gods: a fertility rite inside Cheop's Pyramid? , Discussion in Egyptology, Vol.16, pp. 21-25.

[121] Bauval, R., (1993). Cheop's Pyramid: A new dating using the latest astronomical Data, Discussions in Egyptology, Vol. 26, pp. 5-6.

[122] Bruck, M.T., (1995). Can the Great Pyramid be astronomically dated? , Journal of the British Astronomical Association, Vol.105, No. 4, pp. 161-164.

[123] Imhausen, A., (2016). Mathematics in Ancient Egypt: A Contextual History, Princeton University Press, Oxford.

[124] Laskar, J., (1986). Erratum - Secular Terms of Classical Planetary Theories Using the Results of General Theory, Astronomy and Astrophysics, Vol.157, pp.59-70, and Vol. 164, No.2, p.437.

[125] Edvardsson, S., Karlsson, K. G., and Engholm, M., (2002). Accurate spin axes and solar system dynamics: Climatic variations for the earth and mars, Astronomy and Astrophysics, Vol. 384, No.2, pp.689-701.

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