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Sufficient Conditions for Starlikeness of Univalent Analytic Functions Involving Flett Integral operator

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Abstract

In this paper we determined a condition on M for which $\frac{I_m^n F_\mu(f)(z)}{z} \prec 1+Mz$ implies $f(z) \in S_m^n(\mu,\alpha)$, where I_m^n and $F_\mu(f)(z)$ are respectively, the familiar multiplier transformations and the familiar Bernardi-Libera-Livingston operator.

Keywords: Univalent functions, starlike functions, convex functions, differential subordination, multiplier transformations.

Introduction

Let $A(m)$ denote the class of functions of the form:

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k \qquad (m \in \mathbb{N} = \{1,2,....\}), \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. We note that $A(1) = A$, let $S, S^*(\alpha)$ and $C(\alpha)$ $(0 \le \alpha < 1)$ be the subclasses of functions in A which are, respectively, univalent, starlike of order α and convex of order α in U. We denote by $S^*(0) = S^*$ and $C(0) = C$. If f and g are analytic in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [2], see also [7]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f(z) \in A(m)$ given by (1.1) and $g(z) \in A(m)$ given by

$$g(z) = z + \sum_{k=m+1}^{\infty} b_n z^k \qquad (m \in \mathbb{N}),$$

the Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z)=z+\sum_{k=m+1}^{\infty}a_k b_k z^k=(g*f)(z).$$

Also, for an analytic function $f(z)$ given by (1.1), for all integer values of n and for all $m\in\mathbb{N},$ we define the multiplier transformation $I_m^n f(z)$ by

$$I_m^n f(z)=z+\sum_{k=m+1}^{\infty}k^{-n}a_k z^k \qquad (z\in U). \qquad (1.2)$$

Clearly, the function $I_m^n f(z)$ is analytic in U . We note that

$$I_m^n(I_m^l f(z))=I_m^{n+l}f(z) \qquad (m\in\mathbb{N}; z\in U)$$

for all integers n and $l.$ We also note that :

(i) $I_1^n f(z)=I^n f(z)$ (see Flett [3]);

(ii) $I_1^{-n}f(z)=z+\sum_{k=2}^{\infty}k^n a_k z^k$

$$=D^n f(z)(n\in\mathbb{N}_0=\mathbb{N}\cup\{0\})$$

(see Salagean [9]).

It follows from (1.2) that

$$z(I_m^n f(z))'=I_m^{n-1}f(z), \qquad (1.3)$$

$I_m^0 f(z)=f(z), I_m^{-1}f(z)=zf'(z)$ and

$I_m^{-2}f(z)=z(f'(z)+zf''(z)).$

For a function $f(z)\in A$ (see [1], [4] and [6]) the generalized Bernardi-Libera-Livingston operator $F_\mu : A\to A$ is defined by

$$F_\mu(f)(z)=\frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt$$
$$=z+\sum_{k=2}^{\infty}\frac{\mu+1}{\mu+k}a_k z^k$$
$$=\left(z+\sum_{k=2}^{\infty}\frac{\mu+1}{\mu+k}z^k\right)*f(z)$$
$$=\left[z\ {}_2F_1(1,\mu+1;\mu+2;z)\right]*f(z)$$
$$(\mu>-1; z\in U), \qquad (1.4)$$

where ${}_2F_1$ is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

$$(a,b,c\in\mathbb{C}; c\notin\mathbb{Z}_0=\{0,-1,-2,...\}),$$

and $(d)_k$ denotes the Pochhammer symbol given in terms of the Gamma function Γ, by

$$(d)_k=\frac{\Gamma(d+k)}{\Gamma(d)}\begin{cases}1 & (k=0)\\ d(d+1)...(d+k-1) & (k\in\mathbb{N}).\end{cases}$$

We note that ${}_2F_1$ represents an analytic function in U (see for details [10 ,Ch.14]).

It is easily seen from (1.4) that

$$z(I_m^n F_\mu(f)(z))'=(\mu+1)I_m^n f(z)-\mu\ I_m^n F_\mu(f)(z). \qquad (1.5)$$

Using the operator $I^n f(z)$ Patel and Sahoo [8] introduced and investigated various properties and characteristics in U by using the techniques of Briot-Bouquet differential subordination.

Definition. *A function* $f(z)\in A(m)$ *is said to be in the class* $S_m^n(\mu,\alpha)$ *if and only if*

$$\mathrm{Re}\left\{\frac{z(I_m^n F_\mu(f)(z))'}{I_m^n F_\mu(f)(z)}\right\}>\alpha$$

$$(0\le\alpha<1; n\in\mathbb{Z}; m\in\mathbb{N}; \mu>-1; z\in U). \qquad (1.6)$$

We note that:

(i) $S_m^0(\mu,\alpha)=S_m(\mu,\alpha),$ where $S_m(\mu,\alpha)$ is the class of functions $f(z)\in A(m)$ which satisfy:

$$\mathrm{Re}\left\{\frac{zF_\mu'(f)(z)}{F_\mu(f)(z)}\right\}>\alpha$$
$$(0\le\alpha<1; \mu>-1; z\in U); \qquad (1.7)$$

(ii) $S_m^{-1}(\mu,\alpha)=C_m(\mu,\alpha),$ where $C_m(\mu,\alpha)$ is the class of functions $f(z)\in A(m)$ which satisfy:

$$\mathrm{Re}\left\{1+\frac{zF_\mu''(f)(z)}{F_\mu'(f)(z)}\right\}>\alpha$$
$$(0\le\alpha<1; \mu>-1; z\in U); \qquad (1.8)$$

(iii) $S_m^n(1,\alpha)=S_m^n(\alpha),$ where $S_m^n(\alpha)$ is the class of functions $f(z)\in A(m)$ which satisfy:

$$\mathrm{Re}\left\{\frac{zI_m^n F_1'(f)(z)}{I_m^n F_1(f)(z)}\right\}>\alpha \qquad (0\le\alpha<1; z\in U), \qquad (1.9)$$

where

$$F_1(f)(z)=\frac{2}{z}\int_0^z f(t)\,dt; \qquad (1.10)$$

(iv) $S_m^n(0,\alpha)=\hat{S}_m^n(\alpha),$ where $\hat{S}_m^n(\alpha)$ is the class of functions $f(z)\in A(m)$ which satisfy:

$$\mathrm{Re}\left\{\frac{zI_m^nF_0'(f)(z)}{I_m^nF_0(f)(z)}\right\}>\alpha \qquad (0\le\alpha<1; z\in U), \tag{1.11}$$

where

$$F_0(f)(z)=\int_0^z\frac{f(t)}{t}dt=I_m^1f(z). \tag{1.12}$$

Main Result

Unless otherwise mentioned, we assume throughout this paper that (-1 ≤ B < A ≤ 1; 0 ≤ α < 1; $n\in\mathbb{Z}$; $m\in\mathbb{N}$ and μ > -1).

We now state the following lemma which can be proved analogously to similar result proved by Patel and Sahoo [8, Theorem 3].

Lemma 1. *If* $f(z)\in A(m)$ *satisfies*

$$\frac{I_m^nf(z)}{z}\prec\frac{1+Az}{1+Bz}, \tag{2.1}$$

then

$$\frac{I_m^nF_\mu(f)(z)}{z}\prec q(z)\prec\frac{1+Az}{1+Bz}, \tag{2.2}$$

where

$$q(z)=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+Bz)^{-1}\\ \times{}_2F_1\left(1,1;\mu+m+1;\frac{Bz}{Bz+1}\right) & (B\ne0)\\ 1+\frac{\mu+1}{\mu+m+1}Az & (B=0)\end{cases} \tag{2.3}$$

is the best dominant of (2.2). *Furthermore,*

$$\mathrm{Re}\left\{\frac{I_m^nF_\mu(f)(z)}{z}\right\}>\rho(A,B,\mu,m)$$

$$=\begin{cases}\frac{A}{B}+\left(1-\frac{A}{B}\right)(1+B)^{-1}\\ \times{}_2F_1\left(1,1;\mu+m+1;\frac{B}{B-1}\right) & (B\ne0)\\ 1-\frac{\mu+1}{\mu+m+1}A & (B=0).\end{cases} \tag{2.4}$$

The result is the best possible.

Theorem 1. *Let the operator* $F_\mu(f)(z)$ *defined by (1.4) satisfy the following subordination condition:*

$$\frac{I_m^nF_\mu(f)(z)}{z}\prec1+Mz \qquad (f\in A(m)), \tag{2.5}$$

where

$$M=\frac{(1-\alpha)(1+\frac{m}{\mu+1})}{|\mu+\alpha|+\sqrt{(\mu+1)^2+(\mu+1+m)^2}}. \tag{2.6}$$

Then $f(z)\in S_m^n(\mu,\alpha)$.

Proof. From (1.2) and (1.4), it follows that

$$I_m^nF_\mu(f)(z)=\frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}I_m^nf(t)dt. \tag{2.7}$$

Defining the function $\phi(z)$ in U by

$$\varphi(z)=\frac{I_m^nF_\mu(f)(z)}{z} \qquad (z\in U) \tag{2.8}$$

we see that $\phi(z)=1+p_mz^m+p_{m+1}z^{m+1}+...$ is analytic in U and $\phi(0)=1$. From Lemma 1 with $A=M$ and $B=0$, we have

$$\phi(z)\prec1+\frac{\mu+1}{\mu+1+m}Mz \quad ,$$

which is equivalent to

$$|\phi(z)-1|<\frac{\mu+1}{\mu+1+m}M=N<1 \quad (z\in U). \tag{2.9}$$

Set

$$P(z)=\frac{1}{1-\alpha}\left(\frac{z(I_m^nF_\mu(f)(z))'}{I_m^nF_\mu(f)(z)}-\alpha\right). \tag{2.10}$$

Using the identity (1.5) followed by (2.8), we obtain

$$\frac{I_m^nf(z)}{z}=\left[\left(1-\frac{1-\alpha}{\mu+1}\right)+\left(\frac{1-\alpha}{\mu+1}\right)P(z)\right]\varphi(z). \tag{2.11}$$

In view of (2.11), the hypothesis (2.5) can be written as follows:

$$\left|\left(1-\frac{1-\alpha}{\mu+1}\right)\varphi(z)+\frac{1-\alpha}{\mu+1}P(z)\varphi(z)-1\right|<M$$

$$=\frac{\mu+1+m}{\mu+1}N. \tag{2.12}$$

We need to show that (2.12) yields

$$\mathrm{Re}\{P(z)\}>0 \quad (z\in U). \tag{2.13}$$

If we suppose that $\mathrm{Re}\{P(z)\}\ngtr0$ $(z\in U)$, then there exists a point $z_0\in U$ such that $P(z_0)=ix$ for some $x\in\mathrm{R}$. To prove (2.13), it is sufficient to obtain a contradiction from the following inequality:

$$W = \left|\left(1-\frac{1-\alpha}{\mu+1}\right)\phi(z_0)+\frac{1-\alpha}{\mu+1}P(z_0)\phi(z_0)-1\right|$$
$$\geq M.$$

Let $\phi(z_0)=u+iv$. Then, by using (2.9) and the triangle inequality, we obtain that

$$W^2=\left|\left(1-\frac{1-\alpha}{\mu+1}\right)\varphi(z_0)+\frac{1-\alpha}{\mu+1}P(z_0)\varphi(z_0)-1\right|^2$$
$$=(u^2+v^2)\left(\frac{1-\alpha}{\mu+1}\right)^2x^2+\frac{2(1-\alpha)}{\mu+1}vx$$
$$+\left|\left(1-\frac{1-\alpha}{\mu+1}\right)\phi(z_0)-1\right|^2$$
$$\geq(u^2+v^2)\left(\frac{1-\alpha}{\mu+1}\right)^2x^2+2\frac{(1-\alpha)}{\mu+1}vx$$
$$+\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2.$$

Setting

$$\Psi(x)=W^2-M^2$$
$$=(u^2+v^2)\left(\frac{1-\alpha}{\mu+1}\right)^2x^2+2\frac{(1-\alpha)}{\mu+1}vx$$
$$+\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2$$
$$-\left(\frac{\mu+1+m}{\mu+1}\right)^2N^2,$$

we note that (2.12) holds true if $\Psi(x)\geq 0$ for any $x\in\mathrm{R}$. Since

$(u^2+v^2)(\frac{1-\alpha}{\mu+1})^2>0,$S

the inequality $\Psi(x)\geq 0$ holds true if the discriminant $\Delta\leq 0$; that is

$$\Delta=4\left[\left(\frac{1-\alpha}{\mu+1}\right)^2v^2-\left(\frac{1-\alpha}{\mu+1}\right)^2(u^2+v^2)\right.$$
$$\left.\left\{\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2-\left(\frac{\mu+1+m}{\mu+1}\right)^2N^2\right]\right.$$
$$\leq 0,$$

which is equivalent to

$$v^2\left[1-\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2+\left(\frac{\mu+1+m}{\mu+1}\right)^2N^2\right]$$
$$\leq u^2\left[\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2+\left(\frac{\mu+1+m}{\mu+1}\right)^2N^2\right].$$

Putting $\phi(z_0)-1=\xi e^{i\theta}$ for some real $\theta\in\mathrm{R}$, we get

$$\frac{v^2}{u^2}=\frac{\xi^2\sin^2\theta}{(1+\xi\cos\theta)^2}.$$

Since the above expression attains its maximum value at $\cos\theta=-\xi$, by using (2.9), we obtain

$$\frac{v^2}{u^2}\leq\frac{\xi^2}{1-\xi^2}\leq\frac{N^2}{1-N^2}$$
$$\leq\frac{\left[\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2-\left(\frac{\mu+1+m}{\mu+1}\right)^2N^2\right]}{\left[1-\left(\frac{1-\alpha}{\mu+1}-\left|1-\frac{1-\alpha}{\mu+1}\right|N\right)^2+\left(\frac{\mu+1+m}{\mu+1}\right)^2N^2\right]},$$

which yields $\Delta\leq 0$. Therefore, $W\geq M$, which contradicts (2.12), hence $\mathrm{Re}\{P(z)\}>0$ $(z\in U)$. This proves that $f(z)\in S_m^n(\mu,\alpha)$, which completes the proof of Theorem 1.

Putting $n=0$ in Theorem 1, we obtain the following result.

Corollary 1. *Let the operator* $F_\mu(f)(z)$ *defined by (1.4) satisfy the following subordination condition:*

$$\frac{F_\mu(f)(z)}{z}\prec 1+Mz \quad (f(z)\in A(m)),$$

where M is given by (2.6). Then $f(z)\in S_m(\mu,\alpha)$.

Putting $n=-1$ in Theorem 1, we obtain the following result.

Corollary 2. *Let the operator* $F_\mu(f)(z)$ *defined by (1.4) satisfy the following subordination condition:*

$$F'_\mu(f)(z)\prec 1+Mz \quad (f(z)\in A(m)),$$

where M *is given by (2.6). Then* $f(z)\in C_m(\mu;\alpha)$.

Putting $\mu=1$ in Theorem 1, we obtain the following result.

Corollary 3. *Let the operator* $F_1(f)(z)$ *defined by (1.10) satisfies the following subordination condition*:

$$\frac{I_m^n F_1(f)(z)}{z} \prec 1+Mz \qquad (f \in A(m)),$$

where

$$M = \frac{(1-\alpha)(1+\frac{m}{2})}{(1+\alpha)+\sqrt{4+(2+m)^2}}.$$

Then $f(z) \in S_m^n(\alpha)$.

Putting $\mu = 0$ in Theorem 1, we obtain the following result.

Corollary 4. L*et the operator* $F_0(f)(z)$ *defined by (1.12) satisfies the following subordination condition*:

$$\frac{I_m^n F_0 f(z)}{z} \prec 1+Mz \qquad (f \in A(m)),$$

where

$$M = \frac{(1-\alpha)(1+m)}{\alpha+\sqrt{1+(1+m)^2}}.$$

Then $f(z) \in \hat{S}_m^n(\alpha)$.

Remark 1. *Putting* $n=0$ *in Corollary 4 we obtain the result obtained by Liu* [5, Theorem 2.2 with $\zeta = a = 1$] .

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الملخص العربي

عنوان البحث: شرط ضروري جديد لنجمية الدالة التحليلية وحيدة القيمة المحتوية على مؤثر فلت التكاملي

رابحة محمد الأشوح
قسم الرياضيات، كلية العلوم، جامعة دمياط، مصر

في هذا البحث قمنا بحساب قيمة M بحيث

$$\frac{I_m^n F_\mu(f)(z)}{z} \prec 1+Mz$$

والتي تستلزم $f \in S_m^n(\mu,\alpha)$ حيث I_m^n ، $F_\mu(f)(z)$ هما على الترتيب تحويل المضروب المعروف ومؤثر برنارد - ليبرا - ليفنجستون المعروف.