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Heat conduction equation in physically inhomogeneous moving composite solids

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Abstract

This paper presents a heat conduction problem of inhomogeneous physically moving compound bodies, which consists of two cylinders. Using a sequence of integral transformations(Laplace, Hankel), the Cauchys residue theory, Bessel functions and using the results of the roots of one transcendental equation, which results from the conditions of contact between the two cylinders, a solution in the form of a series is obtained. A special case of alternating zeros of transcendental equation is also given.

Keywords: A moving cylinder; Composite solids; Heat conduction problem.

Introduction

The problems of heat conduction in composite solids are usually solved by the sequence of integral transformations method. Thermal conductivity problem of the moving solid bodies which depends on two cylinders has been considered with by researchers among of them Carslaw and Jaeger [2].
A simple form of the thermal conductivity problem has been initiated almost a century ago, for many of simple moving and simple geometrical regions and all its earlier details have been summarized and given by Carslaw and Jaeger [2] and Özişik [3]. In recent years, a number of the papers investigated the problem of heat conduction in the circular a hollow cylinder using a Maple11 program; see Shahout et al.[4].Analytical solution of the problems thermal conductivity of the moving bodies with finite size (in continuous and hollow cylinders and in parallelepipeds) is obtained by Gasimov et al. [5], Lotarev [6] and Kuznetsova [7]. Kholodovskii [8] considered boundary value problems for linear differential equations in piecewise - homogenous cylinders into two half-cylinders by multilayer film and these problems have a great importance in many engineering fields which intervene in the design of internal combustion engines, material in aviation, and the factories of the production of military weapons. Our motivation in this paper is to investigate the problem of heat propagation in an inhomogeneous composite solid with variable thermic features and with composite region which consists of two cylinders, and moves with velocity ν in the direction of z-axis. We use sequential of integral transformations like Laplace and Hankel transformations. Our idea is to choose appropriate integral transformation with respect to each

variable which allows overcoming the main difficulty to find the desired solution with the help of inverse transformations.

Formulation of the problem

Consider the composite solid which consists of two parts: a hollow cylinder ($a < r < b \ , 0 < z < \ell_1$) and entire cylinder ($0 < r < b \ , \ell_1 < z < \ell_1 + \ell_2$), which moves with velocity v in the direction of the axis *oz* where the heat flow is radial and cylindrically symmetric. At each parts, the initial temperature is given, the interior wall of the hollow cylinder is maintained at zero temperature. It is also sometimes referred to as 'Newton's Law', and from the external surface there happens radiation to the medium with zero temperature. On the joining surface (for $z = \ell_1$) there is no contact resistance.

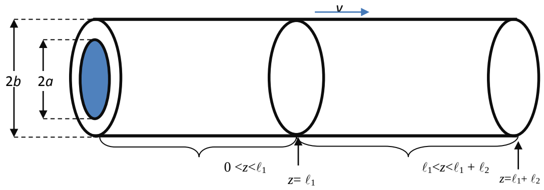

Fig. 1.

The formulation of this problem looks as follows:

$$\frac{\partial T_1}{\partial t} = \varkappa_1\left(\frac{\partial^2 T_1}{\partial r^2} + \frac{1}{r}\frac{\partial T_1}{\partial r} + \frac{\partial^2 T_1}{\partial z^2}\right) - \nu \frac{\partial T_1}{\partial z} \quad , \quad a < r < b \, , 0 < z < \ell_1 \quad , t > 0 \ , \quad (1)$$

$$T_1|_{t=0} = f_1(r,z) \ , \quad (2)$$

$$T_1|_{r=a} = 0 \ , \quad \left(\frac{\partial}{\partial r} + h_1\right) T_1|_{r=b} = 0 \ , \quad (3)$$

$$\left(\frac{\partial}{\partial z} - g_1\right) T_1|_{z=0} = 0 \ ; \quad (4)$$

$$\frac{\partial T_2}{\partial t} = \varkappa_2\left(\frac{\partial^2 T_2}{\partial r^2} + \frac{1}{r}\frac{\partial T_2}{\partial r} + \frac{\partial^2 T_2}{\partial z^2}\right) - \nu \frac{\partial T_2}{\partial z} \quad , \quad 0 < r < b \, , \ell_1 < z < \ell_1 + \ell_2 \quad , t > 0 \ , \quad (5)$$

$$T_2|_{t=0} = f_2(r,z) \ , \quad (6)$$

$$\left(\frac{\partial}{\partial r} + h_2\right) T_2|_{r=b} = 0 \ , \quad (7)$$

$$\left(\frac{\partial}{\partial z} + g_2\right) T_2|_{z=\ell_1+\ell_2} = 0 \ , \quad 0 < r < b \ , \quad (8)$$

$$T_2|_{z=\ell_1} = 0 \ , \quad 0 \le r < a \ . \quad (9)$$

In the first part, if we assume that there is no contact resistance at the surface of separation $z = \ell_1$, the boundary conditions become

$$T_1|_{z=\ell_1} = T_2\big|_{z=\ell_1},$$

$$k_1 \frac{\partial T_1}{\partial z}\bigg|_{z=\ell_1} = k_2 \frac{\partial T_2}{\partial z}\bigg|_{z=\ell_1} \ , \quad a < r < b \ , \quad (10)$$

Where: a and b are the radii of the hollow cylinder and the entire cylinder, respectively. In the first region $0 < z < \ell_1$. T_1, ρ_1, c_1 and $\varkappa_1$ are the temperature, density, specific heat and diffusivity, whereas T_2, ρ_2, c_2 and $\varkappa_2$ for the corresponding quantities in $\ell_1 < z < \ell_1 + \ell_2$ k_1 and k_2 are the thermal conductivities of the substance, that move with velocity v is constant, and h_1, h_2, g_1 and g_2 are the coefficients of surface heat transfer which are numbed to be constants, and $f_1(r,z)$ and $f_2(r,z)$ are given functions .

Solving Method

Using successive transformations, the solution is organized follows.

The first stage: We apply the Laplace transform with respect to t, and Hankel transform in $a < r < b$. The problem is transformed into

$$\varkappa_1\left(-\lambda^2 \bar{\tilde{T}}_1 + \frac{d^2 \bar{\tilde{T}}_1}{dz^2}\right) - \nu \frac{d\bar{\tilde{T}}_1}{dz} - p\bar{\tilde{T}}_1 = -\bar{f}_1(z) \quad , 0 < z < \ell_1 \ , \quad (11)$$

$$\left(\frac{d}{dz} - g_1\right)\bar{\tilde{T}}_1\Big|_{z=0} = 0 \ , \quad (12)$$

We write $\tilde{T}_i$ for the Laplace transform of T_i with respect to t, and $\bar{\tilde{T}}_1$ for the Hankel transform of $\tilde{T}_1$ with respect to r, respectively;

$$\widetilde{T}_i \equiv \widetilde{T}_i(r,z;p) = \int_0^\infty e^{-pt} T_i(r,z,t)dt\,,$$
$$(\mathrm{Re}(p) > 0), \qquad i = 1,2\,.$$

$$\overline{\overline{T}}_1 \equiv \overline{\overline{T}}_1(z;p,\eta) = \int_a^b r\widetilde{T}_1(r,z;p)\,\overline{K}_\eta^{(1)}(r)\,dr\,.$$

$$(\overline{f}_1(z) = \int_a^b rf_1(r,z)\,\overline{K}_\eta^{(1)}(r)\,dr\,).$$

$\overline{K}_\eta^{(1)}(r)$ is considered the solution of the spectral problem of Bessel's equation;

$$\frac{d^2\overline{K}^{(1)}}{dr^2} + \frac{1}{r}\frac{d\overline{K}^{(1)}}{dr} + \lambda^2\overline{K}^{(1)} = 0,$$
$$a < r < b \qquad (13)$$

$$\overline{K}^{(1)}\Big|_{r=a} = 0\ , \qquad \left(\frac{d\overline{K}^{(1)}}{dr} + h_1\overline{K}^{(1)}\right)\Bigg|_{r=b} = 0\ . \qquad (14)$$

The general solution of (13) is:

$\overline{K}^{(1)}(r) = A_0J_0(\lambda r) + B_0Y_0(\lambda r)$;

where J_0, Y_0 are Bessel functions of zero order A_0 and B_0 are arbitrary constants.

The unknowns A_0 and B_0 are to be found from (14), as follows:

$$A_0J_0(\lambda a) + B_0Y_0(\lambda a) = 0\,,$$

$$A_0[\lambda J_0'(\lambda b) + h_1J_0(\lambda b)] + B_0[\lambda Y_0'(\lambda b) + h_1Y_0(\lambda b)] = 0\ . \qquad (15)$$

So, the eigenvalues of the spectral problem (13) and (14) are obtained from the determinant system (15):

$$\begin{vmatrix} J_0(\lambda a) & Y_0(\lambda a) \\ \lambda J_0'(\lambda b) + h_1J_0(\lambda b) & \lambda Y_0'(\lambda b) + h_1Y_0(\lambda b) \end{vmatrix} = 0\ ,$$

or,

$$h_1[J_0(\lambda a)Y_0(\lambda b) - J_0(\lambda b)Y_0(\lambda a)] + \lambda[J_1(\lambda b)Y_0(\lambda a) - J_0(\lambda a)Y_1(\lambda b)] = 0\ . \qquad (16)$$

The roots of (16) are all real and simple; we need to prove the following theorem:

Theorem (Type theorem Dixon's [1]):

If $AD = BC$, then the positive zeros of $AY_0(x) + BxY_0'(x)$ are interlaced with those of $CJ_0(x) + DxJ_0'(x)$.

To prove this theorem, we enter a function in the following form

$$\varphi(x) = \frac{CJ_0(x) + DxJ_0'(x)}{AY_0(x) + BxY_0'(x)}\ ,$$

Let us denote $d \equiv [AY_0(x) + BxY_0'(x)]^2$; and we need to prove that this is an increasing function or is a decreasing function between any two consecutive zeros of the denominator, then

$$\varphi'(x) = \frac{1}{d}\{[(C+D)J_0'(x) + DxJ_0''(x)][AY_0(x) + BxY_0'(x)] - [\,CJ_0(x) + DxJ_0'(x)].$$
$$.\,[(A+B)Y_0'(x) + BxY_0''(x)]\} = \frac{1}{d}\{[(C+D)J_0'(x) - DJ_0'(x) - DxJ_0(x)][AY_0(x) + BxY_0'(x)] -$$
$$-[CJ_0(x) + DxJ_0'(x)][(A+B)Y_0'(x) - BY_0'(x) - BxY_0(x)]\} =$$
$$= \frac{1}{dx}\left\{\frac{2}{\pi}[AC + BDx^2] + [BC - AD][x^2J_0(x)Y_0(x) + J_0'(x)Y_0'(x)]\right\}.$$

$$BC - AD = 0 \ \Rightarrow \exists\, S : \frac{A}{C} = \frac{B}{D} \equiv S \ \Rightarrow A = C.S\,, B = D.S \ \Rightarrow$$

if $S > 0$, then $AC = C^2.S > 0$ and $BD = D^2.S > 0$, if $S < 0$, then $AC < 0$, $BD < 0$.

So form the sign of $\varphi'(x)$, fixed between any two zeros consecutive for the denominator.

Again, we next use this theorem to rewrite (16) as

$$\frac{J_0(\beta x)}{Y_0(\beta x)} = \frac{h_1bJ_0(x) + xJ_0'(x)}{h_1bY_0(x) + xY_0'(x)}\ ,$$

Where $\beta \equiv \dfrac{a}{b}$, $x \equiv \lambda b$.

(17)

Using this theorem for the right hand side of (17) at $A = C = h_1b$, $B = D = 1$, and using the properties of Bessel function J_0, Y_0 [1], all roots of equation (16) are real, simple and have infinite numbers.

Now in the special case for (16), we made our calculations, using program Maple and part of this programming, supposing that the thickness of the cylinder is $10^{-3}m^2$, and considering the center material and the cylinder material are silver ($k_1 = 1.00, h_1 = 0.07$). Calculating the roots of the equation (16) rewriting in Maple language, as well as taking the values to $a = 1, b = 2$, we obtain the following [4] by:

restart;

$$\Phi := unapply\Big(\lambda\cdot\Big(\mathrm{BesselJ}(0,\lambda)\cdot\frac{\partial}{\partial\lambda}\big(\mathrm{BesselY}(0,2\cdot\lambda)\big) - \mathrm{BesselY}(0,\lambda)\cdot\frac{\partial}{\partial\lambda}\big(\mathrm{BesselJ}(0,2\cdot\lambda)\big)\Big) + 0.07\cdot\big(\mathrm{BesselJ}(0,\lambda)\cdot\mathrm{BesselY}(0,2\cdot\lambda) - \mathrm{BesselY}(0,\lambda)\cdot\mathrm{BesselJ}(0,2\cdot\lambda)\big)\Big) : \Phi(0);$$

$\lambda\,(-2\,\mathrm{BesselJ}(0,\lambda)\,\mathrm{BesselY}(1,2\,\lambda) + 2\,\mathrm{BesselY}(0,\lambda)\,\mathrm{BesselJ}(1,2\,\lambda)) + 0.07\,\mathrm{BesselJ}(0,\lambda)\,\mathrm{BesselY}(0,2\,\lambda) - 0.07\,\mathrm{BesselY}(0,\lambda)\,\mathrm{BesselJ}(0,2\,\lambda)$

$plot(\Phi(1,\lambda), \lambda = 0..10);$

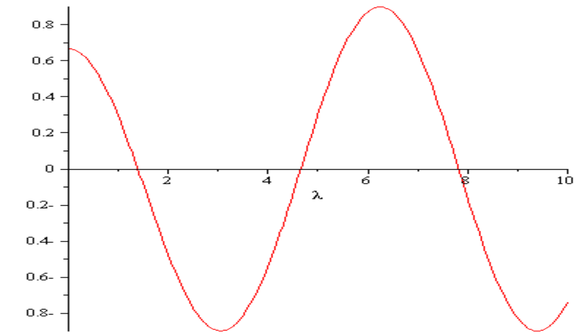

Fig (2)

$\mathrm{lambda}[1,1] := fsolve(\Phi(1), \lambda, 1..2)$

1.388691949

for k **from** 2 **to** 10 **do** $\mathrm{lambda}[1,k] := fsolve(\Phi(1), \lambda, \% + \pi .. \% + 2\cdot\pi)$ **od**

4.653498141 , 7.818655693 , 10.97034008 , 14.11753953 , 17.26270039 , 20.40676348 , 23.55016784 , 26.69314596 , 29.83583241

The two solutions of the equations can be found as:

$$A_0 = \lambda_{1,\eta} Y_0'(\lambda_{1,\eta} b) + h_1 Y_0(\lambda_{1,\eta} b) \ ,$$
$$B_0 = -\lambda_{1,\eta} J_0'(\lambda_{1,\eta} b) - h_1 J_0(\lambda_{1,\eta} b) \ .$$

Therefore one can consider the following functions as eigen functions for problems (13)-(14):

$$\Phi_{1,\eta}(r) = [\lambda_{1,\eta} Y_0'(\lambda_{1,\eta} b) + h_1 Y_0(\lambda_{1,\eta} b)] J_0(\lambda_{1,\eta} r) - [\lambda_{1,\eta} J_0'(\lambda_{1,\eta} b) + h_1 J_0(\lambda_{1,\eta} b)] Y_0(\lambda_{1,\eta} r) \ ,$$

and the kernel can be found from a relation:

$$\bar{K}_\eta^{(1)}(r) = \frac{1}{N_{1,\eta}} \Phi_{1,\eta}(r) \ ,$$

where

$$N_{1,\eta} = \int_a^b r[\Phi_{1,\eta}(r)]^2 dr \ .$$

Thus, our solution for (19)-(20) is in the form

$$\bar{\bar{T}}_{1,\eta}(z;p,\eta) = e^{\nu z/(2\varkappa_1)} \left[\frac{2}{\omega_1 - q_{1,\eta}} (\omega_1 \sinh q_{1,\eta} z - q_{1,\eta} \cosh q_{1,\eta} z) A_{1,\eta} - \frac{1}{\varkappa_1 q_{1,\eta}} \int_0^z e^{-\nu\zeta/(2\varkappa_1)} \sinh q_{1,\eta}(z - \zeta) . \bar{f}_{1,\eta}(\zeta) d\zeta \right] ,$$

where

$$q_{1,\eta}^2 \equiv \left[p + \left(\varkappa_1 \lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1} \right) \right] / \varkappa_1 \ , \ \omega_1 \equiv \frac{\nu}{2\varkappa_1} - g_1 \ ,$$

$A_{1,\eta}$ are arbitrary constants .

The second stage: Applying a sequence of the Laplace transformation and the zero-order Hankel transform with respect to r to (5)-(8), gives

$$\varkappa_2 \left(-\lambda^2 \bar{\bar{T}}_2 + \frac{d^2 \bar{\bar{T}}_2}{dz^2} \right) - \nu \frac{d\bar{\bar{T}}_2}{dz} - p\bar{\bar{T}}_2 = -\bar{f}_2(z) \ , \quad \ell_1 < z < \ell_1 + \ell_2 \ , \tag{18}$$

$$\left(\frac{d}{dz} + g_2 \right) \bar{\bar{T}}_2 \Big|_{z=\ell_1+\ell_2} = 0 \ ,$$

where $\bar{\bar{T}}_2$ is the Hankel transform, by kernel $\bar{K}_\eta^{(2)}(r)$, for $\tilde{T}_2$ namely:

$$\bar{\bar{T}}_2 \equiv \bar{\bar{T}}_2(z;p,\eta) = \int_0^b r\tilde{T}_2(r,z;p)\, \bar{K}_\eta^{(2)}(r)\, dr \ .$$

$$(\bar{f}_2(z) = \int_0^b r f_2(r,z)\, \bar{K}_\eta^{(2)}(r)\, dr \ ).$$

$\bar{K}_\eta^{(2)}(r)$ is considered as a solution of the spectral problem:

$$\frac{d^2\bar{K}^{(2)}}{dr^2} + \frac{1}{r}\frac{d\bar{K}^{(2)}}{dr} + \lambda^2 \bar{K}^{(2)} = 0, \quad 0 < r < b, \tag{20}$$

$$\bar{K}^{(2)}\Big|_{r=0} < \infty \ , \quad \left(\frac{d\bar{K}^{(2)}}{dr} + h_2 \bar{K}^{(2)} \right)\Big|_{r=b} = 0 \ . \tag{21}$$

The eigenvalues $\lambda_{2,\eta}^2$ for the problems (20)-(21), can be obtained from the equation:

$$\lambda J_0'(\lambda b) + h_2 J_0(\lambda b) = 0 \quad . \tag{22}$$

It is known that at [1], the roots of (22) are all real, simple and have infinite numbers.

The kernel $\bar{K}_\eta^{(2)}(r)$ is calculated from the relation:

$$\bar{K}_\eta^{(2)}(r) = \frac{1}{N_{2,\eta}} J_0(\lambda_{2,\eta} r) \ ,$$

where

$$N_{2,\eta} = \int_0^b r J_0^2(\lambda_{2,\eta} r) dr = \frac{b^2}{2\lambda_{2,\eta}^2} (h_2^2 + \lambda_{2,\eta}^2) J_0^2(\lambda_{2,\eta} b) \ .$$

Thus, the solution of the problem (18)-(19) takes the form:

$$\overline{\overline{\mathrm{T}}}_{2,\eta}(z;p,\eta) = e^{\nu z/(2\varkappa_2)}\left\{\frac{2e^{(\ell_1+\ell_2)q_{2,\eta}}}{\omega_2 - q_{2,\eta}}\left[\omega_2 \sinh q_{2,\eta}(z-(\ell_1+\ell_2)) - q_{2,\eta}\cosh q_{2,\eta}(z-(\ell_1+\ell_2))\right].A_{2,\eta} + \frac{1}{\varkappa_2 q_{2,\eta}}\int_z^{\ell_1+\ell_2} e^{-\nu\zeta/(2\varkappa_2)}\sinh q_{2,\eta}(z-\zeta).\bar{f}_{2,\eta}(\zeta)d\zeta\right\},$$

where

$q_{2,\eta}^2 \equiv \left[p + \left(\varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}\right)\right]/\varkappa_2$, $\omega_2 \equiv \frac{\nu}{2\varkappa_2} + g_2$, $A_{2,\eta}$ are arbitrary constants .

The third stage: Using the inverse Hankel transform, we find two solutions of the two problems (11)-(12) and (18)-(19) in the form:

$$\tilde{T}_1(r,z;p) = 2e^{\nu z/(2\varkappa_1)}\sum_{\eta=1}^{\infty}\frac{A_{1,\eta}}{\omega_1 - q_{1,\eta}}(\omega_1 \sinh q_{1,\eta}z - q_{1,\eta}\cosh q_{1,\eta}z)\Phi_{1,\eta}(r) - -\frac{1}{\varkappa_1}\sum_{\eta=1}^{\infty}\left[\int_0^z e^{\nu(z-\zeta)/(2\varkappa_1)}\sinh q_{1,\eta}(z-\zeta).\bar{f}_{1,\eta}(\zeta)d\zeta\right]\frac{1}{q_{1,\eta}}\Phi_{1,\eta}(r),$$

$$\tilde{T}_2(r,z;p) = 2e^{\nu z/(2\varkappa_2)}\sum_{\eta=1}^{\infty}\frac{A_{2,\eta}}{\omega_2 - q_{2,\eta}} e^{(\ell_1+\ell_2)q_{2,\eta}}\left[\omega_2\sinh q_{2,\eta}\left(z-(\ell_1+\ell_2)\right) - q_{2,\eta}.\,.\cosh q_{2,\eta}\left(z-(\ell_1+\ell_2)\right)\right]\Phi_{2,\eta}(r) + \frac{1}{\varkappa_2}\sum_{\eta=1}^{\infty}\left[\int_z^{\ell_1+\ell_2} e^{\nu(z-\zeta)/(2\varkappa_2)}\sinh q_{2,\eta}(z-\zeta).\bar{f}_{2,\eta}(\zeta)d\zeta\right]\frac{\Phi_{2,\eta}(r)}{q_{2,\eta}}, \quad (24)$$

where $\Phi_{2,\eta}(r) = J_0(\lambda_{2,\eta}r)$.

When $a < r < b$, the constants $A_{1,\eta}$ and $A_{2,\eta}$ are found from

$$\tilde{T}_1\Big|_{z=\ell_1} = \tilde{T}_2\Big|_{z=\ell_1}, \quad k_1\frac{\partial\tilde{T}_1}{\partial z}\Big|_{z=\ell_1} = k_2\frac{\partial\tilde{T}_2}{\partial z}\Big|_{z=\ell_1}, \quad a < r < b, \quad (25)$$

in the following forms

$$A_{1,\eta} = (C_{2,2}.G_{1,\eta} - C_{1,2}.G_{2,\eta})/(\Delta_\eta.\Phi_{1,\eta}), \quad (26)$$

$$A_{2,\eta} = (C_{1,1}.G_{2,\eta} - C_{2,1}.G_{1,\eta})/(\Delta_\eta.\Phi_{2,\eta}), \quad (27)$$

where

$$C_{1,1} \equiv C_{1,1}(\eta) = \frac{2}{\omega_1 - q_{1,\eta}}\left(\omega_1\sinh q_{1,\eta}\ell_1 - q_{1,\eta}\cosh q_{1,\eta}\ell_1\right).e^{\nu\ell_1/(2\varkappa_1)},$$

$$C_{1,2} \equiv C_{1,2}(\eta) = \frac{2}{\omega_2 - q_{2,\eta}}\left(\omega_2\sinh q_{2,\eta}\ell_2 + q_{2,\eta}\cosh q_{2,\eta}\ell_2\right).e^{\nu\ell_1/(2\varkappa_2)}.e^{(\ell_1+\ell_2)q_{2,\eta}},$$

$$C_{2,1} \equiv C_{2,1}(\eta) = \frac{1}{\omega_1 - q_{1,\eta}}\left[\left(\frac{\nu\omega_1}{\varkappa_1} - 2q_{1,\eta}^2\right)\sinh q_{1,\eta}\ell_1 + q_{1,\eta}\left(2\omega_1 - \frac{\nu}{\varkappa_1}\right)\cosh q_{1,\eta}\ell_1\right]e^{\nu\ell_1/(2\varkappa_1)},$$

$$C_{2,2} \equiv C_{2,2}(\eta) = \frac{k}{\omega_2 - q_{2,\eta}}\left[\left(\frac{\nu\omega_2}{\varkappa_2} - 2q_{2,\eta}^2\right)\sinh q_{2,\eta}\ell_2 - q_{2,\eta}\left(2\omega_2 - \frac{\nu}{\varkappa_2}\right)\cosh q_{2,\eta}\ell_2\right].\,.e^{\nu\ell_1/(2\varkappa_2)}.e^{(\ell_1+\ell_2)q_{2,\eta}},$$

$$G_{1,\eta} \equiv G_{1,\eta}(p,r) = \bar{F}_{2,\eta}^{(s)}(p).\Phi_{2,\eta}(r) + \bar{F}_{1,\eta}^{(s)}(p).\Phi_{1,\eta}(r),$$

$$G_{2,\eta} \equiv G_{2,\eta}(p,r) = k\left[\frac{\nu}{2\varkappa_2}\bar{F}_{2,\eta}^{(s)}(p) + \bar{F}_{2,\eta}^{(c)}(p)\right]\Phi_{2,\eta}(r) + \left[\frac{\nu}{2\varkappa_1}\bar{F}_{1,\eta}^{(s)}(p) + \bar{F}_{1,\eta}^{(c)}(p)\right]\Phi_{1,\eta}(r), \quad (23)$$

$$\Delta_\eta \equiv \Delta_\eta(p) = C_{1,1}.C_{2,2} - C_{1,2}.C_{2,1}, \quad k \equiv k_2/k_1,$$

$$\bar{F}_{1,\eta}^{(s)} = \frac{1}{\varkappa_1 q_{1,\eta}}\int_0^{\ell_1} e^{\nu(\ell_1-\zeta)/(2\varkappa_1)}\sinh q_{1,\eta}(\ell_1-\zeta).\bar{f}_{1,\eta}(\zeta)d\zeta,$$

$$\bar{F}_{1,\eta}^{(c)} = \frac{1}{\varkappa_1}\int_0^{\ell_1} e^{\nu(\ell_1-\zeta)/(2\varkappa_1)}\cosh q_{1,\eta}(\ell_1-\zeta).\bar{f}_{1,\eta}(\zeta)d\zeta,$$

$$\bar{F}_{2,\eta}^{(s)} = \frac{1}{\varkappa_2 q_{2,\eta}}\int_{\ell_1}^{\ell_1+\ell_2} e^{\nu(\ell_1-\zeta)/(2\varkappa_2)}\sinh q_{2,\eta}(\ell_1-\zeta).\bar{f}_{2,\eta}(\zeta)d\zeta,$$

$$\bar{F}_{2,\eta}^{(c)} = \frac{1}{\varkappa_2}\int_{\ell_1}^{\ell_1+\ell_2} e^{\nu(\ell_1-\zeta)/(2\varkappa_2)}\cosh q_{2,\eta}(\ell_1-\zeta).\bar{f}_{2,\eta}(\zeta)d\zeta.$$

When $0 < r < a$, the constants $A_{2,\eta}$ are obtained from the relation $\tilde{T}_2\Big|_{z=\ell_1} = 0$, as

$A_{2,\eta} = \overline{F}^{(s)}_{2,\eta}(p)/C_{1,2}(\eta)$.
(28)

The fourth stage: we look for the zeros of the function Δ_η (p).

Using these relations for $C_{i,j}(i,j = 1,2)$, we write the equation $\Delta_\eta(p) = 0$ in the form:

$$\frac{\nu}{2}\left(\frac{k}{\varkappa_2} - \frac{1}{\varkappa_1}\right)(\omega_1 \sinh q_{1,\eta}\ell_1 - q_{1,\eta}\cosh q_{1,\eta}\ell_1)(\omega_2 \sinh q_{2,\eta}\ell_2 + q_{2,\eta}\cosh q_{2,\eta}\ell_2) - -q_{1,\eta}(\omega_1 \cosh q_{1,\eta}\ell_1 - q_{1,\eta}\sinh q_{1,\eta}\ell_1)(\omega_2 \sinh q_{2,\eta}\ell_2 + q_{2,\eta}\cosh q_{2,\eta}\ell_2) - -kq_{2,\eta}(\omega_1 \sinh q_{1,\eta}\ell_1 - q_{1,\eta}\cosh q_{1,\eta}\ell_1)(\omega_2 \cosh q_{2,\eta}\ell_2 + q_{2,\eta}\sinh q_{2,\eta}\ell_2) = 0 . \qquad (29)$$

if $\varkappa_1\lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1} > \varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}$, then the equation (29)takes the form :

$$\frac{\nu}{2}\left(\frac{k}{\varkappa_2} - \frac{1}{\varkappa_1}\right)(\omega_1 \sin\beta_\eta\ell_1 - \beta_\eta\cos\beta_\eta\ell_1)\left(\omega_2 \sin\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,\ell_2 + \sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\cos\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,\ell_2\right) - -\beta_\eta(\omega_1 \cos\beta_\eta\ell_1 + \beta_\eta\sin\beta_\eta\ell_1)\left(\omega_2 \sin\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,\ell_2 + \sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\cos\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,\ell_2\right) - k\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,(\omega_1 \cos\beta_\eta\ell_1 + \beta_\eta\sin\beta_\eta\ell_1)\left(\omega_2 \cos\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,\ell_2 - \sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\sin\sqrt{\varkappa\beta_\eta^2 + \gamma_\eta^2}\,\ell_2\right) = 0 , \qquad (30)$$

where

$$\varkappa \equiv \varkappa_1/\varkappa_2 \quad , \gamma_\eta^2 \equiv \left(\varkappa_1\lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1}\right) - \left(\varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}\right).$$

The relation (30) is an algebraic equation for β_η, we proved that the equation has only real, simple root and infinite numbers

The fifth stage: we apply the inverse Laplace transform with respect to (t) of the two relations (17), (27).

Substituting the relations (23),(27) and (28) into the relations (17),(27)and using Cauchy's theory of residues, on the basis of the results of the fifth stage, we obtain a solution to the problem under study in the form:

$$T_1(r,z,t) = e^{-\nu(\ell_1 - z)/(2\varkappa_1)}\sum_{\eta=1}^{\infty}\sum_{m=1}^{\infty}\varphi_1(z,\beta_{\eta m})\left\{\varphi_2(\ell_2,\beta_{2,\eta m})\left[\frac{1}{\varkappa_1}\beta_{2,\eta m}\int_0^{\ell_1}\left(\frac{\nu}{2}\left(\frac{k}{\varkappa_2} - \frac{1}{\varkappa_1}\right)R^s(\zeta;\beta_{\eta m}) - R^c(\zeta;\beta_{\eta m})\right)\bar{f}_{1,\eta}(\zeta)d\zeta\Phi_{1,\eta}(r) - \frac{k}{\varkappa_2}\beta_{\eta m}\int_{\ell_1}^{\ell_1+\ell_2}R_2^c(\zeta;\beta_{2,\eta m})\bar{f}_{2,\eta}(\zeta)d\zeta.\,\Phi_{2,\eta}(r)\right] - k\varphi'_{2,z}(\ell_2,\beta_{2,\eta m})Q(r,\beta_{\eta m})\right\}S_{\eta m}(t) ,$$

$$T_2(r,z,t) = e^{-\nu(\ell_1 - z)/(2\varkappa_2)}\sum_{\eta=1}^{\infty}\sum_{m=1}^{\infty}\varphi_2(\ell_1+\ell_2 - z,\beta_{\eta m})\left\{\varphi_1(\ell_1,\beta_{\eta m})\frac{1}{\varkappa_2}\,.\left[-\beta_{2,\eta m}\int_0^{\ell_1}R^c(\zeta;\beta_{\eta m})\bar{f}_{1,\eta}(\zeta)\,d\zeta\,.\,\Phi_{1,\eta}(r) + \beta_{\eta m}\int_{\ell_1}^{\ell_1+\ell_2}\left(\frac{\nu}{2}\left(\frac{1}{\varkappa_1} - \frac{k}{\varkappa_2}\right)R_2^s(\zeta;\beta_{2,\eta m}) - R_2^c(\zeta;\beta_{2,\eta m})\right).\bar{f}_{2,\eta}(\zeta)d\zeta.\,K_{2,\eta}(r)\right] - \varphi'_{1,z}(\ell_1,\beta_{\eta m})Q(r,\beta_{\eta m})\right\}.\,S_{\eta m}(t) ,$$

when $a < r < b$,

$$T_2(r,z,t) = -2e^{-\nu(\ell_1 - z)/(2\varkappa_2)}\sum_{\eta=1}^{\infty}\sum_{j=1}^{\infty}\frac{\Phi_{2,\eta}(r)\varphi_2(\ell_1+\ell_2 - z,\beta_{2,\eta j})}{\varkappa_2\beta_{2,\eta j}\varphi'_{2,\beta}(\ell_2,\beta_{2,\eta j})}.$$

$$. \int_{\ell_1}^{\ell_1+\ell_2} R_2^s(\zeta;\beta_{2,\eta j})\,\bar{f}_{2,\eta}(\zeta)d\zeta\,.\,e^{-\varkappa_2\left(\beta_{2,\eta j}^2+\lambda_{2,\eta}^2+(\nu/(2\varkappa_2))^2\right)t}\,,$$

when $0 < r < a$, where

$$\varphi_1(z,\beta) \equiv \omega_1 \sin\beta z - \beta\cos\beta z\ ,$$
$$\varphi_2(z,\beta) \equiv \omega_2 \sin\beta z + \beta\cos\beta z\ ,$$
$$R^s(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_1)}\sin\beta(\ell_1 - z)\,,$$
$$R_2^s(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_2)}\sin\beta(\ell_1 - z)\,,$$
$$R^c(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_1)}\beta\cos\beta(\ell_1 - z)\,,$$
$$R_2^c(z;\beta) \equiv e^{\nu(\ell_1-z)/(2\varkappa_2)}\beta\cos\beta(\ell_1 - z)\,,$$

$$Q(r,\beta_{\eta m}) \equiv -\frac{1}{\varkappa_1}\beta_{2,\eta m}\int_0^{\ell_1} R^s(\zeta;\beta_{\eta m})\,\bar{f}_{1,\eta}(\zeta)d\zeta.\,\Phi_{1,\eta}(r) - \frac{1}{\varkappa_2}\beta_{\eta m}\int_{\ell_1}^{\ell_1+\ell_2} R_2^s(\zeta;\beta_{2,\eta m})\,\bar{f}_{2,\eta}(\zeta)d\zeta.\,\Phi_{2,\eta}(r)\ ,$$

if $\left(\varkappa_1\lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1}\right) > \left(\varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}\right)$, *then* $\beta_{\eta m}$ are the positive roots of the equation

$$\delta(\beta_\eta) \equiv \left[\frac{\nu}{2}\left(\frac{k}{\varkappa_2} - \frac{1}{\varkappa_1}\right)\varphi_1(\ell_1,\beta_\eta) - \varphi'_{1,z}(\ell_1,\beta_\eta)\right]\varphi_2\left(\ell_2,\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\right) - k.\,\varphi_1(\ell_1,\beta_\eta)\varphi'_{2,z}\left(\ell_2,\sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\right) = 0,$$

$$\beta_{2,\eta m} \equiv \sqrt{\varkappa\beta_\eta^2+\gamma_\eta^2}\ ,$$

$$\gamma_\eta^2 \equiv \frac{1}{\varkappa_2}\left[\left(\varkappa_1\lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1}\right) - \left(\varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}\right)\right]$$

$$S_{\eta m}(t) = \frac{2}{\beta_{\eta m}\,\beta_{2,\eta m}\delta'(\beta_{\eta m})}\exp\{-\varkappa_1(\beta_{\eta m}^2 + \lambda_{1,\eta}^2 + (\nu/(2\varkappa_1))^2)t\}\ ,$$

if $\left(\varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}\right) > \left(\varkappa_1\lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1}\right)$, *then* $\beta_{2,\eta m}$ are the positive roots of the equation

$$\delta_2(\beta_{2,\eta}) \equiv \left[\frac{\nu}{2}\left(\frac{k}{\varkappa_2} - \frac{1}{\varkappa_1}\right)\varphi_1\left(\ell_1,\sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\right) - \varphi'_{1,z}\left(\ell_1,\sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\right)\right]\varphi_2(\ell_2,\beta_{2,\eta}) - k.\,\varphi_1\left(\ell_1,\sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\right)\varphi'_{2,z}(\ell_2,\beta_{2,\eta}) = 0\ ,$$

$$\beta_{\eta m} \equiv \sqrt{\frac{1}{\varkappa}\beta_{2,\eta}^2+\gamma_{2,\eta}^2}\ ,$$

$$\gamma_{2,\eta}^2 \equiv \frac{1}{\varkappa_1}\left[\left(\varkappa_2\lambda_{2,\eta}^2 + \frac{\nu^2}{4\varkappa_2}\right) - \left(\varkappa_1\lambda_{1,\eta}^2 + \frac{\nu^2}{4\varkappa_1}\right)\right],$$

$$S_{\eta m}(t) = \frac{2}{\beta_{\eta m}\,\beta_{2,\eta m}\delta'_2(\beta_{2,\eta m})}\exp\{-\varkappa_2(\beta_{2,\eta m}^2 + \lambda_{2,\eta}^2 + (\nu/(2\varkappa_2))^2)t\}\ ,$$

$\beta_{2,\eta j}$ are the positive roots of the equation $\omega_2\sin\beta_{2,\eta}\ell_2 + \beta_{2,\eta}\cos\beta_{2,\eta}\ell_2 = 0$ ..

Conclusion

In this paper, the spread heat in a body which in two cylinders is studied. The solution behavior of the boundary problem is given as an infinite series using two sequences of integral transformation forms, Bessel function theory and the conditions of contact between the two cylinders. Hence we get a transcendental equation and using the theorem of type Dixon, we proved that this equation has infinitely, simple and real roots. The behavior of the eigenvalues spectral problem is determined. The obtained Eigen functions are the kernel of integral transformations of first kind Bessel function. This problem has numerous engineering applications, such as, the administration motion of bodies which consists of two cylinders or more. Also, machining, welding, grinding, internal combustion engines, as well as in the factories for the production of military weapons, are all others practical examples.

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الملخص العربي

عنوان البحث: معادلة التوصيل الحرارى فى وسط غير متجانس للأجسام الصلبة المركبة المتحركة

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يهتم البحث بمسألة التوصيل الحرارى فى وسط غير متجانس للأجسام المركبة التى تتكون من اسطوانتين باستخدام سلسلة من تحويلات (لابلاس ،هينكيل) ونظرية كوشى ودوال بسى والجذور الناتجة من معادلة الحركة للشرط الحدى بسبب الأسطوانتين. الحل المشتق عبارة عن متسلسلة. أيضا تم اعطاء حالة خاصة لتبادل أصفار معادلة الحركة