Analytical Study of Muqarnas Formations in Islamic Architecture According to Digital Simulation

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Abstract

The history of Islamic architecture gives great features to the rich Islamic heritage through muqarnas, which formed interior spaces with their intricate and brilliant architectural ornaments. Muqarnas plays a central and vital role in aesthetic architecture, an element, which belongs to architecture and ornamentation. The geometric and ornamental analysis of Islamic art raises new interest in digital and algorithmic modeling, as well as new horizons in muqarnas formation. Muqarnases incorporate among their elements harmonious blocks and lines, and mathematical design, which perform a specific architectural function and an aesthetically ornamental role. The research seeks to link the originality of the past and the modernity of the future, in addition to studying the role of digital design, algorithmic systems, and various generative geometries in the evolution of the muqarnas, which is one of the progressive features of Islamic architecture. This study attempts to translate it into a modern language with innovative patterns and the effectiveness of their connection with digital and algorithm architecture. The paper presents an analytical study of traditional muqarnas patterns in Islamic art and their evolution by determining the logic and rules of muqarnas patterns using digital algorithms through digital design. In addition to controlling classical ornamental patterns of muqarnas by coding and programming in computational design. Thus, overcoming the gap between classical ornamental patterns and the flow of the digital design. The main concern is digital software which can be used to facilitate the design process and visualization of virtual models of muqarnas. The formation of three-dimensional muqarnases may provide new insights for mathematics to analyze the shape and create innovative patterns in the digital age.

The problem with the study is that it does not clarify modern methods of digital algorithms in the discovery of a modern architectural style for Islamic art, which contributes significantly to enhancing muqarnas. This analysis encompasses the studies of muqarnases through concept and historical evolution as well as structural and algorithmic analysis, on the other hand.

The research supplies a new approach to the characteristics of muqarnas through computational logic and establishes digital modeling of muqarnas through digital algorithms to produce different design patterns. The main concern of the research is to study the theoretical principles of digital algorithms for the identity of muqarnas through algorithmic rules and digital patterns. Therefore, we conclude the value of analyzing digital algorithms, which are often used to develop muqarnas, since Islamic architecture has endless possibilities for future design innovations. Therefore, this study firstly reviews earlier approaches to the analysis of geometrical and conceptual principles of muqarnas.

Secondly, the research interprets the recommended scheme through a series of digital experiments and algorithmic forms. During these experiments, this study asks whether mathematical and algorithmic equations contribute to developing muqarnas. It can also be influential in the transfer and revival of Islamic structural philosophy to other civilizations'

countries. It is therefore very important to stimulate the geometric approach of traditional Islamic architecture with modern digital tools in order to keep up Islamic heritage with the evolution of computational design thought and adapt it to reality.

Keywords

Islamic architecture, Islamic culture, Islamic pattern, Muqarnas, Digital modeling, computational design, Islamic ornaments, Coding, Geometrical design, Algorithmic systems, Digital algorithm, Analysis of Muqarnas, Modeling Muqarnas, Computer Generated Muqarnas.

الملخص

إن تاريخ العمارة الاسـلامية يعطى معالم رائعة للتراث الاسـلامى الغنى من خلال المقرنصـات التى شـكلت الفراغات الداخلية بزخارفها الهندسـية المعقدة والرائعة. وتلعب المقرنصـات دورا أسـاسـيا ومحوريا في الجماليات المعمارية وهو عنصـر ينتمي إلى العمارة والزخرفة. إن التحليل الهندسـى والزخرفى للفن الإسـلامي يثير اهتماما جديدا فيما يتعلق بالنظم الجديدة للنمذجة الرقمية، إلى جانب فتح آفاق جديدة في تشـكيل المقرنصـات ونمذجتها. وتجمع المقرنصـات بين عناصـرها خطوط وكتل متناغمة، رياضيّة التصميم، تؤدي وظيفة معمارية محددة، ودَوراً زخرفياً جمالياً يتجاوز كلّ الحدود. ويسعى البحث لربط أصالة الماضى وحداثة المستقبل ودراسة دور التصميم البارامترى والانظمة الخوارزمية والهندسات التوليدية المختلفة فى تطور العمارة الاسلامية ومن أهمها المقرنص، فهو واحد من الملامح المبدعة فى العمارة الإسلامية. ويحاول البحث ترجمتها إلى لغة حديثة ذات أنماط مبتكرة ودراســة مدى فعالية ارتباطها بالعمارة الرقمية والخوارزمية. تقدم هذه الورقة دراسـة تحليلية لأنماط المقرنصـات التقليدية فى العمارة الاسـلامية وتطورها من خلال تحديد منطق وقواعد تكوين وانماط المقرنص بإستخدام الخوارزمية الرقمية من خلال التصميم البارامترى. بالاضافة الى التحكم في الأنماط الزخرفية الكلاسـيكية للمقرنصـات من خلال الاكواد والبرمجة فى التصـميم الحسـابى. وبالتالى التغلب على الفجوة بين الأنماط الزخرفية الكلاسـيكية وانسـيابية التصـميم الرقمى مما يؤدي إلى تطوير عدة أنماط، مسـتمدة من هندسـات توليدية مختلفة. يمكن استخدام البرمجيات الرقمية لتسـهيل عملية التصـميم، والتصـور، وحتى إعادة بناء المقرنصـات والنماذج الافتراضـية للمقرنصـات. فإن تشـكيل المقرنص ثلاثي الأبعاد قد يوفر رؤى جديدة للرياضـيات لتحليل الشـكل خارج الصـفات الجمالية والزينة في العصـر الرقمي. إن تشـكيل المقرنصـات لديها القدرة على خلق الانماط المبتكرة بسـلسـلة من القواعد أو الخوارزميات الرقمية.

وتتضح اشكالية الدراسة فى عدم وضوح الاساليب الحديثة من العمارة الرقمية والخوارزميات الرقمية فى إكتشاف اسلوب معمارى حديث للفن الاسـلامى والذى يسـاهم بشـكل كبير فى تطور المقرنصـات. وتشـمل الدراسـة تحليل المقرنصـات من حيث المفهوم والتطور التاريخى وايضـا التحليل الهيكلى والخوارزمى من جهة اخرى. وتحليل العناصـر الهندسـية المعقدة للمقرنصات ويضيف البحث رؤية جديدة لسمات وخصائص المقرنصات من خلال المنطق الحسابى. وهناك يحاول البحث للاسـتفادة من المقرنصـات في التكوين العام للمباني، وخلق تكوينات زُخرفيّة مسـتوحاة منه أو فيها محاكاة للمقرنصـات التقليديّة بمواد وخطوط تتناسـب مع العمارة الحديثة. وتطوير نمذجة للمقرنصـات من خلال الخوارزميات الرقمية وانتاج أنماط تصـميمية مختلفة. يهدف البحث إلى دراسـة هوية العمارة الاسـلامية من خلال القواعد والأنماط الرقمية ، وفتح افاق جديدة فى تقنيات التصـميم الرقمى من خلال المقرنصـات . وتطوير أداة للنمذجة الرقمية للمقرنصـات، مما يتيح إمكانية إنتاج متغيرات مختلفة. وبالتالى إستنتاج أهمية تحليل معايير التصميم البارامترى التى يمكن استخدامها لتطوير المقرنصات

حيث أن الهندسة المعمارية الإسلامية لها إمكانيات لا نهاية لها لابتكارات التصميم المستقبلية. وهكذا، فإن هذه الدراسة تستعرض أولا النهج السابقة لتحليل المبادئ الهندسية والإنشاءية للمقرنصات. وثانيا، فإن البحث يفسر المخطط المقترح من خلال سلسلة من التجارب الرقمية والاشكال الخوارزمية. في هذه التجارب، نتسائل هل المعادلات الرياضية والخوارزمية، تسهم في تطور المقرنصات ومدى استخدام العمارة الرقمية في الهندسة المعمارية الاسلامية وهل يمكن أن يكون مؤثرا في نقل فلسفة العمارة الإسلامية ؟ والهدف من التحقيق الحالي هو دراسة المبادئ النظرية للخوارزميات الرقمية للحصول على نهج جديد لتطوير العمارة التقليدية الإسلامية مع نهج هندسي متطور من خلال تحليل معايير العمارة الرقمية. ولذلك قد يكون من المهم للغاية محاولة تنشيط النهج الهندسي للهندسة المعمارية الاسلامية التقليدية مع هذه الأدوات الرقمية الجديدة من أجل مواكبة التراث مع تطور الفكر التصميمى الحالى وتكييفه مع الواقع.

الكلمات الدالة:
العمارة الإسلامية ، الثقافة الإسلامية ، النمط الإسلامي ، المقرنصات ، النمذجة الرقمية ، التصميم الحسابي ، الزخارف الإسلامية ، الكود ، التصميم الهندسي ، النظم الحسابية ، الخوارزمية الرقمية ، تحليل المقرنصات ، نمذجة المقرنصات ، المقرنصات التى يتم انتاجها رقميا.

1. Introduction

Heritage and culture are fundamental elements of the lives of societies and peoples, including Islamic culture. [1] Islamic architecture features a unique artistic character, which is unparalleled in different art fields. The depth of the connection between architecture and art is reflected in an organic unity, which is reflected in the formulation concept of architectural elements, and that is clear from the dual function of muqarnas in Islamic architecture, which is both an architectural element and a decorative element in the style and character of Islamic art. [2] Therefore, geometry is a feature of mathematical holiness and one of the distinguishing characters in Islamic art.

The main research approach is to analyze muqarnas as a transition between several shapes which aligned on top of each other through digital tools. In order to overcome muqarnas' complexity, designs were studied through the two-dimensional projections of muqarnas' structure, which these flat layers were used to design and construct muqarnas. Unfortunately, currently, there are only a few practitioners who have learned the creative skill of muqarnas design from their earlier masters. As a result, muqarnas is on the verge of extinction. The major goal of this study is to revive the traditional art of muqarnas by examining its geometry and offering a set of algorithms to model muqarnas algorithmically. As a result of these methodologies, designers and architects can create software tools for modeling muqarnas and, as a result, conserve this cultural treasure for future generations.[3] Modeling muqarnas is also an important step toward gaining a deeper understanding of these complicated geometrical formations.

2. Challenges and problems

- The design of muqarnas using advanced digital tools is a challenge. Each piece must be individually and properly shaped. Each piece depends on what is next to it. In addition, there is usually a level of inaccuracy and asymmetry in muqarnas structure. This could be due to

building restrictions or the surface characteristics of the muqarnas. As well as in symmetry and partial replication, making the problem of modeling muqarnas more difficult to deal with. Thus, creating advanced elements and patterns of traditional muqarnas through challenging equations and modeling digital where modern methods of digital design and digital algorithms are unclear in discovering a modern architectural style of Islamic art, including muqarnas.

- Traditional Mathematical inherited methods are followed in the construction and formation of muqarnas, which exposes them to loss and extinction.
- Following traditional methods in the formation of muqarnas, they are characterized by rigidity and lack of development.
- Widening the gap between traditional methods of forming muqarnas and digital formations.

3. Objectives

- Provide methods and digital tools to enhance the digital modeling process and muqarnas modeling.
- Production of virtual models from muqarnas and providing better insight for analysis of muqarnas.
- Use new approaches and methodologies for muqarnas which can enrich the designer's ability to generate heritage designs with an innovative idea, which expresses Islamic art authenticity.

4. Research Methods

The study encompasses a historical approach to reach the development of muqarnas stages and an analytical descriptive approach through studying the foundations of muqarnas design and its aesthetics. In addition, the analytical approach in architectural studies of muqarnas and its ornamental and architectural forms through advanced digital solutions.[4]

5. Framework for research and hypotheses

The research assumes: Relying on digital simulation in the formation of muqarnas patterns keeps them from disappearing and opens new horizons for innovation. The geometric study, based on Islamic art for ornamentation, raises new interest in new systems of digital modeling in computer art, as well as opening new horizons in construction techniques with new materials. This research focuses on the relationship between the three-dimensional structure of muqarnas and its two-dimensional projection plane through computational design. The research asks how the digital logic of muqarnas can abide by irregular surfaces. In addition, what types of patterns and ornaments may be produced by digital algorithms? Thus, the muqarnases come with a system of regular gradients which break simple geometric shapes and surfaces, leading to the development of several patterns derived from different geometrical structures, construction techniques, and materials.

6. Muqarnas formations in Islamic architecture

Muqarnas is an iconic element in Islamic architecture and displays great beauty and captures viewers' attention through its intricate shape and exquisite patterns. [5] It is a three-dimensional digital architectural ornament with complex geometries, which was a common vault in Islamic architecture. [6] It has been used throughout the Islamic world. Muqarnas is a fascinating 3D feature of Islamic architecture with complicated geometry.

Muqarnas is a type of ornamental vaulting in Islamic architecture that is known as Ahoopy (Persian:) in Iranian architecture and Mocárabe in Iberian architecture. [7] It is the quintessential form of Islamic architecture, and it is an essential part of the Islamic building vernacular. [8] The muqarnas structure arose from the squinch.

Muqarnas, also known as "honeycomb vaulting" or "stalactite vaulting," is designed to create a smooth, beautiful transition zone in an otherwise barren, structural area. This construction serves as a transition from a room's walls to a domed ceiling, allowing the main components of the building to be distinguished. [9]

Muqarnas is a three-dimensional architectural embellishment that may be found all over the world, from Spain to India. It is made up of superimposed ranks of miniature geometric recesses that are often interspersed with pendant parts. [10] The muqarnas evolved from a structural shape utilized in niches and squinches under domes to a highly distinctive style of architectural adornment in the Islamic world. [10] These designs are made up of various basic structures that are layered on top of each other to create complex 3D surfaces. Abstract muqarnas (stalactite vault) is a distinct structural and tectonic feature of Islamic architecture. They are used to provide a transition from a square plan to a circular circumference, which allows it to be established above a square or rectangle layout.

So that, the research presents a new method for interactively modeling muqarnas based on their layered structure in this work. Floor layouts are utilized as a guide for the modeling workflow to update and produce new forms automatically. [3]

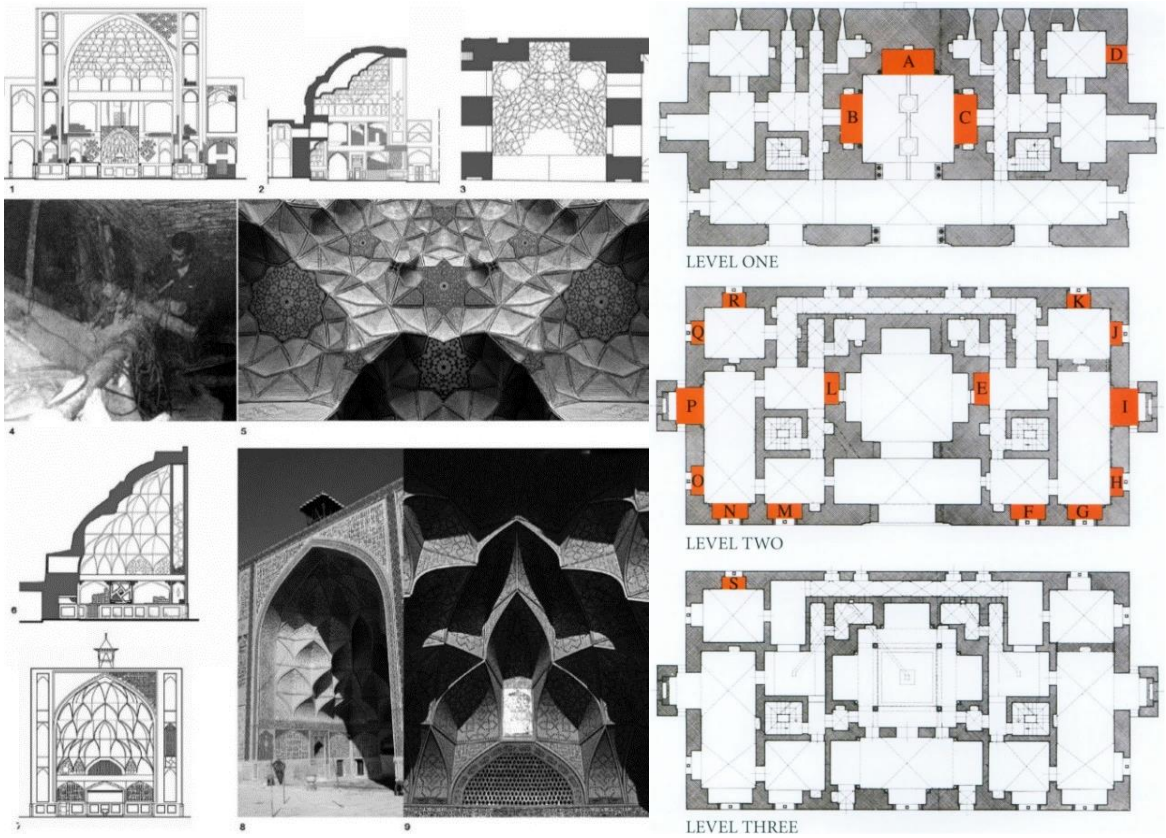

Figure 1. On the left: Comparing two muqarnases in the west and east iwan of the Great Mosque of Esfahan. 1,2,3. Plan, Elevation, and Section of muqarnas in the East Iwan. Muqarnas tensile elements. 5.Takht in the muqarnas of the East Iwan. 6,7,8. Plan, Elevation, and Section of muqarnas in the West Iwan of the Great Mosque of Esfahan. On the right: the location of muqarnases on the three levels of the Zisa, Palermo.

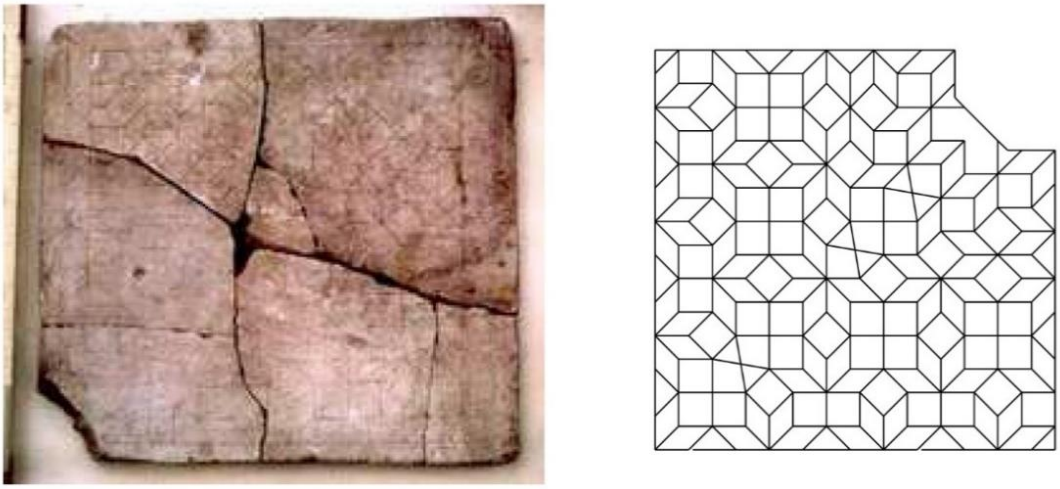

Figure 2. On the left: a picture of the plate found at Takhti Sulayman with a muqarnas design incised on it. On the right: the lines of the plate as recognized by Harb.[11]

7. The design characteristics of muqarnas

In the mid-tenth century, muqarnases began to develop in northeastern Iran and North Africa. As well, in the Muslim world, muqarnases spread throughout the 11th century. The muqarnas is a three-dimensional shape and its visual function is to provide a gradual transition between two levels, two volumes, or two shapes from the rectangular foundation of the building to the vaulted ceiling. It is sometimes considered a kind of sculpture on the wall and involves three-dimensional architectural ornamentation consisting of symmetrical elements in layers which usually cover the sides of arches. The two-dimensional projection of muqarnas consists of a small variety of simple geometric elements. [6] Geometric patterns play an important role in Islamic architecture. Including muqarnas, a distinctive element of the Islamic architectural style. The muqarnas combine elements, harmonious lines, mathematical design, a specific architectural function, and an ornamental role, like "surreal" sculptures. [12]

Thus, virtual modeling of muqarnas can be very difficult and requires a lot of time and effort. Consequently, the research provides an interactive framework for muqarnas design using digital software to create a basis for the application of digital muqarnas modeling. Therefore, the structures of muqarnases are often ornamented in different forms. The aim is to provide an analysis of muqarnas and their ornamental patterns, as well as a range of digital methods and tools which can facilitate muqarnas modeling and create new possibilities in digital design to overcome the complexity of muqarnas design.

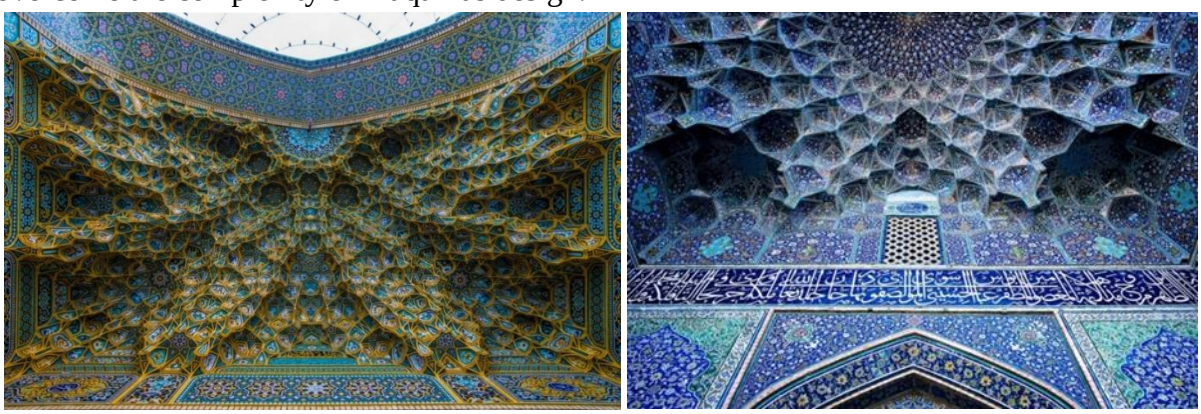

Figure 3. Various applications of muqarnas.[13]

The overlap of science and digital scientific developments has become an important element in the present era. Recent years have seen a growing interest in mathematics and computational science, including digital design. The aim is to enhance digital and algorithm design as a way of innovation in Islamic architecture, in addition, the use of digital tools which contribute to the transfer of Islamic architecture concepts, based on the evolution of the current era. In addition, it can be influential in the flourishing of Islamic architectural philosophy. Thus, studying the role of digital architecture to obtain a new approach in the development of muqarnas to form equations of traditional Islamic architecture with an innovative and sophisticated approach. [2]

Figure 4. Wonderful formations of muqarnas and mathematical approach in traditional Islamic architecture.

The muqarnas combinations have a unique beauty, which is distinct from traditional two-dimensional architecture. However, the formation of three-dimensional muqarnas provides new insights into the mathematical analysis of shape which exceeds its aesthetic and ornamental features in the digital age by a series of rules and algorithms. [14] Although the rapid growth of digital design software has brought a lot of innovations and freedom of use to the architecture world.

Figure 5. Muqarnas formation in the Court of the Lions in the Alhambra, Spain.[15]

In this paper, we present a new approach to the muqarnas model and study it as a transition between layers of different forms. Besides, the study introduces innovative modern methods which could generate new digital models of muqarnas. [16] By starting to analyze the existing additional Islamic geometry variables and testing them, we can derive a new unique geometry from them. [17] Through using digital software, it is possible to generate Islamic patterns with algorithm technology to devolve new vision in muqarnas design. [12] This study provides a modification of the muqarnas pattern in Islamic architecture by using digital algorithms through digital design, [18] as well as studying the control of classical ornamental patterns within coding

and programming in computational design. Over the last two decades, research into the computerization of muqarnas has aimed to drive the evolution of this geometry in new directions. [19]

Figure 6. On the left: the transition between layers of different muqarnas forms. In the middle, on the right: Madrasah Chahar Bagh (Four Garden School)-Isfahan, Iran.

Besides, this will help overcome the gap between classical patterns and the flow of digital future design to enhance a better understanding of various modern methods to control the geometrical of muqarnas.[20] Accordingly, computational programs are tools, which help produce an actual form for visualizing, improving the muqarnas design quality, and overcoming muqarnas complexity. Recently, digital programs have emerged to contribute to the solution of this complexity and create new forms of muqarnas. [19]

The main purpose of the study is to generate new three-dimensional muqarnas geometry and patterns in algorithmic modeling software and scripting environments. The use of two-dimensional projections of muqarnas plays an essential role in obtaining digital patterns and exploring new rules and concepts through the algorithmic form of muqarnas. [13] These properties are used to identify muqarnas in a more mathematical approach. [11] The research seeks to express a central role of the computer-aided design process by applying algorithm design methods, which control a set of rules to develop muqarnas. [14] Digital algorithms were used in muqarnas research to create new evolutions in form.[19]

Figure 7. Muqarnas details the kaleidoscopic worlds of these mosque ceilings.[21]

New prospects for muqarnas design have given rise to the widespread use of digital models and manufacturing techniques, which have given rise to new prospects for the design of muqarnas. [22] The formation of three-dimensional muqarnas provides new insights for mathematics to analyze the shape outside aesthetic and ornamentation features in the digital age. The emerging digital design approach and computational thinking explore innovative possibilities of muqarnas in forming spaces and developing tourism. Therefore, it may be very important to activate the geometrical approach to muqarnas and traditional Islamic art patterns with new digital tools in order to keep up with the evolution of concept design and adapt them to reality. The study focuses on the aesthetics of muqarnas ornamental design and touches on an important aspect of Islamic ornamentation elements in which muqarnas represent geometrical formations. Thus, it contains an aesthetic standard law that is different from other decorative elements, geometric or ornamental. If muqarnases have this importance, which depends on uniqueness, diversity, and richness in Islamic architecture, it is no doubt they will be of the same importance in the Islamic ornamentation field, especially if the geometric rules and their aesthetic values are framed in digital rules. [4] The study is based on the development of rules and methodologies for muqarnas elements through digital algorithms using innovative applications with the heritage basis of Islamic ornamentation in digital and algorithm design through study standards of digital algorithms and their impact on muqarnas parameters.

The importance of the study is that it deals with a fundamental architectural element in Islamic architectural styles. It contains a rich possibility of achieving relationships and textures based on shade and light. The main concern is its analysis of the basic design for different types of muqarnas, which contains its movement from the transition between its components in Islamic architecture spaces, as is verified in the transition from square to circle or appraiser.[4]

Figure 8. On the left: muqarnas elements in the Imam al-Dur Dome.[23] On the right: the muqarnas decoration on Qazveh Jame mosque in Central Asia. Muqarnas is made with small square tiles, mosaics, calligraphy, which are integrated into the muqarnas.[24]

8. The form and function of muqarnas

There are two types of Islamic geometric decorations: two-dimensional patterns and three-dimensional patterns. The first category includes ornaments, which are used to decorate flat or curving objects like domes or walls. Muqarnas, a sort of 3D ornamental construction, is the other category. Muqarnas is the Arabic name for stalactite vault, a fascinating structure made up of repeating units stacked in tiers, each supporting another corbel above it. [3]

Figure 9. On the left: Magnificent muqarnas of symmetrical palaces.[25] On the middle: the facade of a Madrassa (Religious School) named Tilla-Kari.[26] On the right: geometrical elements of three-dimensional muqarnas to ornament surfaces and it represents blocks of parameters and algorithms to shape a unique muqarnas.

9. The methods of construction of muqarnas

Understanding the construction methods of muqarnas is an excellent place to start when learning about the notion of muqarnas. Muqarnas are usually built in one of three ways: corbeled, overlaid, or suspended. [3]

Throughout history, the transition from square planning to circuit planning in three-dimensional has been a difficult subject not only for craftsmen but also for philosophers, mathematicians, and designers. [22] This study enhances a comprehensive understanding of muqarnases with their historical, philosophical, and conceptual backgrounds, on the one hand, structural and algorithmic principles, to reveal the internal logic of these complex geometric elements. [22]

Figure 10. On the left: A transition layer from a dome to a right-angle corner. In the middle: Masjid Jame, Isfahan, Iran. On the right: The muqarnas of the central niche of the iwan in the Zisa, Palermo. [5]

10. The types of muqarnas in terms of composition

Since its beginning, the Muqarnas building has been codified with mathematics, geometry, and algorithms. Introducing spatial and geometric relations (angles, dimensions, adjacency), rules (mathematical definitions, construction rule/order), and vocabulary elements, manuscripts provide core information for craftspeople (units, cells, roofs, filling elements). Algorithms have the potential to encode those first assumptions and definitions. [27] Because of its sophisticated geometry, the muqarnas is a difficult style of architectural embellished vaulting to model. It is

the partition of a cupola or corbel into a vast number of miniature squinches, resulting in a cellular structure known as honeycomb vaults due to their likeness. [28]

The North African/Middle Eastern style, which consists of a sequence of downward triangle projections, and the Iranian style, which consists of linked tiers of segments, are the two main forms of muqarnas. Many scientists have had great and wonderful studies on muqarnas, for instance (Al-Kashi and Shiro Takahashi). One of the scientific results of their mathematical and engineering efforts has been to create innumerable combinations and highly sophisticated formations from muqarnas, which cleverly intertwine with each other to ornament Islamic architecture. [6] Al-Ksh and Shiro Takahashi provide two different forms of Muqarnas compositions. [29] The surface of a Muqarnas has been approximated, and it was the oldest definition of Muqarnas.

Table 1. Al-Kashi and Shiro Takahashi and their impact on evolution muqarnas

First	**1. Al-Kashi**

[Al-Kashi] is among the greatest mathematicians and astronomers in the Islamic world. He gave the earliest mathematical approach to muqarnas and studied the measurement of muqarnas. In order to measure muqarnas, al-Kashi provides an overview of the different muqarnas elements. He described the elements' surfaces through angles and circular arcs. This makes it possible to give a perfectly accurate calculation of surface area.[11]

1. A barley kernel	2. A jug	3. A small biped	4.An almond (deltoid)	5.A half rhombus	6.A square

Muqarnas is divided into four categories by Al-Kashi. [29] [30]

1. **The simple muqarnas, which consist solely of planar surfaces.**
2. **All tiers of the clay plastered muqarnas do not have the same height.**
3. **Triangles and quadrilaterals make up the curved muqarnas - curved surfaces and the plan.**
4. **Other polygons such as pentagons, hexagons, octagons, and multi-pointed stars are found in the Shirazi muqarnas**. [2]

According to al-Kashi, an element's plane (horizontal) projection is made up of simple geometric forms: [29] [30]

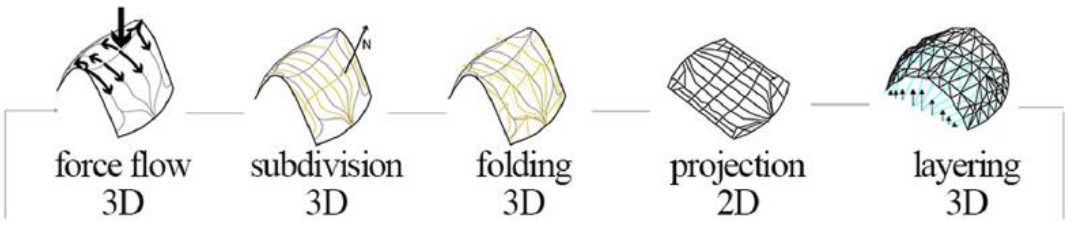

Figure 11: Muqarnas methods diagram.

Muqarnas shapes

Second	**2. Shiro Takahashi**

Another point of differentiation (1943) is Shiro Takahashi's muqarnas. The muqarnas is divided into three categories:

- **Muqarnas with square lattices.**
- **The muqarnas of the pole table.**
- **'Other style' muqarnas are those that do not fit into the first two categories.**

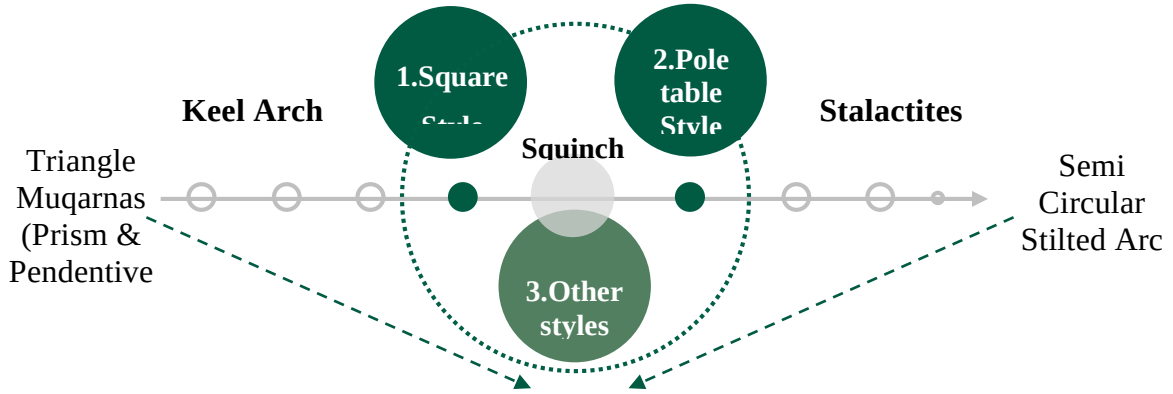

Figure 12. Muqarnas form diagram.

11. Physical and digital muqarnas modelling

11.1 Muqarnas' mathematics

The rich geometry of muqarnas, symmetrical design, and ornamental surface which fascinates it, is a beautiful element for design buildings. The form of these structures creates a sense of stability and balance. The modeling and analysis of these structures can reveal valuable information for architects and artists to provide a better understanding of these mathematical elements. [16] Abstract forms of mathematical patterns have been used extensively in ornament architecture for thousands of years. Because of the use of these geometric patterns, Islamic art has a close relationship with mathematics, especially geometry. Many artists and designers of traditional Islamic art were great mathematicians. [16] The mathematics-based design of muqarnas is very different, although, a computer algorithm is used to generate muqarnas and motivate designers to innovate in muqarnas' forms. [11]

11.2 Modeling parameters for muqarnas

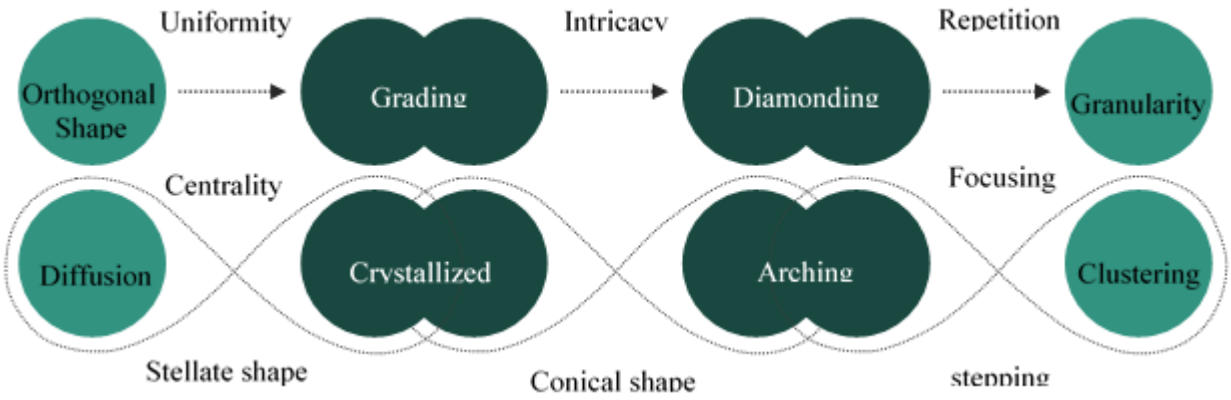

Figure 13. Diagram of muqarnas parameter analysis.

12. Development of muqarnas modeling

12.1 The Muqarnas projection plan

There is an algorithm for reconstructing muqarnas, and there are various muqarnases reconstructed with the same projection plane. [11] Muqarnas elements do not overlap in-plane projection, making it possible to design muqarnas by drawing their projection planes. However, such two-dimensional designs do not directly include all three-dimensional information. [11] The purpose of the reconstruction algorithm is to analyze muqarnas structure. Therefore, the formulation of digital programming can project a three-dimensional plane for the muqarnas. A muqarnas is designed by drawing the plane projection of the muqarnas elements. It consists of layers, which contain elements, and there is a large and diverse array of these elements. Among these basic elements are cells, which are small pieces of the vault, and the most important element in muqarnas building, because it forms the body and structure of muqarnas. The reconstruction of three-dimensional digital programs of muqarnases directly from their designs through digital algorithms allows one to build an innovative design from a three-dimensional model of muqarnas in computational programs as well as design unique patterns to ornament them. Hence, an analysis of its impact on Islamic architecture, virtual tourism, and other fields is needed.

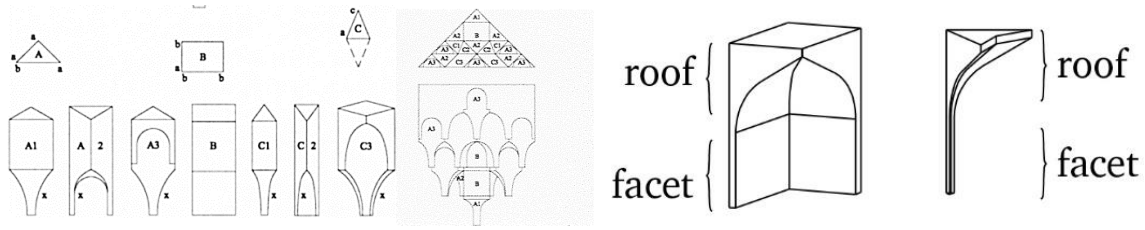

Figure 14. Simple elements of muqarnas (roof and facet), Plans, Elevations, Sections, and Details of the Alhambra. [5]

12.2 The analysis of muqarnas using a digital algorithm

The muqarnas are created from different cells. These cells are the building blocks on which the muqarnas were built. Besides cells, there is a space that needs to be filled with another type of building block, so-called intermediate elements. Traditional and digital surveying; CAD modeling (e.g., direct parametric modeling or reverse modeling based on digital surveying); mathematical computation (e.g., using algorithms for calculation of theoretical reconstruction); and a combination of these are the main techniques and tools used to achieve these goals using a commercial 3D CAD program. [31] The research describes a procedural way of decoding a 3D pattern (from its 3D actual geometry to its analysis) while also clarifying the rules that govern the construction of 3D patterns (from the 2D drawing to the 3D geometry). The research utilizes photogrammetric methodology for reconstruction and CAD-based modeling. [31] The photographs must depict the object's geometry from several angles of view. Photogrammetry is used to create a 3D textured model using a free or open-source software tool. The systematic

graphical examination of a 2D drawing of the plan projection of 3D elements in a CAD environment results in the identification of the key geometric elements and their modular pattern in the plane XY. To unambiguously specify the position, the 3D textured model is employed as a datum point. [31] The software should be able to export the 3D model in a file format, which can be read by a CAD modeler (e.g., in .Stl file format). [31]

Image selection	Photogrammetry	2D model analysis	Integration of 2D elements and 3D model and development of final CAD model.
No resolution requirements. Different points of view. If necessary, copyright the images. Preprocessing (cutting	3D geometric model (cloud of points). 3D textured model.	Identification of patterns and module. Import in a CAD environment. If not available,	

Figure 15. 3D pattern analysis and reconstruction of the muqarnas. [31]

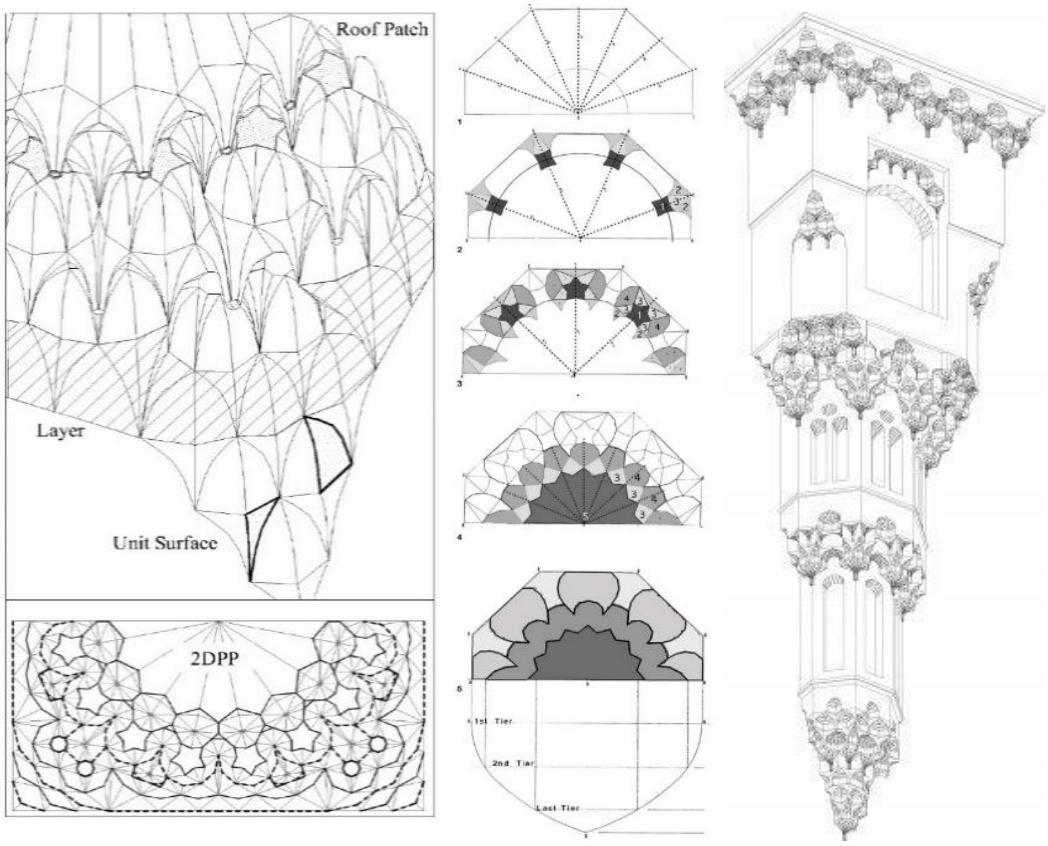

Figure 16. On the left: The basic parts of the muqarnas using traditional CAD tools and 3D modeling software. In the middle: the formation stages of Muqarnas in the context of half an octagon. On the right, a traditional CAD perspective of the muqarnas projecting bay window for the International Medical Center in Jeddah, Saudi Arabia.[32]

12.3 The stages of analysis of muqarnas in the digital environment

- The first images selected from muqarnas for 3D photogrammetry reconstruction are of the muqarnas themselves.
- Using 123D Catch, a free Autodesk tool, for its versatility and efficiency.
- Photogrammetry-based software works by uploading all the photographs to a cloud-based

environment and then returns a textured 3D point cloud model.

- By zooming, rotating, and measuring element distances, the textured model enables a comprehensive investigation of its elements.
- Furthermore, it relates the final model to each photograph used in its reconstruction, just as other photogrammetry applications, but only the elements visible in greater than two to three images are constructed.
- Because 123D Catch was designed primarily for rapid prototyping, the output is in the Obj file format. However, the file may be quickly exported in the most common neutral file format utilizing another free Autodesk product called MeshMixer (e.g., .stl file format).
- The analysis was conducted using a 2D plan projection.
- Muqarnas was divided into angular sectors. [31]
- While identifying the tiers in a 2D projection is impossible, using a 3D virtual model and rotating and zooming on major pieces, it is possible to determine the levels they are on.
- Each flat element's surface was modeled in the CAD environment.
- The integration of the 2D pieces with the 3D model in the CAD system is the fourth phase. The 2D pieces must first be modeled as surfaces on a base plane, then offset to their true position on the muqarnas. model in Stl.
- On the muqarnas 3D model, the offset of the 2D elements to their true position.
- Once all the pieces of a module have been modeled, the process can be repeated as needed to produce the dome/full vault's geometry.
- The final phase generates a surface model in CAD format. [31]

12.4 Muqarnas' computational potential

Digital design, or computational design, is different from traditional design in its terminology, concepts, and processes. The research focuses on the exploration of muqarnas through digital and physical modeling using Photogrammetry 3d model software and CAD tools to create 3d models of that with 3D RHINO or 3D MAX software + Fbx and explores three-dimensional form-finding algorithms. [14] In the translation process from physical to digital model, the focus was on determining the variables of the muqarnas unit.

In addition, two-dimensional architecture patterns can be the basis of a digital design unit. [2] The research focuses on the relationship between the three-dimensional structure and its two-dimensional plane projection. [11]

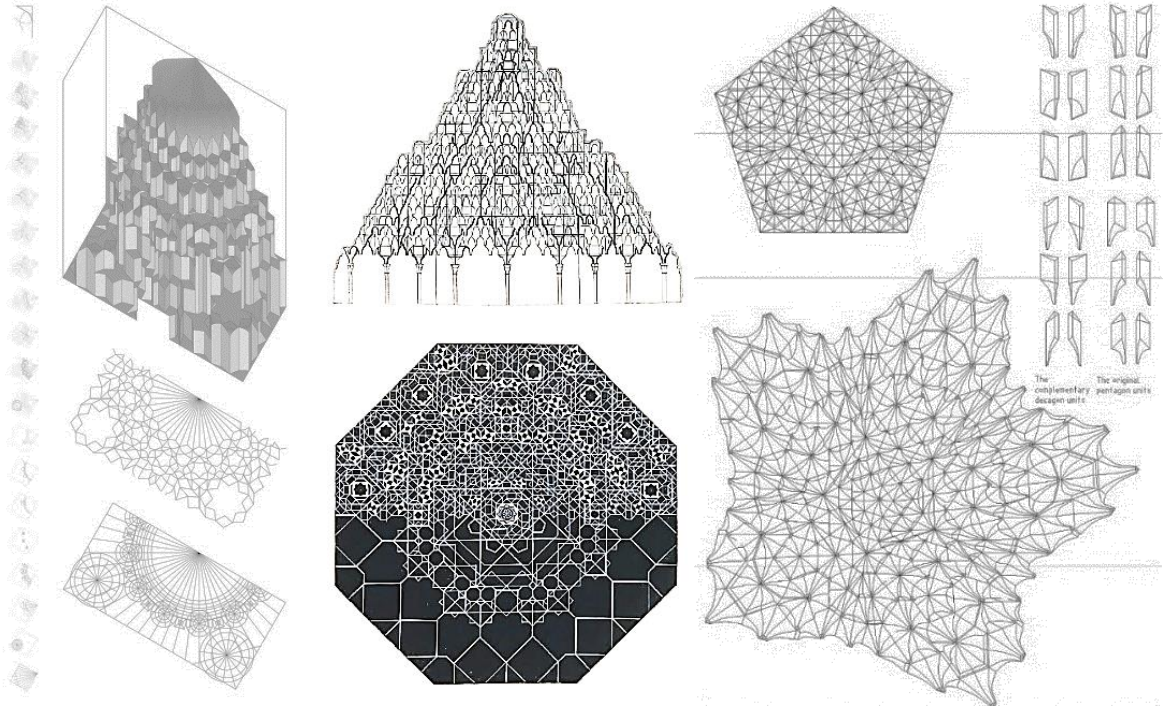

Figure 17. On the left: steps of the formation of muqarnas through projection. [33] In the middle: plan an elevation of muqarnas. On the right: a muqarnas in a pentagonal space.

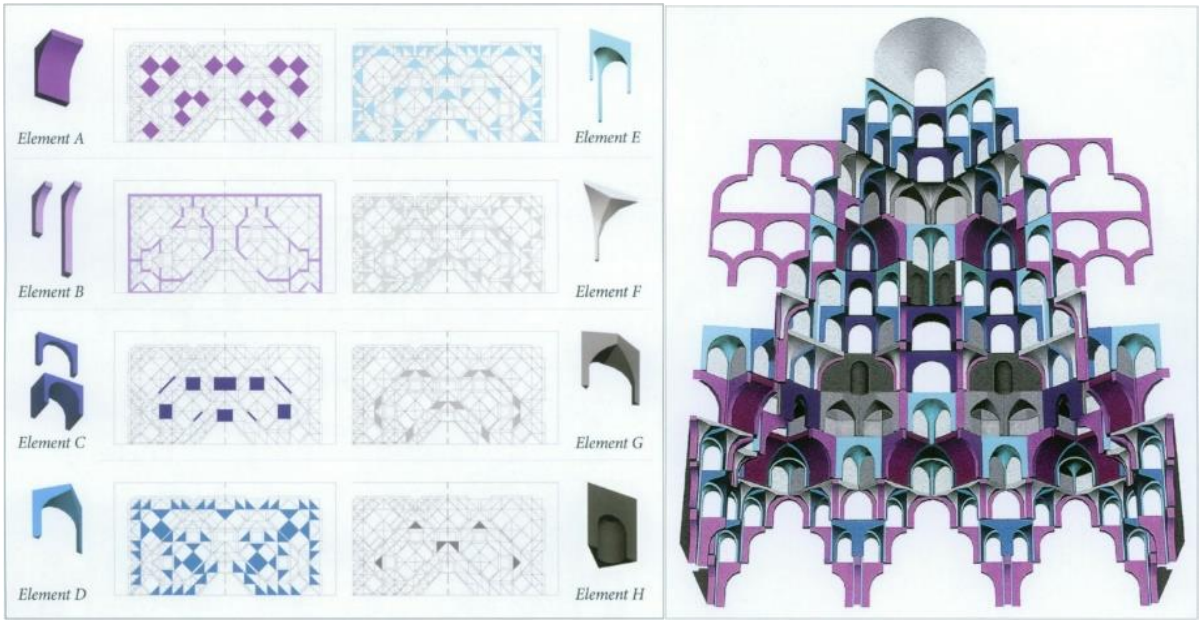

Figure 18. On the left: Geometric analysis using traditional CAD tools of the twelve registers of the muqarnas of the central niche of the iwan in the Zisa, Palermo. This is the simplified profile of the register, in which each element is a square, a triangle, or a rhombus. On the right: a virtual model of muqarnas using traditional CAD tools for producing new elements at the Zisa in Palermo. [5]

13. The types of muqarnas in terms of function

The use of muqarnas has become ubiquitous in Islamic architecture. In addition, it has been applied structurally and decoratively over the main areas in mosques, shrines, facades, arches, cornices, and entrance gates. [34] Muqarnas was later used in various portions of a structure, including enormous domes, cupolas, niches, arches, and as a nearly flat decorative frieze. [3] Muqarnas in various forms. Portal, mihrab, dome, walls, flat ceiling, corners, the base of a

dome, niche, and minaret (from right to left, top to bottom). [3] The research classifies these ceiling layouts into three different patterns by aligning them in a time-space matrix. [35]

Muqarnas in three different patterns:

1) square style.

2) pole table style.

3) Circular polar and star polygons.

Figure 19. On the left: Natanz Jaame Mosque, Natanz, Iran, Modular stucco Muqarnas. [2] In the middle: Muqarnas examples in Damascus. On the right: Muqarnas examples in Syria and Cairo, Egypt. [36] The traditional art of muqarnas can be reinterpreted by using a computational approach. Consequently, virtual modeling of muqarnas has been widely used in history, as well as algorithm modeling, which has played a huge role in the understanding of the three-dimensional structure of muqarnases.[12] The virtual models of muqarnas can help designers as well as scientists and historians to achieve a better understanding of the muqarnas to develop their designs. Thus, virtual tourism may be one of the most important applications of these models. The architecture of muqarnas spreads throughout Asia and Europe. Through the virtual tour, many people around the world can enjoy the beauty of these buildings. Besides, by using large projection domes, a realistic experience can be achieved by visiting the actual building. [16]

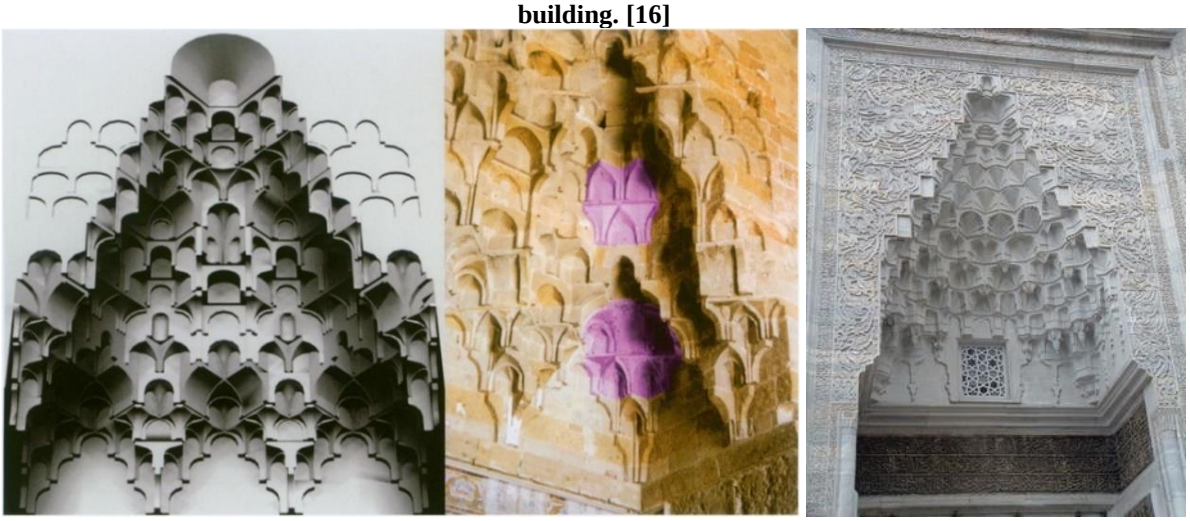

Figure 20. On the left: The stone muqarnas of the central niche of the iwan in the Zisa, Palermo (the virtual model). In the middle: A photograph of the right-hand corner, with registers 3 and 4, and 6 and 7 highlighted in lavender. [37] On the right: the magnificent muqarnas of the City Walls. [38] Thus, we realize the importance of muqarnas through its varied presence, forms, types, organization of its parts, and its compatibility in arches, crowns of columns, entrances, openings, cornices, ceilings, domes, minarets, and niches, as well as other rich and distinctive functions. This is more influential than its geometric formulation, which represents pure beauty and gives harmonious, rhythmic, and consistent muqarnas in Islamic architecture.[2]

Figure 21. On the left: a view from the front. In the middle: A view from underneath. On the right: plane projections of muqarnas. [11]

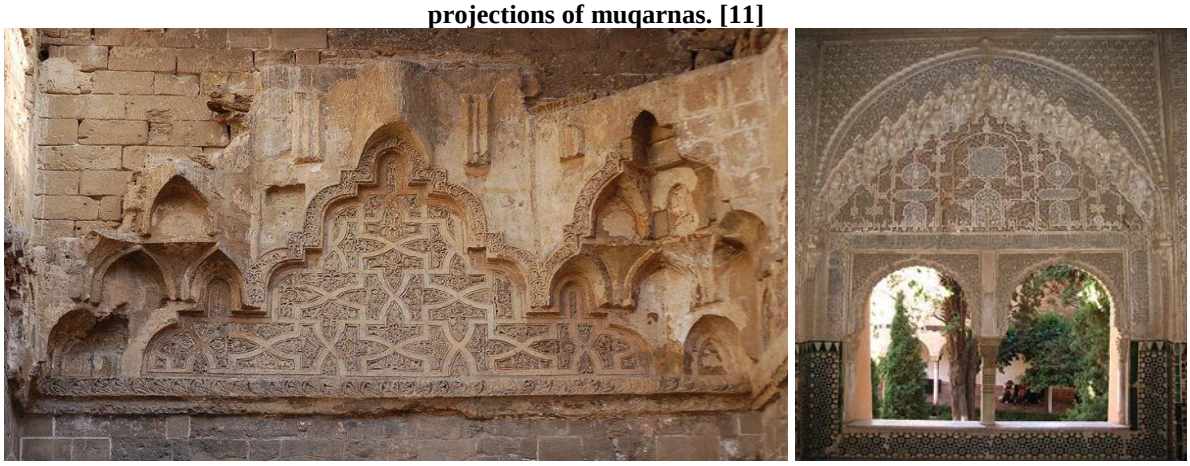

Figure 22. On the left: The ornamentation which decorates the pattern, below the muqarnas in Palermo, Italy.[39] On the right, the Mocárabe of Al-Andalus' palaces and mosques, as well as the Central Asian Muqarnas. The ornamentation ranges from tailings to stone carving to calligraphy in Alhambra, Spain. [40]

14. The design structure of the muqarnas

Muqarnas is challenging to describe because of their complicated shapes, geometry, and repeated structures such as (symmetry-repetition-diversity-accumulation - diversity in scale). The work becomes even more difficult due to the variety of styles. [3] One of the main characteristics of muqarnas is the three-dimensional unit which can be represented as a two-dimensional scheme. The research studies the use of scripting and coding with a two-dimensional geometry in morphological methods of muqarnas, in addition, the development of a digital algorithm for the reconstruction of three-dimensional muqarnas.

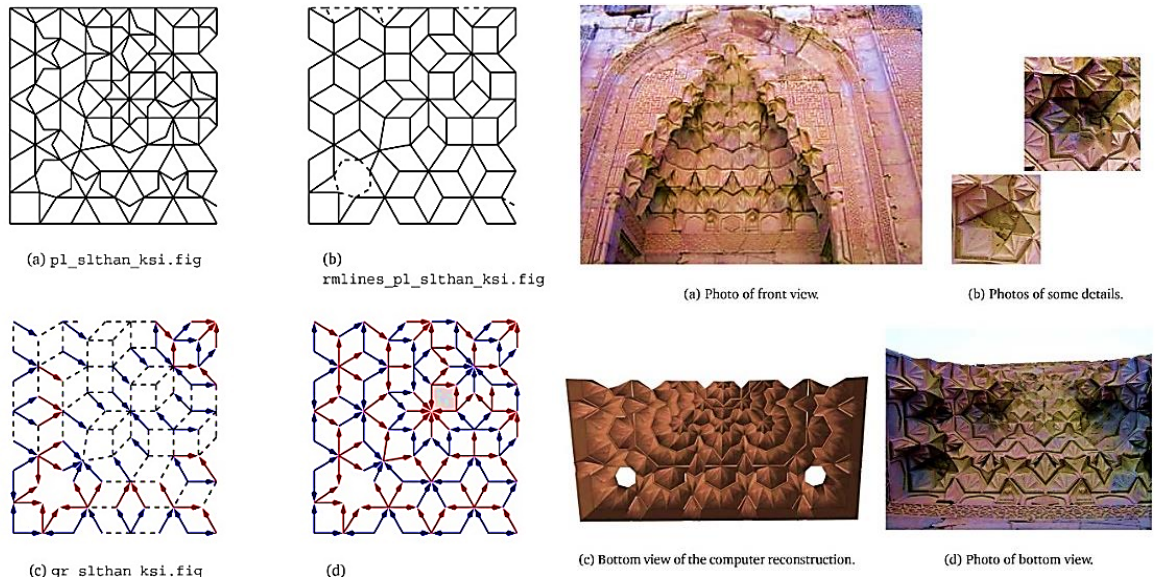

Figure 23. On the left: plan analysis through coding of muqarnas in the entrance. On the right: development of a digital model of the entrance.

14.1 The muqarnas' symmetry

Muqarnas is known for their beautiful symmetry, which is one of their most distinguishing features. [3] Muqarnas geometry is complex through repetitive and symmetrical shapes which appear in its design. Muqarnas structures are generally designed based on radiational symmetry. Radiation symmetry is a symmetrical arrangement of ornaments around a center.

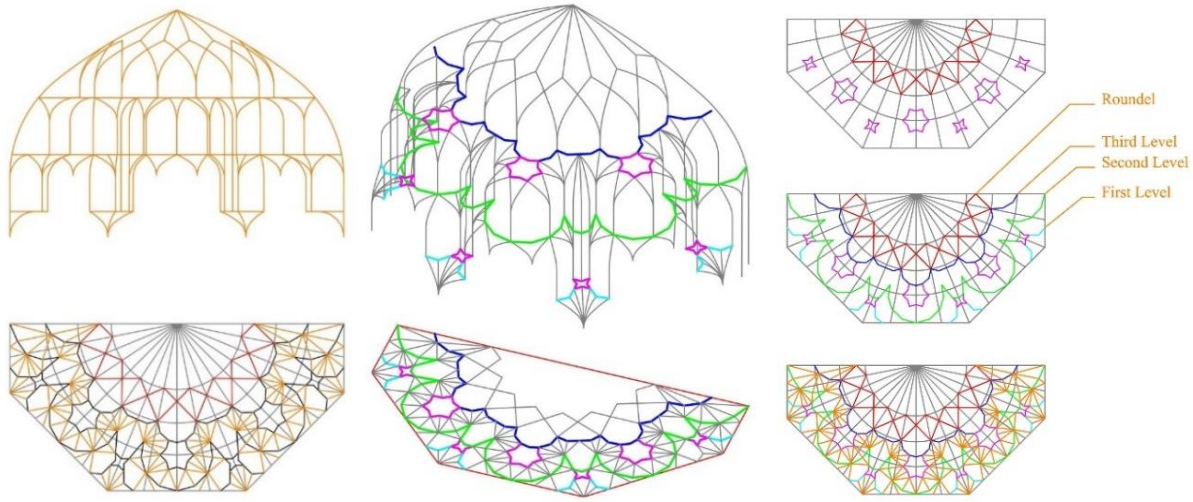

Figure 24. Muqarnas projections to reach the final digital model.[41]

14.2 The layered structure of the muqarnas

Muqarnas is a transition between two forms of different sizes and shapes. For example, a dome is a transition from a circle to a single point. Muqarnas is typically moved in multiple steps. This means that the starting form is converted to several other shapes before reaching its final form. The layered structure plays an important role in the digital modeling approach. We can design the muqarnas model as a group of surfaces, which fill the area between the layer lines. [16]

Figure 25. On the left: Muqarnas above the main entrance of the Al-Azhar Mosque in Cairo, Egypt. [42] On the right: Stone muqarnas with hanging elements at the Zisa, Palermo. [43]

14.3 Muqarnas generated computationally

The architect can create a computational model through a set of rules. However, in the case of muqarnas, the presence of geometric characteristics and symmetry helps designers achieve digital design through the computer without using the rules of form. [16]

A possible way to model muqarnas in computer graphics using Photogrammetry 3d model software and CAD tools is solid modeling, which involves constructing a solid shape and deleting sphere-like elements from it. [3] The designer identifies vector graphs derived from the muqarnas schema, which contains structure information for muqarnas. From this chart, we can identify muqarnas elements. Then, the process of calculating muqarnas is divided into two steps: first, a vector graph containing structure information for muqarnas from the muqarnas schema. Second, we extend this diagram to the structure of three-dimensional muqarnas. [11] The research seeks to explain the theory underlying computational design to use digital trends in the evolution of traditional forms and patterns of muqarnas.

15. The materials used in the construction of the muqarnas

The muqarnas are made of small pieces which are simple shapes and together form these simple blocks of complex blocks. [16] Muqarnases are complex concave hollow structures that work as ornamental tools used in Islamic architecture. As well as the materials used to build muqarnas, these include bricks, plaster, mosaics, ceramic tiles, frescoes, mirrors, wood, and paint. Its importance comes from a special aesthetic in its geometry and its three-dimensional formations, as abstract sculptures which simulate nature with an Islamic conceptual methodology and play a structural role in the transition from one geometric form to another and from one roof to another.

Muqarnas is encased in tiles or plaster and are composed of brick, stone, stucco, or wood. The shape and medium of these creatures differ depending on where they are found. In North Africa, they're usually made of plaster and wood, while in Iran and Iraq, the muqarnas dome is made of plastered bricks or ceramic clay. Plaster stucco, mosaic panels, ceramic plates, and stone are among the materials employed.[35] From the 12th century, muqarnas decoration in stucco, brick, wood, and stone was applied routinely to cornices, squinches, pendentives, vault interior surfaces, and other architectural components that can be found all over the Islamic world. [44]

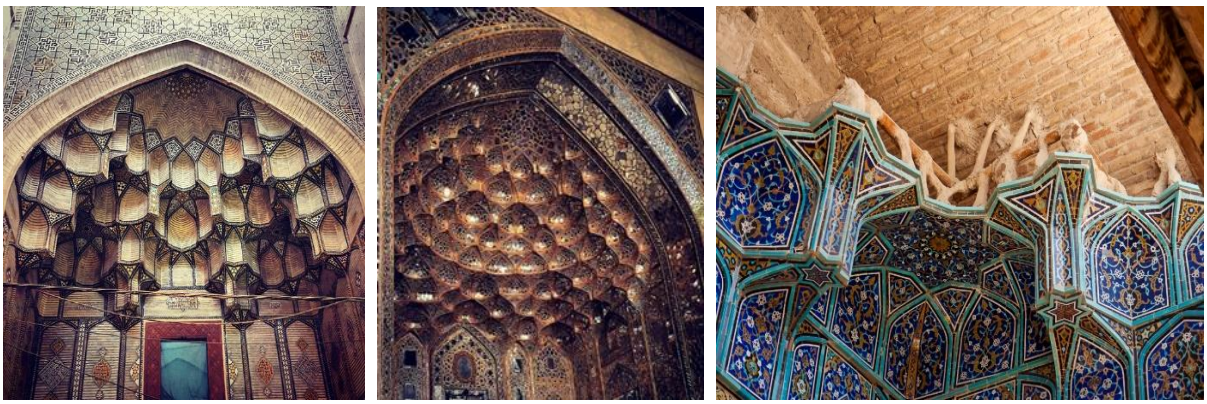

Figure 26. On the left: Muqarnas in Jammeh's mosque. In the middle: Muqarnas in the Palace of Shah Abbas Safavi. [45] On the right: Suspended muqarnas in the process of repair. There are obvious attachment ribs and space between the muqarnas surface and the walls.

Figure 27. On the left: Old muqarnas through the stone.[46] In the middle: corbeled muqarnas. On the right: superimposed muqarnas.

16. The installation methods and industry of the muqarnas

16.1.1 Muqarnas form-finding

In the past, it was difficult to apply the process of form-finding a limited form of advanced science from its time, although now the choice of digital tools is of great importance. At present, the strength of algorithms and programming codes is enhanced by the science of designing shapes and spaces. [2]

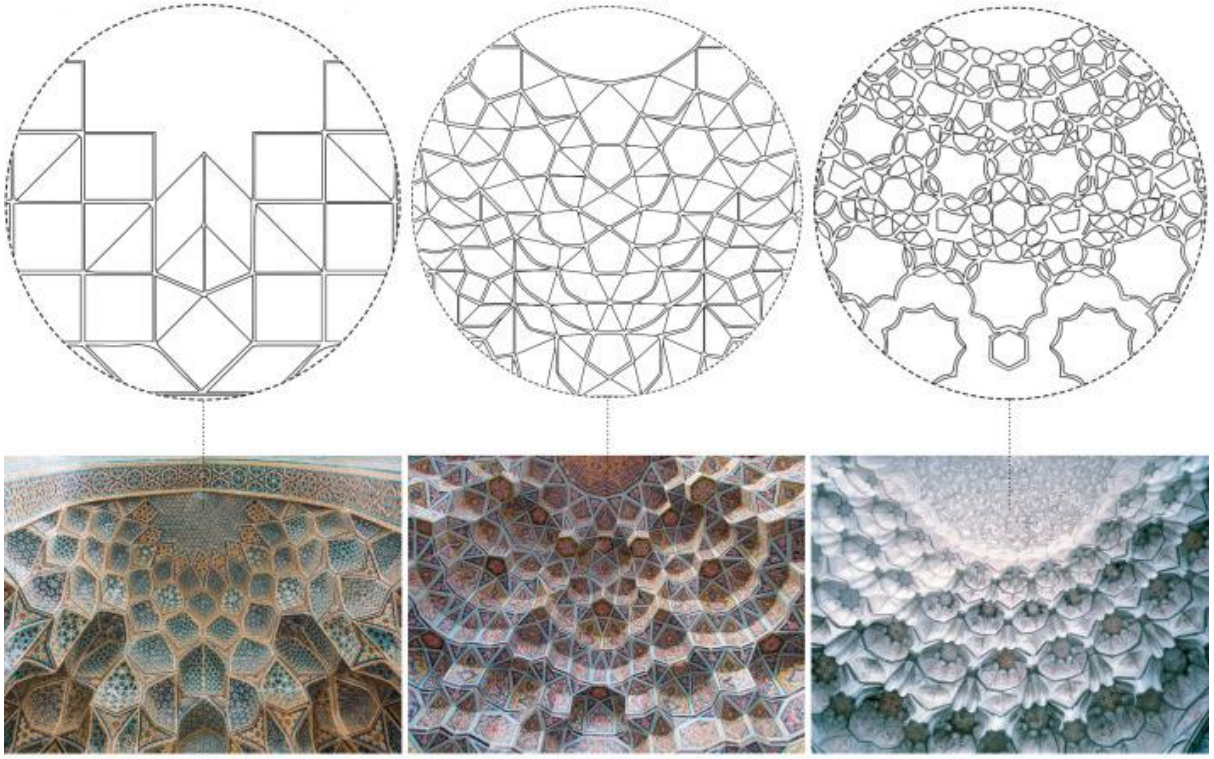

Figure 28. Form-finding geometric plan views and classification of muqarnas according to Shiro Takahashi. On the left: Shaykh Abd al-Samad Mosque, Natanz, Iran; detail shot of the khanqah doorway. On the right: a detailed view of the Pishtaq entrance. [47]

16.2 Muqarnas and geometric patterning

16.2.1 Muqarnas modeling

It is important to identify digital muqarnas in software using traditional CAD tools, algorithms, and programming codes. In addition, to take advantage of the muqarnas layer structure for easier modeling. The designer chooses ornaments, which are repeated to create shapes in computational design. [16] In order to develop accurate models of muqarnas, the geometric method of drawing two-dimensional projections is studied to achieve the best procedure for drawing muqarnas. Then, the two-dimensional digital modeling is performed by the computational software. [12] The digital model allows architects and designers to compare and produce innovative outputs by controlling dynamic calculations of digital modeling and design variables for muqarnas. The research develops a digital and algorithm design approach to develop muqarnas and gives a new vision for the preservation of Islamic heritage. [48]

In this context, digital design procedures are introduced as a methodology, which enhances the design capability of the digital model to perform design differences using shapes as parameters in digital design. Thus, the digital model becomes a flexible tool, which allows for changes at topological and engineering levels of muqarnas. [49] The properties of muqarnas are controlled by a non-geometric component called the variable. The existence of an algorithm can analyze the structure of the muqarnas. Consequently, digital software integration provides rapid analysis, which is an important feature compared with old methods.

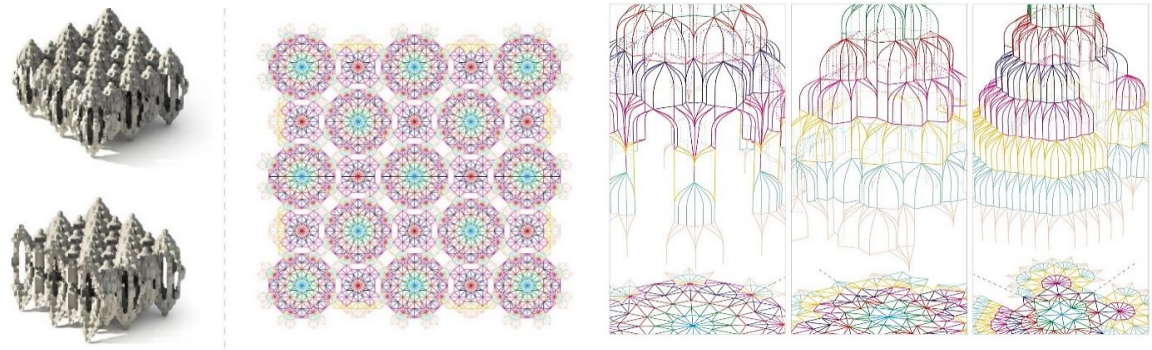

Figure 29. Geometric organization of muqarnas.[50]

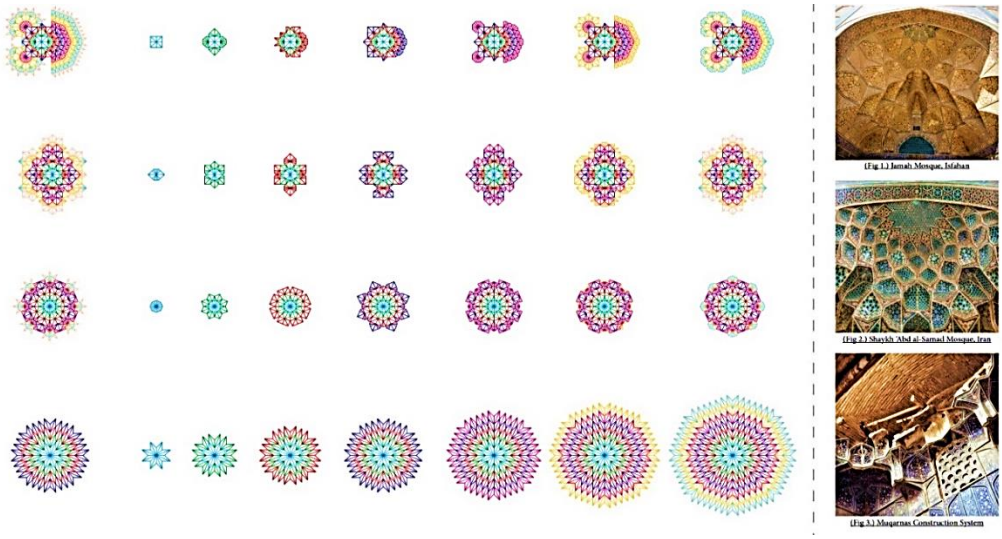

Figure 30. Surfacing and subdivision of muqarnas in the form-finding process.[50]

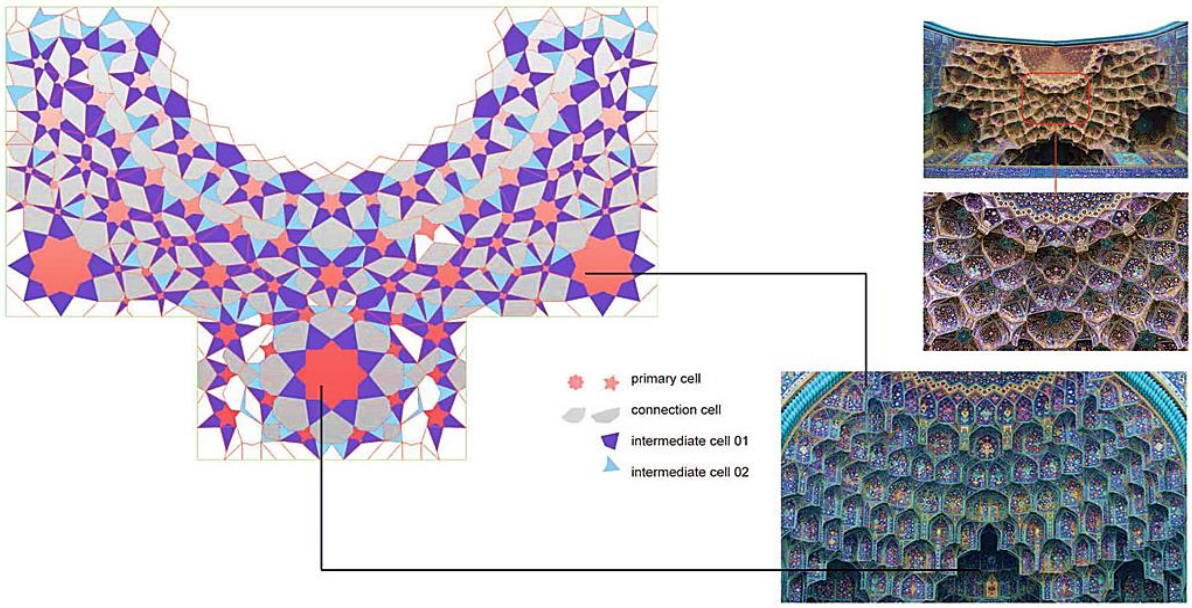

Figure 31. Experiments with digital modelling muqarnas. [51]

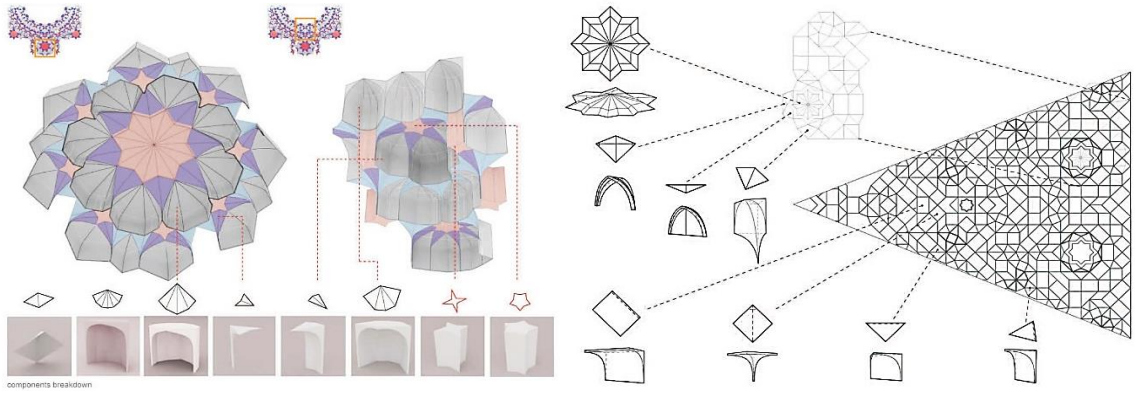

Figure 32. Form-finding muqarnas through rules of the digital algorithm.

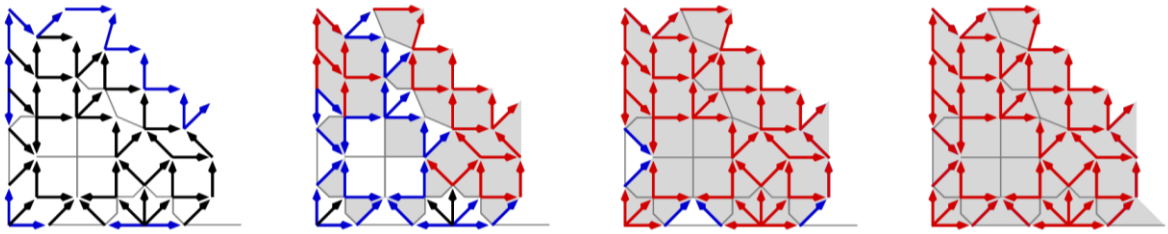

Figure 33. Finding the faces in the muqarnas graph that correlate to the muqarnas in the basement of the Friday Mosque in Natanz's north iwan.

Contemporary digital software and tools help designers and architects to design complex shapes in the creation of modular forms which are called algorithm software. "Digital design" is related to advanced digital techniques. The systems of digital design allow for the formation of complex design, conceptual structures through the implementation of a simple set of processes and variables. This represents the emergence of creative approaches to design thought. [12]

16.2.2 Algorithm for Reconstructing Muqarnas

In the calculation of the muqarnas unit, Al-Kashi analyzes many forms of muqarnas in its basic units. The horizontal projections of muqarnas have been used, and designers use these planes to study muqarnas forms. The shape of muqarnas is defined through the composition of

symmetry, the shape of its layers, and how these layers are attached. When changing these factors and their variants, the designer can create innovative designs. [16] Hence, the study of mathematical and geometrical characteristics of muqarnas through algorithm design. [22] There are various techniques, which are represented during digital modeling involving flow force, division, projection techniques, and layers to generate computing from new structural capabilities and mathematical algorithms performed by digital software. [48]

Figure 34. On the left: the portal of the Jame mosque in Isfahan, Iran. [16] In the middle: The colored lines represent the orbits in the muqarnas graph. On the right: Overlay of virtual model and muqarnas.

16.2.3 Coding for evaluation of muqarnas

For most designers and architects, the computer is an advanced device, which allows them to have better control over the production of complex designs. [2] Therefore, the designer studying the function of mathematical formulas must incarnate and express forms based on rules in the process. Thus, the application of mathematical formulas in the design of construction can be used effectively to control design by utilizing computational technology with traditional heritage, and we can implement muqarnas modification processes through digital algorithm steps. [20] The research shows how to form-finding of muqarnas, which is derived from chart muqarnas, through the conversion process. This process is divided into two parts. First, we define nodes, which define the muqarnas element together. Then, these combinations of nodes are converted into a group of muqarnas elements side by side. In some cases, it is possible to find the structure of the source directly from the muqarnas chart, without using the muqarnas scheme. [11]

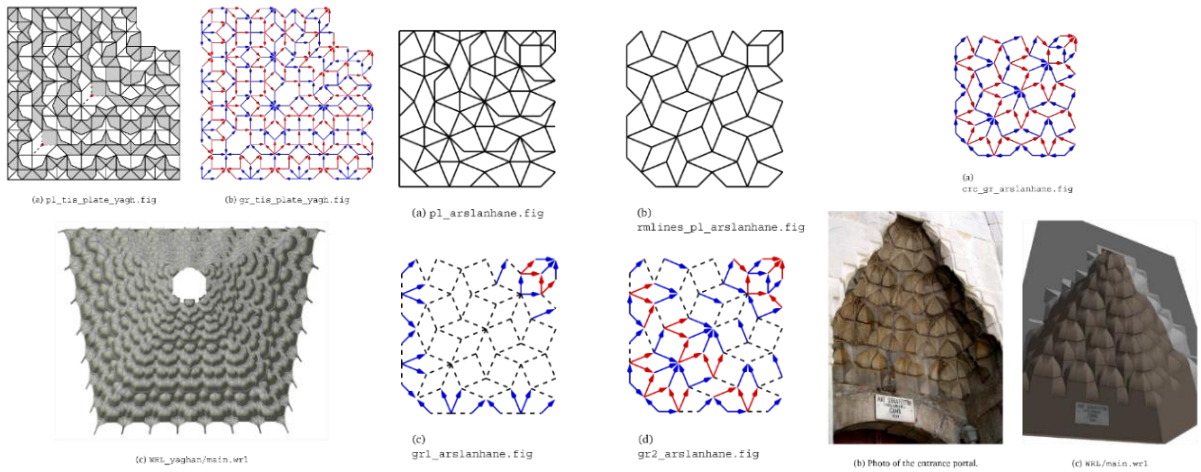

Figure 35. On the left: the reconstruction muqarnas proposal of Yaghan. On the right: On the middle: Plan analysis of the muqarnas vault in the entrance portal of the Arslan-Hane Mosque in Ankara. On the left: reconstruction of the entrance portal.

17. Discussion and Results

Research begins by describing the problem in a historical and mathematical context by reconstructing and developing muqarnas and study software tools, which can analyze muqarnas designs. The research presented concepts and theories behind many studies of muqarnas which led to the evolution of its form at many levels. We started with an introduction to muqarnas and their role in Islamic architecture. Second, the theory of muqarnas evolution was presented by projecting their planes of three-dimensional design and introducing digital applications in the muqarnas field. These include new forms which evolve from horizontal projections and two-dimensional projection layers, as well as new forms of structural value, and a new vision from the inside out of muqarnas. Third, the theory of digital algorithms was introduced as a design trend in the evolution of three-dimensional muqarnas modeling.

Islamic art is distinct because of its mathematical nature, and therefore the formation of three-dimensional muqarnas may provide new insights into mathematics. Digital modeling patterns can be used to devise and interpret formal generation algorithms of traditional Islamic patterns in creating innovative elements of muqarnas. Consequently, the study concluded new systems of digital through complex shapes and muqarnas patterns.

Finally, the applications of computational theory and digital algorithms were presented in the muqarnas field. The digital modeling tools for muqarnas allow the possibility of producing different digital simulations for Islamic patterns. The result was the preservation of cultural heritage through virtual modeling of muqarnas which formed elements and innovative patterns of Islamic art. These findings can be generalized to future studies in this field.

18. Conclusion

The research concluded the application of innovative digital techniques used to identify, select and analyze information about the muqarnas. Therefore, these will involve the development of cultural heritage and the encouragement of tourism through the virtual modeling of muqarnases with their attractive and fascinating patterns. Hence, the use of digital technologies makes it possible to develop historical ideas and traditional Islamic elements which are possible, sophisticated, and creative. Besides, the two-dimensional projection contains all the information for the three-dimensional muqarnas structure. The search provides many different tools and approaches for editing muqarnas of digital models. Consequently, the construction of muqarnas carries the ability to be connected to mathematics, geometry, and algorithm approaches. Digital algorithms are created by using Photogrammetry 3d model software , CAD tools and coding which can give a quick analysis of designs and results in a better understanding of the structure and evolution of muqarnas. Finally, the nature of digital tools harmonizes with Islamic art elements and the principles of Islamic architecture. Thus, the transfer of Islamic architecture culture contributes to the confrontation of violence and the recognition of the important role of Islamic architecture as an effective stage in challenging racist concepts.

19. References

[1] S. A. G. and ’Mustafa K. Murat Yakar, Haci Murat Yilmaz, “Advantage of Digital Close Range Photogrammetry in Drawing of Muqarnas in Architecture,” Journal, Inf. Technol., vol. 8, no. (2), pp. 202–207, 2009.

[2] M. G. and M. A. Ahad Nejad Ebrahimi, “Parametric Design pattern Language and Geometric Patterns in Historical Domes in Persian Architecture,” vol. 29, pp. 234–256, 2014.

[3] N. Hamekasi, F. Samavati, and A. Nasri, Interactive Modeling of Muqarnas. 2011.

[4] E.-B. H, “The Muqarnas AGenuine characteristic of Islamic art, Its Early use, and Development in Domes Minbar Al Islam,” vol. 1, p. 34, 1965.

[5] V. GAROFALO, “A methodology for studying muqarnas: The extant examples in Palermo,” Muqarnas journal, Publ. by Brill, vol. 27, pp. 357–406, 2010, [Online]. Available: http://www.jstor.org/stable/25769702.

[6] H. S. and G. M. NEGIN DADKHAH, “Traditional complex modularity in Islamic and Persian architecture: Interpretations in muqarnas and patkane crafts, focusing on their prefabricated essence,” in ACSA Fall Conference, 2012, pp. 130–138.

[7] S. Mulder, The Shrines of the ’Alids in Medieval Syria : Sunnis, Shi'is and the architecture of coexistence. Edinburgh: Edinburgh University Press, 2014.

[8] “http://www.virtualani.org/glossary/index.htm.”

[9] J. M. Bloom, “The introduction of the muqarnas into Egypt,” Muqarnas, vol. 5. pp. 21–28, 1988, DOI: 10.1163/22118993-90000219.

[10] “https://www.oxfordreference.com/view/10.1093/oi/authority.20110803100216945.”

[11] V. Von, “Algorithmic Computer Reconstructions of Stalactite Vaults - Muqarnas - in Islamic Architecture,” Dipl.–Math. Silvia Harmsen aus Ermelo (die Niederlande) Tag, 2006.

[12] I. Sayah, “Creating a parametric muqarnas utilizing algorithmic software,” Int. J. Rev. life Sci., vol. 6, no. 1, pp. 47–53, 2016, [Online]. Available: www.ijrls.com.

[13] “Various applications of muqarnas.” pbs.twimg.com.

[14] S. O. Z. G. Alacam, “Computational Interpretations of 2D Muqarnas Projections in 3D Form FINDING,” in Parametricism Vs. Materialism: Evolution of Digital Technologies for Development [8th ASCAAD Conference Proceedings London (United Kingdom), 2016, no. January, pp. 421–430, DOI: ascaad2016_043.

[15] “Court of the Lions in the Alhambra, Spain.” landlopers.com.

[16] N. Hamekasi, “Interactive Design of Muqarnas,” The University of Calgary, department of computer science, 2013.

[17] C. B. Mostafa Alani, “A parametric metamorphosis of Islamic geometric patterns : The extraction of new from traditional,” Futur. Archit. Res. 2015, pp. 442–449, 2015.

[18] I. I. Notkin, “Decoding Sixteenth-Century Muqarnas Drawings,” Muqarnas journal, Publ. by Brill, vol. 12, no. 1995, pp. 148–171, 1995, [Online]. Available: http://www.jstor.org/stable/1523229.

[19] M. A. Yaghan, “The evolution of architectural forms through computer visualization : muqarnas example,” pp. 113–120.

[20] J. Lee, S. Kim, and Y. Jeon, “Study of the Control of Geometric Pattern Using Digital Algorithm (with Focus on Analysis and Application of the Islamic Star Pattern),” Adv. Mater. Sci. Eng. Publ. Corp., vol. 2015, pp. 1–15, 2015, [Online]. Available: http://dx.doi.org/10.1155/2015/950232.

[21] "Muqarnas details in mosque ceilings." http://doorofperception.com/2016/10/islamic-architecture-mosque-ceilings/.
[22] S. Alaçam, O. Z. Güzelci, E. Gürer, and S. Zeynep, "Reconnoitring computational potentials of the vault-like forms : Thinking aloud on muqarnas tectonics," Int. J. Archit. Comput. 2017, vol. 15, no. (4), pp. 285–303, 2017, DOI: 10.1177/1478077117735019.
[23] "The Imam al-Dur Dome." https://www.flickr.com/photos/8349545@N03/galleries/72157622816287546?rb=1.
[24] "Muqarnas decoration on a mosque." https://starsinsymmetry.wordpress.com/category/gaga-over-geometry-monday/.
[25] "Magnificent muqarnas of symmetrical palaces." http://tumblr.photojojo.com/post/39063150492/magnificent-photos-of-symmetrical-palaces-in-iran.
[26] "The facade of a Madrassa." www.fotopedia.com/.
[27] S. Alaçam and O. Güzelci, Computational Interpretations of 2D Muqarnas Projections in 3D Form Finding. 2016.
[28] E. Imani, "Historical and Geometrical Analysis of Muqarnas and Prospect of Its Reflection on Today's Architecture," Master of Science, Middle East Technical University, 2017.
[29] Y. D.-S. and S. L. Harmsen, "The Muqarnas Plate Found at Takht-I Sulayman: A New Interpretation," Muqarnas journal, Publ. by Brill, vol. 22, pp. 85–94, 2005, [Online]. Available: http://www.jstor.org/stable/25482424.
[30] H. M. Al Jumaily, "Muqarnas Form Efficiency in Diffusing Sound Waves Within the Space," Int. J. Innov. Sci. Technol., vol. 12, no. 4, pp. 131–150, [Online]. Available: online as open access.
[31] F. Gherardini and F. Leali, "A Framework for 3D Pattern Analysis and Reconstruction of Persian Architectural Elements," Nexus Netw. J., vol. 18, no. 1, pp. 133–167, 2016, DOI: 10.1007/s00004-015-0287-z.
[32] "Perspective of the muqarnas projecting." http://www.bonner-design.com/islamic-ornamental-design/muqarnas/.
[33] "formation muqarnas through projection." http://bluecarpetcollective.blogspot.com.eg/2010/11/muqarnas.htm.
[34] M. Kashef, "Bahri Mamluk muqarnas portals in Egypt : Survey and analysis," Front. Archit. Res., vol. 6, no. 4, pp. 487–503, 2017, DOI: 10.1016/j.foar.2017.09.004.
[35] S. Takahashi, "http://www.shiro1000.jp/muqarnas/default-.htm," Muqarnas: A Three-Dimensional Decoration of Islamic Architecture.
[36] "Muqarnas examples in Damascus, Syria, and Cairo." http://islamicarchitecturebydxx.blogspot.com.eg/2015/02/basic-definitions-muqarnas.html.
[37] "The virtual model." Vincenza Garofalo.
[38] "Muqarnas in the City Walls." https://www.flickr.com/photos/8349545@N03/galleries/72157622816287546?rb=1.
[39] "Muqarnas in Palermo." https://starsinsymmetry.wordpress.com/category/gaga-over-geometry-monday/.
[40] "The Central Asian Muqarnas." wikipedia.org.

[41] “A simple Persian Muqarnas design.” https://i.pinimg.com/originals/f8/a7/5e/f8a75e0275056060e7c874f385cf23a8.jpg.
[42] “Al - Azhar Mosque, Cairo, Egypt.” http://islamicarchitecturebydxx.blogspot.com.eg/2015/02/basic-definitions-muqarnas.html.
[43] “Stone muqarnas.” Garofalo, Vincenza.%0A[5]%09V. Von, “Algorithmic Computer Reconstructions of Stalactite Vaults - Muqarnas - in Islamic Architecture,” Dipl.–Math. Silvia Harmsen aus Ermelo (die Niederlande) Tag, 2006%0A.
[44] Y. Tabbaa, “Muqarnas [Arab. muqarnas; muqarnaṣ; muqarbaṣ; Sp. mocárabes],” Grove Art Online, 2003, [Online]. Available: https://doi.org/10.1093/gao/9781884446054.article.T060413.
[45] “Muqarnas details.” http://www.scienceplus2ch.com.
[46] “Old muqarnas through the stone.” https://www.flickr.com/photos/8349545@N03/galleries/72157622816287546?rb=1.
[47] “Classification of Muqarnas.” https://spatialexperiments.wordpress.com/2016/09/16/useful-geometry/.
[48] F. A. Bukhari, “A Hierarchical Evolutionary Algorithmic Design (HEAD) System for Generating and Evolving Building Design Models,” Queensland University of Technology, 2011.
[49] O. ALRAWI, “Regenerating architecture elements using AI, the case of Muqarnas,” in 3rd Int’l ASCAAD Conference on Em‘body’ing Virtual Architecture [ASCAAD-07, Alexandria, Egypt], 2005, pp. 273–274.
[50] “Geometric Organization of muqarnas.” http://www.monthihan.com/Computational-Muqarnas.
[51] “Parametric Muqarnas.” pp. 1–8, [Online]. Available: https://archinect.com/features/article/100296/student-works-stalactile-tessellated-manifolds.