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Enumerating Spanning Trees of Some Advanced Families of Graphs

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ABSTRACT

In designing communication networks (graphs), number of spanning trees plays a vital and significant role, as the more quality and perfect the network, the greater the number of trees spanning this network, and this leads to greater possibilities for the connection between two vertices, and this ensures good rigidity and resistance. In this work, we derive an obvious formula for the number of spanning trees (complexity) graphs generated by duplicating edge by a vertex of the path, cycle, and wheel graphs. Also, clear expressions of complexity of duplicating a vertex by an edge of path and cycle graphs. The eigenvalues of the Laplacian matrix of a graph are known as the Laplacian spectrum. Furthermore, by using the spectrum of Laplacian matrix, we deduce an evident formula of the complexity of the shadow graph of the path graph, cycle graph and complete graphs. These explicit formulas have been found out by utilizing techniques from linear algebra, matrix theory, and orthogonal polynomials

1. Introduction

A graph is a formal mathematical illustration of a network since any network can be modeled by a graph G where nodes are represented by vertices $V(G)$ and links are represented by edges $E(G)$. Let $|E(G)|$ be the cardinality of $E(G)$ and $|V(G)|$ be that of $V(G)$. We deal with finite and undirected with multiple edges and loops permitted. The degree of a vertex $x \in V(G)$, is the number of edges incident with the vertex, while the average degree of a graph is applied to measure the number of edges compared to the number of vertices which calculates by dividing the summation of all vertex degrees by the total number of nodes. As in the case with most mathematical entities, one always tries to get new structures from given ones, this also applies to the field of graphs, where one can generate new graphs from a given set of graphs [1].

Definition 1.1. Duplication of an edge $e = uv$ in a graph G by a new vertex w produces a new graph G' such that $N(w) = \{u, v\}$. Duplication of a vertex v_i by a new edge $e = v_i' v_i''$ in a graph G produces a new graph G' such that $N(v_i') \cap N(v_i'') = \{v_i\}$. In Fig. 1 we present duplicating every edge by a vertex in the path P_n. In Fig. 2 we present duplicating every vertex by an edge in the path P_n.

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A spanning tree of any graph is a communication subgraph that guarantees connectivity between all vertices in the original graph with a minimum number of edges. In other words, a spanning tree ensures the existence and uniqueness of a connection between any pair of vertices. The number of spanning trees $\tau(G)$ is equal to the total number of various spanning subgraphs of G that are trees, this quantity is also known as complexity $\tau(G)$ of G.

There exist several methods for finding this number. The celebrated Matrix tree theorem of Kirchhoff [2] , tells us that: the complexity $\tau(G)$ of a graph G is equal any cofactor of Laplace matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees of G and $A(G)$ is the adjacency matrix of $G. \tau(G)$ also can be calculated from the eigenvalues of the Laplace matrix L. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n (= 0)$ denote the eigenvalues of L matrix.

Kelmans and Chenlnokov [3], have shown that

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i \qquad (1.1)$$

After that Timperley [4], has shown that:

$$\tau(G) = \frac{1}{n^2} \det(H + I) \qquad (1.2)$$

where I is the $n \times n$ matrix all of whose elements are unity. From Timperley's equation (1.2) it is easy to prove the following lemma.

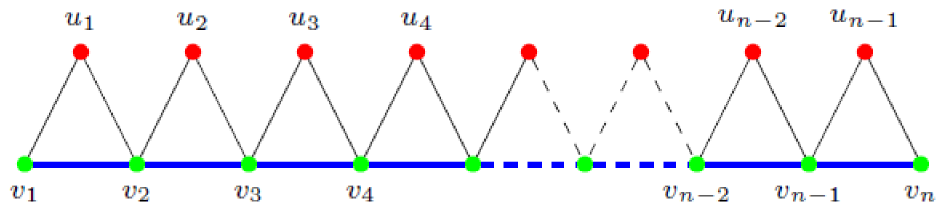

Fig. 1: Duplicating every edge by a vertex in the path P_n.

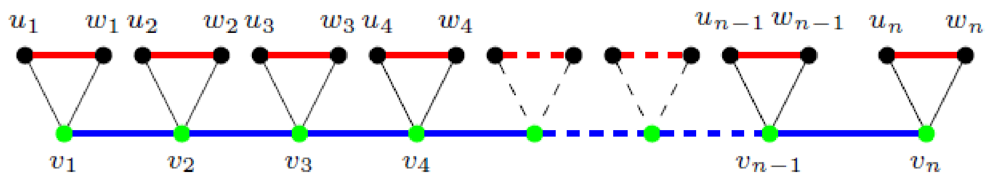

Fig. 2: Duplicating every vertex by an edge in the path P_n.

Lemma 1.2. Let G be a graph with n vertices and $\bar{D}, \bar{A}$ are the degree and adjacency matrices, respectively, of $\bar{G}$, the complement of G. Then,

$$\tau(G) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A}) \qquad (1.3)$$

Proof. Since $H = D - A$, then $D = H + A$ and $\bar{D} = (n-1)I - D$, then $D = (n-1)I - \bar{D}$, then

$$H + A = (n-1)I - \bar{D} \qquad (1.4)$$

Since $A + \bar{A} = J - I$, then

$$J - A = I + \bar{A} \qquad (1.5)$$

From Eq. 1.4 and Eq. 1.5 we have $H + J = nI - \bar{D} + \bar{A}$ substitution in $[1.2)$ then,

$$\tau(G) = \frac{1}{n^2}\det(nI - \bar{D} + \bar{A}.)$$

The senior advantage of formula (1.3) is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors or eigenvalues.

For some special classes of graphs, there exist, simple closed formulas that make it much easier to calculate and determine the number of corresponding spanning trees, especially when these numbers are very large. Cayley showed that a complete graph K_n has $n^{n-2}, n \geq 2$ spanning trees. Another result is due to Sedlacek [5], he derived a formula for the wheel with $n+1$ vertices, W_{n+1}, has a number of spanning trees $\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2, n \geq 2$. Zeen El Deen et al. [6,7] have derived many explicit formulas for the number of spanning trees of some duplicating graphs. Recently, several closed formulas have been published on counting and maximizing the number of spanning trees for some families of graphs (see, e.g., [8-12])

2. Basic proof tools

The Chebyshev Polynomials of the first kind are defined as the solution of the recursion relation

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0;\ T_0(x) = 1,\ T_1(x) = x \qquad (2.1)$$

Using standard methods for solving the recursion (2.1) getting

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right],\ n \geq 1 \qquad (2.2)$$

Also, the Chebyshev Polynomials of the second kind are defined as the solution of the recursion relation

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x);\ U_0(x) = 1,\ U_1(x) = 2x \qquad (2.3)$$

Furthermore, by using standard methods for solving the recursion (2.3) one obtains the explicit formula

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right],\ n \geq 1 \qquad (2.4)$$

where the identity is true for all complex x (except at $x = \pm 1$). we have $U_{n-1}\left(\cos\frac{i\pi}{n}\right) = 0, i = 1,2,\dots n-1.$ Hence

$$U_{n-1}(x) = 2^{n-1}\prod_{i=1}^{n-1}\left(x - \cos\frac{i\pi}{n}\right) \qquad (2.5)$$

One further notes that:

$$U_{n-1}(-x) = 2^{n-1}\prod_{i=1}^{n-1}\left(x + \cos\frac{i\pi}{n}\right) \qquad (2.6)$$

These two results yields

$$U_{n-1}^2(x) = 4^{n-1}\prod_{i-1}^{n-1}\left(x^2 - \cos^2\frac{i\pi}{n}\right) \qquad (2.7)$$

$$U_{n-1}^2\left(\sqrt{\frac{x+2}{4}}\right) = \prod_{i=1}^{n-1}\left(x - 2\cos\frac{2i\pi}{n}\right) \qquad (2.8)$$

One further note that:

$$\prod_{i-1}^{n-1}\left(2 - 2\cos\frac{i\pi}{n}\right) = n, \quad n \geq 2 \qquad (2.9)$$

$$\prod_{i=1}^{n-1}\left(2 - 2\cos\frac{2i\pi}{n}\right) = n^2,\ n \geq 2 \qquad (2.10)$$

Polynomials $T_n(x)$ and $U_{n-1}(x)$ are related by the following identity

$$U_{n-1}^2(x) = \frac{1}{2(x^2-1)}\left[T_n(2x^2-1) - 1\right] \qquad (2.11)$$

By using the substitution $x = \sqrt{\frac{4-\mu}{4}}$ we have:

$$2\left[T_n\left(\frac{2-\mu}{2}\right)-1\right] = -\mu\, U_{n-1}^2\left(\sqrt{\frac{4-\mu}{4}}\right) \qquad (2.12)$$

Chebyshev polynomials of the first and second kind have a strong relation with determinants, we will use. it in our computations **[13]**. Let $A_n(x)$ be $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & 2x & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & 2x & -1 & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 2x & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2x \end{pmatrix}$$

where all other elements are zeros.

From this recursion relation and by expanding det $A_n(x)$ one gets

$$U_n(x) = \det[A_n(x)], \;\; n \geq 1 \qquad (2.13)$$

Lemma 2.1. [14] Suppose P, Q, R, and S are block matrices of dimension $j \times j, j \times k, k \times j$ and $k \times k$, respectively. Then, when P and S are nonsingular,

$$\det\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det(P) \times \det(S - RP^{-1}Q)$$

$$= \det(S) \times \det(P - QS^{-1}R)$$

Lemma 2.2. [14] Suppose P, Q are block matrices Then det $\begin{pmatrix} P & Q \\ Q & P \end{pmatrix} = \det(P-Q) \times \det(P+Q)$

Lemma 2.3. [15, 16]

(i) If

$$B_n(x) = \begin{pmatrix} x & -1 & 0 & 0 & \dots & \dots & 0 \\ -1 & (x+1) & -1 & 0 & \dots & \dots & 0 \\ 0 & -1 & (x+1) & -1 & \dots & \dots & 0 \\ \vdots & \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 & (x+1) & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & x \end{pmatrix},$$

then $\det[B_n(x)] = (x-1)U_{n-1}\left(\frac{1+x}{2}\right)$

(ii) $\forall\, x \geq 3$, if

$$C_n(x) = \begin{pmatrix} x & -1 & 0 & \dots & \dots & -1 \\ -1 & x & -1 & \dots & \dots & 0 \\ 0 & -1 & x & \dots & \dots & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & x & -1 \\ -1 & 0 & 0 & \dots & -1 & x \end{pmatrix}, \quad \text{then} \quad \det[C_n(x)] = 2\left[\, T_n\left(\frac{x}{2}\right) - 1 \,\right]$$

Lemma 2.4. [17] If

$$E_n(x)=\begin{pmatrix} x & 1 & 1 & 1 & \dots & 1\\ 1 & x & 1 & 1 & \dots & 1\\ 1 & 1 & x & 1 & \dots & 1\\ \vdots & \vdots & \dots & \ddots & \ddots & \vdots\\ 1 & 1 & 1 & \dots & x & 1\\ 1 & 1 & 1 & \dots & 1 & x \end{pmatrix}, \quad \text{then} \quad \det\left(E_n(x)\right) = (x+n-1)(x-1)^{n-1}$$

Lemma 2.5. Let A, B and C be matrices of dimension $n \times n$ if B and C commute, then

$$\det\begin{pmatrix} A & B & B\\ B & C & B\\ B & B & C \end{pmatrix} = [\det(B-C)]\det[2B^2 - A\times B - A\times C]$$

Proof. Using the properties of determinants and matrix row and column operations yields:

$$\begin{pmatrix} A & B & B\\ B & C & B\\ B & B & C \end{pmatrix} = \begin{pmatrix} A-B & O & B\\ B-C & C-B & B\\ 0 & B-C & C \end{pmatrix} = \begin{pmatrix} A-B & O & B\\ B-C & O & B+C\\ O & B-C & C \end{pmatrix}$$

$$\det\begin{pmatrix} A & B & B\\ B & C & B\\ B & B & C \end{pmatrix} = -\det(B-C)\times\det\begin{pmatrix} A-B & B\\ B-C & B+C \end{pmatrix}$$

$$= -\det(B-C)\times\det\begin{pmatrix} A-B & A\\ B-C & 2B \end{pmatrix}$$

When B and C commute (i.e., $B\times C = C\times B$), so $(B-C)$ and $2B$ commute, then

$$\begin{pmatrix} A & B & B\\ B & C & B\\ B & B & C \end{pmatrix} = -[\det(B-C)]\det[(A-B)\times 2B - A\times(B-C)]$$

$$= [\det(B-C)]\det[2B^2 - A\times B - A\times C]$$

Lemma 2.6. Let A, B and C be matrices of dimension $n \times n$, then

$$\det\begin{pmatrix} p & Q & R & Q\\ Q & P & Q & R\\ R & Q & P & Q\\ Q & R & Q & P \end{pmatrix} = [\det(P-R)]^2\det(P+R+2Q)\det(P+R-2Q)$$

Proof. Using the properties of determinants and matrix row and column operations [?] yields:

$$\begin{pmatrix} P & Q & R & Q\\ Q & p & Q & R\\ R & Q & P & Q\\ Q & R & Q & p \end{pmatrix} = \begin{pmatrix} p-R & O & R & Q\\ O & P-R & Q & R\\ R-P & O & P & Q\\ O & R-P & Q & p \end{pmatrix} = \begin{pmatrix} p-R & O & R & Q\\ O & P-R & Q & R\\ O & O & p+R & 2Q\\ O & O & 2Q & P+R \end{pmatrix}$$

$$\det\begin{pmatrix} P & Q & R & Q\\ Q & P & Q & R\\ R & Q & P & Q\\ Q & R & Q & P \end{pmatrix} = \det\begin{pmatrix} P-R & O\\ O & P-R \end{pmatrix}\times\det\begin{pmatrix} P+R & 2Q\\ 2Q & P+R \end{pmatrix}$$

$$= [\det(P-R)]^2\det(P+R+2Q)\det(P+R-2Q)$$

3. Complexity of duplication edge by vertex graphs

Theorem 3.1. For $n \geq 2$, the number of the spanning trees of the graph α_n obtained by duplication of every edge of the path P_n by a vertex is given by $\tau(\alpha_n) = 3^{n-1}$

Proof. The graph α_n obtained by duplication of every edge of the path P_n by a vertex has number of vertices, $|V(\alpha_n)|$ and edges $|E(\alpha_n)|$ are $|V(\alpha_n)| = 2n-1$ and $|E(\alpha_n)| = 3(n-1)$, $n = 2,3,\cdots$, see **Fig. 1**.

Applying Lemma (1.2), we have:

$$\tau(\alpha_n) = \frac{1}{(2n-1)^2}[(2n-1)I - \bar{D} + \bar{A}]$$

$$\tau(\alpha_n) = \frac{1}{(2n-1)^2} \det \begin{pmatrix}
3 & 0 & 1 & \cdots & \cdots & \cdots & 1 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\
0 & 5 & 0 & 1 & \cdots & \cdots & 1 & 0 & 0 & 1 & 1 & \cdots & \cdots & 1 \\
1 & 0 & 5 & 0 & \cdots & \cdots & 1 & 1 & 0 & 0 & 1 & \cdots & \cdots & 1 \\
1 & 1 & 0 & 5 & \cdots & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & \cdots & 1 \\
\vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & \cdots & 0 & 5 & 0 & 1 & 1 & \cdots & \cdots & \cdots & 0 & 0 \\
1 & 1 & \cdots & \cdots & 1 & 0 & 3 & 1 & 1 & \cdots & \cdots & \cdots & 1 & 0 \\
0 & 0 & 1 & 1 & \cdots & \cdots & 1 & 3 & 1 & 1 & \cdots & \cdots & \cdots & 1 \\
1 & 0 & 0 & 1 & \cdots & \cdots & 1 & 1 & 3 & 1 & \cdots & \cdots & \cdots & 1 \\
1 & 1 & 0 & 0 & \cdots & \cdots & 1 & 1 & 1 & 3 & \cdots & \cdots & \cdots & 1 \\
1 & 1 & 1 & 0 & \cdots & \cdots & 1 & 1 & 1 & 1 & 3 & \cdots & \cdots & 1 \\
\vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \ddots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & \cdots & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & \cdots & 3 & 1 \\
1 & 1 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & \cdots & 1 & 3
\end{pmatrix}$$

$$= \frac{1}{(2n-1)^2} \det \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

From Lemma (2.1), then

$$\tau(\alpha_\pi) = \frac{1}{(2n-1)^2}\det(C)\det(A - BC^{-1}B^t)$$
$$= \frac{1}{(2n-1)^2}(n+2)(2)^{n-1}\frac{(2n-1)^2(3)^{n-1}}{(n+2)(2)^{n-1}} = 3^{n-1}$$

Theorem 3.2. For $n \geq 3$, the number of the spanning trees of the graph Ω_n obtained by duplication of each edge of the cycle C_n by a vertex is given by:

$$\tau(\Omega_n) = 2n3^{n-1}.$$

Proof. The graph Ω_n obtained by duplication of each edge of the cycle C_n by a vertex has number of vertices $|V(G_n)|$ and edges $|E(Gn)|$ are $|V(G_n)| = 2n$ and $|E(G_n)| = 3n$ $n = 3,4 \ldots$, see **Fig. 3.**

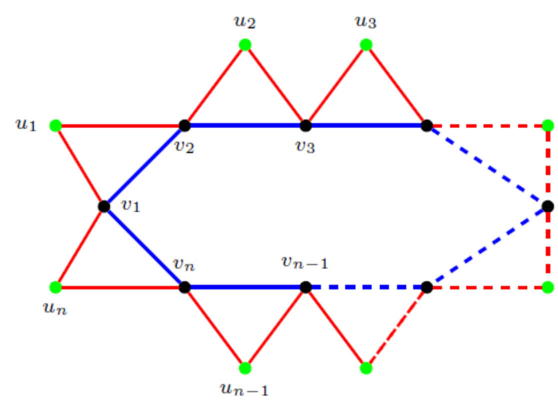

Fig. 3: The graph Ω_n obtained by duplication of each edge of the cycle C_n by a vertex

let us apply Lemma (1.2), we have: $\tau(\Omega_n) = \frac{1}{(2n)^2}[2nI - \bar{D} + \bar{A}]$

$$\tau(\Omega_n) = \frac{1}{(2n)^2} \det \begin{pmatrix}
5 & 0 & 1 & \dots & \dots & 1 & 0 & 0 & 1 & 1 & \dots & \dots & \dots & 1 \\
0 & 5 & 0 & 1 & \dots & \dots & 1 & 0 & 0 & 1 & 1 & \dots & \dots & 1 \\
1 & 0 & 5 & 0 & \dots & \dots & 1 & 1 & 0 & 0 & 1 & \dots & \dots & 1 \\
1 & 1 & 0 & 5 & \dots & \dots & 1 & 1 & 1 & 0 & 0 & \dots & \dots & 1 \\
\vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots \\
1 & 1 & \dots & \dots & 0 & 5 & 0 & 1 & 1 & \dots & \dots & \dots & 0 & 0 \\
0 & 1 & \dots & \dots & 1 & 0 & 5 & 1 & 1 & \dots & \dots & \dots & 1 & 0 \\
0 & 0 & 1 & 1 & \dots & \dots & 1 & 3 & 1 & 1 & \dots & \dots & \dots & 1 \\
1 & 0 & 0 & 1 & \dots & \dots & 1 & 1 & 3 & 1 & \dots & \dots & \dots & 1 \\
1 & 1 & 0 & 0 & \dots & \dots & 1 & 1 & 1 & 3 & \dots & \dots & \dots & 1 \\
1 & 1 & 1 & 0 & \dots & \dots & 1 & 1 & 1 & 1 & 3 & \dots & \dots & 1 \\
\vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots & \dots & \dots & \ddots & \ddots & \vdots & \vdots \\
1 & 1 & \dots & \dots & \dots & 0 & 1 & 1 & 1 & 1 & \dots & \dots & 3 & 1 \\
0 & 1 & \dots & \dots & \dots & 1 & 0 & 1 & 1 & 1 & \dots & \dots & 1 & 3
\end{pmatrix}$$

$$= \frac{1}{(2n)^2}\, det \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \text{ Here we have } B^t \times C = C \times B^t, \text{ then}$$

$$\tau(\Omega_n) = \frac{1}{(2n)^2}\, det(A \times C - B \times B^t)$$

$$\tau(\Omega_n) = \frac{1}{(2n)^2} det \begin{pmatrix}
14 & 5 & 8 & 8 & 8 & \dots & \dots & 8 & 5 \\
5 & 14 & 5 & 8 & 8 & \dots & \dots & 8 & 8 \\
8 & 5 & 14 & 5 & 8 & \dots & \dots & 8 & 8 \\
8 & 8 & 5 & 14 & 5 & \dots & \dots & 8 & 8 \\
\vdots & \vdots & \dots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\
8 & 8 & 8 & 8 & 8 & \dots & 5 & 14 & 5 \\
5 & 8 & 8 & 8 & 8 & \dots & 8 & 5 & 14
\end{pmatrix}$$

$$= \frac{2\ (3)^{n-1}}{n^2}\, det \begin{pmatrix}
3 & 0 & 1 & 1 & 1 & \dots & \dots & 1 & 0 \\
0 & 3 & 0 & 1 & 1 & \dots & \dots & 1 & 1 \\
1 & 0 & 3 & 0 & 1 & \dots & \dots & 1 & 1 \\
1 & 1 & 0 & 3 & 0 & \dots & \dots & 1 & 1 \\
\vdots & \vdots & \dots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & 1 & 1 & \dots & 0 & 3 & 0 \\
0 & 1 & 1 & 1 & 1 & \dots & 1 & 0 & 3
\end{pmatrix}$$

straightforward induction using the properties of determinants, we obtain:

$$\tau(\Omega_n) = \frac{23^{n-1}}{n^2}.n^3 = 2n3^{n-1}$$

Theorem 3.3. For $n \geq 3$, the number of the spanning trees of the graph Γ_n obtained by duplication of each edge of the wheel W_n by a vertex is given by

$$\tau(\Gamma_n) = 3^n\left[(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2^{n+1}\right]$$

Proof. Let W_n be the wheel with vertex set $V(W_n) = \{v_0, v_1, v_2, \cdots, v_n\}$ with the hub vertex v_0. The graph Γ_n obtained by duplication of each edge of the wheel W_n by a vertex has number of vertices $|V(\Gamma_n)| = 3n+1$ and edges $|E(\Gamma n)| = 6n, n = 3,4,\cdots$, see **Fig. 4**

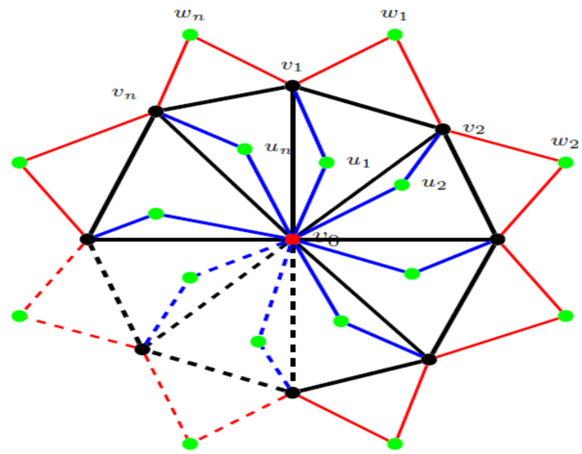

Fig. 4: The graph Γ_n obtained by duplication of each edge by a vertex of the wheel W_n

The Kirchhoff matrix H associated to the graph Γ_n is

$$H = \begin{pmatrix}
2n & -1 & -1 & -1 & \cdots & \cdots & -1 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & -1 & -1 & \cdots & \cdots & -1 & -1 \\
-1 & 2 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\
-1 & 0 & 2 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & 0 & 2 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & \cdots & -1 & 0 \\
-1 & 0 & 0 & \cdots & \cdots & 0 & 2 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & -1 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 2 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 2 & 0 & 0 & \cdots & \cdots & 0 & 0 & -1 & -1 & \cdots & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 2 & 0 & \cdots & \cdots & 0 & 0 & 0 & -1 & -1 & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 2 & 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & -1 \\
0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 2 & -1 & 0 & 0 & \cdots & \cdots & 0 & -1 \\
-1 & -1 & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & \cdots & -1 & 6 & -1 & 0 & 0 & \cdots & \cdots & -1 \\
-1 & 0 & -1 & 0 & \cdots & \cdots & 0 & -1 & -1 & 0 & 0 & \cdots & \cdots & 0 & -1 & 6 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 0 & 0 & -1 & \cdots & \cdots & 0 & 0 & -1 & -1 & 0 & \cdots & \cdots & 0 & 0 & -1 & 6 & -1 & \cdots & \cdots & 0 \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \ddots & \ddots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots & \vdots \\
-1 & 0 & 0 & \cdots & \cdots & -1 & 0 & 0 & 0 & 0 & \cdots & \cdots & -1 & -1 & 0 & 0 & 0 & \cdots & \cdots & 6 & -1 \\
-1 & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & \cdots & -1 & -1 & -1 & 0 & 1 & \cdots & \cdots & -1 & 6
\end{pmatrix}_{3n+1.3n+1}$$

Thus, we get:

$$\tau[\Gamma_n] = \det\begin{pmatrix} 2I & O & -I \\ O & 2I & B \\ -I & B^t & A \end{pmatrix}_{3n.3n} = \det\begin{pmatrix} I & O & O \\ B & 2I & 2B \\ A-I & B^t & 2A-I \end{pmatrix}_{3n.3n}, \text{where}$$

$$A=\begin{pmatrix}6&-1&0&\dots&\dots&0&-1\\-1&6&-1&0&\dots&\dots&0\\0&-1&6&-1&\dots&\dots&0\\\vdots&\dots&\dots&\dots&\ddots&\ddots&\vdots\\0&0&\dots&\dots&-1&6&-1\\-1&0&1&\dots&\dots&-1&6\end{pmatrix}\text{ and }B=\begin{pmatrix}-1&-1&0&\dots&\dots&0&0\\0&-1&-1&0&\dots&\dots&0\\0&0&-1&-1&\dots&\dots&0\\\vdots&\dots&\dots&\ddots&\ddots&\ddots&\vdots\\0&0&\dots&\dots&0&-1&-1\\-1&0&0&\dots&\dots&0&-1\end{pmatrix}$$

$$\text{Since}\quad A\times B^t=B^t\times A\Rightarrow\tau[\Gamma_n]=\det[\,4A\quad-2I-2B^tB]$$

$$\tau[\Gamma_n]=\det\begin{pmatrix}18&-6&0&\dots&\dots&0&-0\\-6&18&-6&0&\dots&\dots&0\\0&-6&18&-6&\dots&\dots&0\\\vdots&\vdots&\dots&\dots&\ddots&\ddots&\vdots\\0&0&\dots&\dots&\dots&18&-6\\-6&0&0&\dots&\dots&-6&18\end{pmatrix}_{n.n}$$

$$=6^n\ \det\begin{pmatrix}3&-1&0&\dots&\dots&0&-1\\-1&3&-1&0&\dots&\dots&0\\0&-1&3&-1&\dots&\dots&0\\\vdots&\vdots&\dots&\dots&\ddots&\ddots&\vdots\\0&0&\dots&\dots&\dots&3&-1\\-1&0&0&\dots&\dots&-1&3\end{pmatrix}$$

$$\tau[\Gamma_n]=6^n2\left[T_n\left(\tfrac{3}{2}\right)-1\right]=3^n[(3+\sqrt{5})^n+(3-\sqrt{5})^n-2^{n+1}].$$

4. Complexity of duplication vertex by edge graphs

Theorem 4.1. For $n\geq 2$, the number of the spanning trees of the graph $D\big(E(P_n)\big)$ obtained by duplication of a vertex of the path P_n by an edge is given by:

$$\tau[D(E(P_n)]=3^n$$

Proof. The graph $D\big(E(P_n)\big)$ obtained by duplication of a vertex of the path P_n by an edge has total number of vertices $|V\big(D\big(E(P_n)\big)|=3n$ and edges $|E\big(D\big(E(P_n)\big)|=4n-1, n=2,3,\dots$ see **Fig. 2**

Let us apply Lemma (1.2) , we have:

$$\tau\big[D\big(E(P_n)\big)\big]=\frac{1}{(3n)^2}\det[3nI-\bar{D}+\bar{A}]=\frac{1}{(3n)^2}\det\begin{pmatrix}A&B&B\\B&C&B\\B&B&C\end{pmatrix}$$

Where

$$A=\begin{pmatrix}4&0&1&\dots&\dots&1&1\\0&5&0&1&\dots&\dots&1\\1&0&5&0&\dots&\dots&1\\\vdots&\dots&\dots&\dots&\ddots&\ddots&\vdots\\1&1&\dots&\dots&0&5&0\\1&1&1&\dots&\dots&0&4\end{pmatrix},\ B=\begin{pmatrix}0&1&1&\dots&\dots&1&1\\1&0&1&1&\dots&\dots&1\\1&1&0&1&\dots&\dots&1\\\vdots&\dots&\dots&\ddots&\ddots&\ddots&\vdots\\1&1&\dots&\dots&1&0&1\\1&1&1&\dots&\dots&1&0\end{pmatrix}\text{ and }C=\begin{pmatrix}3&1&1&\dots&\dots&1&1\\1&3&1&1&\dots&\dots&1\\1&1&3&1&\dots&\dots&1\\\vdots&\dots&\dots&\ddots&\ddots&\ddots&\vdots\\1&1&\dots&\dots&1&3&1\\1&1&1&\dots&\dots&1&3\end{pmatrix}$$

Applying Lemma (2.5)], we have:

$$\tau[D(E(P_n))] = \frac{1}{(3n)^2}[\det(B-C)]\det[2B^2 - A\times B - A\times C]$$

$$\tau[D(E(P_n))] = \frac{1}{(3n)^2}\det\begin{pmatrix} -3 & 0 & 0 & \dots & \dots & \dots & 0 \\ 0 & -3 & 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & -3 & 0 & \dots & \dots & 0 \\ \vdots & & & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & \dots & -3 & 0 \\ 0 & 0 & \dots & \dots & \dots & 0 & -3 \end{pmatrix}$$

$$\times \det\begin{pmatrix} -10 & -8 & -9 & \dots & \dots & \dots & -9 \\ -8 & -11 & -8 & -9 & \dots & \dots & -9 \\ -9 & -8 & -11 & -8 & \dots & \dots & -9 \\ \vdots & \dots & \dots & \dots & \ddots & \ddots & \vdots \\ -9 & -9 & \dots & \dots & -8 & -11 & -8 \\ -9 & -9 & \dots & \dots & \dots & -8 & -10 \end{pmatrix}$$

Using the properties of determinants, we obtain

$$\tau[D(E(P_n))] = \frac{1}{(3n)^2}\times(-3)^n\times(-1)^n 9\det\begin{pmatrix} 2 & 0 & 1 & 1 & \dots & \dots & 1 \\ 0 & 3 & 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & 3 & 0 & \dots & \dots & 1 \\ \vdots & \vdots & \dots & \dots & \ddots & \dots & \vdots \\ 1 & 1 & \dots & \dots & \dots & 3 & 0 \\ 1 & 1 & \dots & \dots & 1 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{(3n)^2}9n^2 3^n = 3^n$$

Theorem 4.2. For $n \geq 3$, the number of the spanning trees of the graph $D(E(C_n))$ obtained by duplication of a vertex of the cycle C_n by an edge is given by $\tau[D(E(C_n))] = n3^n$

Proof. The graph $D(E(C_n))$ obtained by duplication of a vertex of the cycle C_n by an edge has total number of vertices $|V(D(E(C_n))| = 3n$ and edges $|E(D(E(C_n))| = 4n$, $n = 3,4,\cdots$, see **Fig. 5**.

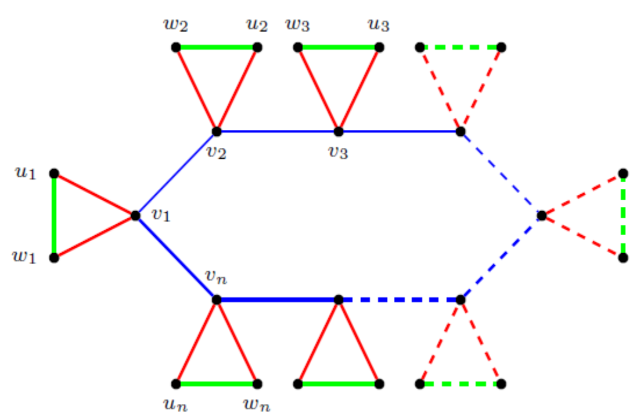

Fig. 5: A graph $D(E(C_n))$ obtained by duplication of a vertex of C_n by an edge.

let us apply Lemma (1.2), we have:

$$\tau[D(E(C_n))] = \frac{1}{(3n)^2}\det[3nI - \bar{D} + \bar{A}] = \frac{1}{(3n)^2}\det\begin{pmatrix} A & B & B \\ B & C & B \\ B & B & C \end{pmatrix}$$

where

$$A=\begin{pmatrix}5&0&1&\cdots&\cdots&1&0\\0&5&0&1&\cdots&\cdots&1\\1&0&5&0&\cdots&\cdots&1\\\vdots&\cdots&\cdots&\cdots&\ddots&\ddots&\vdots\\1&1&\cdots&\cdots&0&5&0\\0&1&1&\cdots&\cdots&0&5\end{pmatrix},B=\begin{pmatrix}0&1&1&\cdots&\cdots&1&1\\1&0&1&1&\cdots&\cdots&1\\1&1&0&1&\cdots&\cdots&1\\\vdots&\cdots&\cdots&\ddots&\ddots&\ddots&\vdots\\1&1&\cdots&\cdots&1&0&1\\1&1&1&\cdots&\cdots&1&0\end{pmatrix}\text{ and }C=\begin{pmatrix}3&1&1&\cdots&\cdots&1&1\\1&3&1&1&\cdots&\cdots&1\\1&1&3&1&\cdots&\cdots&1\\\vdots&\cdots&\cdots&\ddots&\ddots&\ddots&\vdots\\1&1&\cdots&\cdots&1&3&1\\1&1&1&\cdots&\cdots&1&3\end{pmatrix}$$

Applying Lemma (2.5) we have:

$$\tau[D(E(C_n))]=\frac{1}{(3n)^2}[\det(B-C)]\det[2B^2-A\times B-A\times C]$$

$$\tau[D(E(C_n))]=\frac{1}{(3n)^2}\times\det\begin{pmatrix}-3&0&1&1&\cdots&\cdots&1\\0&-3&0&1&\cdots&\cdots&1\\1&0&-3&0&\cdots&\cdots&1\\\vdots&\vdots&\cdots&\cdots&\cdots&\cdots&\vdots\\\vdots&\vdots&\cdots&\cdots&\ddots&\cdots&\vdots\\1&1&\cdots&\cdots&\cdots&-3&0\\1&1&\cdots&\cdots&1&0&-3\end{pmatrix}$$

$$\times\det\begin{pmatrix}-11&-8&-9&1&\cdots&\cdots&-8\\-8&-11&-8&-9&\cdots&\cdots&-9\\-9&-8&-11&-8&\cdots&\cdots&-9\\\vdots&\vdots&\cdots&\cdots&\cdots&\cdots&\vdots\\\vdots&\vdots&\cdots&\cdots&\ddots&\cdots&\vdots\\-9&-9&\cdots&\cdots&\cdots&-11&-8\\-8&-9&\cdots&\cdots&-9&-8&-11\end{pmatrix}$$

$$\tau[D(E(C_n))]=\frac{1}{(3n)^2}\times(-3)^n\times(-1)^n9\det\begin{pmatrix}3&0&1&1&\cdots&\cdots&1\\0&3&0&1&\cdots&\cdots&1\\1&0&3&0&\cdots&\cdots&1\\\vdots&\vdots&\cdots&\cdots&\cdots&\cdots&\vdots\\\vdots&\vdots&\cdots&\cdots&\ddots&\cdots&\vdots\\1&1&\cdots&\cdots&\cdots&3&0\\1&1&\cdots&\cdots&1&0&3\end{pmatrix}$$

$$=\frac{1}{(3n)^2}9n^3 3^n=n\ 3^n$$

5. Enumerating spanning trees of the shadow graph using Laplacian spectrum

The spectrum of a finite graph G is the set of eigenvalues of its adjacent matrix A together with their multiplicities. The eigenvalues of the Laplacian matrix of a graph are known as the Laplacian spectrum. From the Laplace spectrum of a graph one can determine the number of spanning trees (which will be nonzero only if the graph is connected). While the involved calculations are complicated, and this method is not valid for calculating the number of spanning trees for larger graphs.

Definition 5.1 The shadow graph $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G_1 and G_2. Join each vertex v_i in G_1 to the neighbors of the corresponding vertex u_i in G_2. The shadow graph $D_2(G)$

can be obtained from the splitting graph $S'(G)$ by adding edge between any two new vertices u_i and u_j if the corresponding original vertices v_i and v_j are adjacent.

5.1 Complexity of the Shadow of the path

Theorem 5.1. For $n \geq 2$, the number of the spanning trees of the shadow graph $D_2(P_n)$ of the path P_n is given by: $\tau[D_2(P_n)] = 2^{3n-4}$

Proof. The shadow graph $D_2(P_n)$ of the path P_n is obtained by taking two copies of P_n, say P_n^v, P_n'' and joining each vertex u' of P_n' to the neighbors of the corresponding vertex u'' of P_n''. According to the construction, the number of total vertices $|V(D_2(P_n)|= 2n, n = 2,3,...,$ see **Fig. 6**

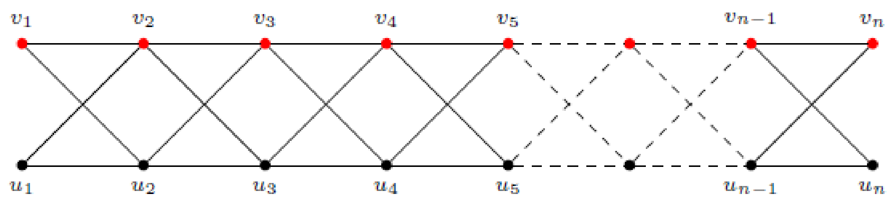

Fig. 6: The shadow graph $D_2(P_n)$.

Let $L[D_2(P_n)] = D[D_2(P_n)] - A[D_2(P_n)]$ be the Laplacian matrix of the shadow of the path P_n, then

$$\det [L[D_2(P_n)] - \lambda I] = \det \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det (A - B) \times \det (A + B)$$

where

$$A - \begin{pmatrix} 2-\lambda & -1 & 0 & \dots & 0 & 0 \\ -1 & 4-\lambda & -1 & \dots & 0 & 0 \\ 0 & -1 & 4-\lambda & \dots & 0 & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 4-\lambda & -1 \\ 0 & 0 & \dots & \dots & -1 & 2-\lambda \end{pmatrix} \text{ and } \boldsymbol{B} = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 & 0 \\ -1 & 0 & -1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

$$\det[L[D_2(P_n)] - \lambda I] = \det \begin{pmatrix} 2-\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 4-\lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 4-\lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \dots & \cdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4-\lambda & 0 \\ 0 & 0 & 0 & \cdots & 0 & 2-\lambda \end{pmatrix} \times$$

$$det \begin{pmatrix} 2-\lambda & -2 & 0 & \dots & 0 & 0 \\ -2 & 4-\lambda & -2 & \dots & 0 & 0 \\ 0 & -2 & 4-\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 4-\lambda & -2 \\ 0 & 0 & 0 & \dots & -2 & 2-\lambda \end{pmatrix}$$

straightforward induction using the properties of determinants and Lemma. 2.3. we obtain:

$$\det [L[D_2(P_n)] - \lambda I] = (2-\lambda)^2 (4-\lambda)^{n-2} \times 2^{n-1} (-\lambda) U_{n-1} \left(\frac{2-\frac{\lambda}{2}}{2} \right)$$

Using Eq. 2.5 then

$$\det\left[L[D_2(P_n)]-\lambda I\right]=(2-\lambda)^2(4-\lambda)^{n-2}\times 2^{2n-2}(-\lambda)\prod_{i=1}^{n-1}\left(\frac{4-\lambda}{4}-\cos\frac{i\pi}{n}\right)$$

When $\det\left[L[D_2(P_n)]-\lambda I\right]=0$, then the eigenvalues λ_i are:

$$0,\ 2^{(2)},\ 4^{(n-2)},\ 2^{n-1}\left(2-2\cos\frac{i\pi}{n}\right)\quad,\quad i=1,2,\cdots,n-1$$

Using Eq. (2.9) and From Eq. (1.1), then $\tau[D_2(P_n)]=\frac{1}{2n}\prod_{i=1}^{2n-1}\lambda_i=2^{3n-4}$.

5.2 Complexity of the Shadow of the cycle

Theorem 5.2. For $n\geq 3$, the number of the spanning trees of the shadow graph $D_2(C_n)$ of cycle C_n given by:

$$\tau[D_2(C_n)]=n2^{3n-2}$$

Poof. The shadow graph $D_2(C_n)$ of the cycle C_n is obtained by taking two copies of C_n, say C_n^r, C_n^n and joining each vertex u' of C_n' to the neighbors of the corresponding vertex u'' of C_n''. According to the construction, the number of total vertices| $V(D_2(C_n)]=2n, n=3,4,\dots\dots$ see **Fig. 7**. Let $L[D_2(C_n)]=D[D_2(C_n)]-A[D_2(C_n)]$ be the Laplacian matrix of the shadow of the cycle C_n, then

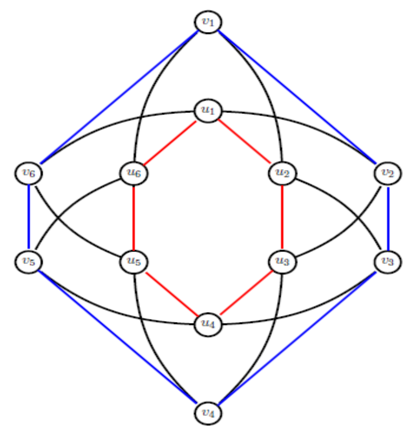

Fig. 7: The shadow graph $D_2(C_6)$

$$\det\left[L[D_2(C_n)]-\lambda I\right]=\det\begin{pmatrix}A & B\\ B & A\end{pmatrix}=\det(A-B)\times\det(A+B)$$

where

$$A-\begin{pmatrix}4-\lambda & -1 & 0 & \dots & 0 & -1\\ -1 & 4-\lambda & -1 & \dots & 0 & 0\\ 0 & -1 & 4-\lambda & \dots & 0 & 0\\ \vdots & \dots & \dots & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & \dots & 4-\lambda & -1\\ -1 & 0 & \dots & \dots & -1 & 4-\lambda\end{pmatrix}\text{ and } B=\begin{pmatrix}0 & -1 & 0 & \dots & 0 & -1\\ -1 & 0 & -1 & \dots & 0 & 0\\ 0 & -1 & 0 & \dots & 0 & 0\\ \vdots & \dots & \dots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 0 & -1\\ -1 & 0 & 0 & \dots & -1 & 0\end{pmatrix}$$

$$\det[L[D_2(C_n)]-\lambda I]=\det\begin{pmatrix}4-\lambda & 0 & 0 & \dots & 0 & 0\\ 0 & 4-\lambda & 0 & \dots & 0 & 0\\ 0 & 0 & 4-\lambda & \dots & 0 & 0\\ \vdots & \vdots & \dots & \dots & \vdots & \vdots\\ \vdots & \vdots & \dots & \dots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 4-\lambda & 0\\ 0 & 0 & 0 & \dots & 0 & 4-\lambda\end{pmatrix}$$

$$\times \det \begin{pmatrix} 4-\lambda & -2 & 0 & \dots & 0 & -2 \\ -2 & 4-\lambda & -2 & \dots & 0 & 0 \\ 0 & -2 & 4-\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4-\lambda & -2 \\ -2 & 0 & 0 & \dots & -2 & 4-\lambda \end{pmatrix}$$

straightforward induction using the properties of determinants and Lemma. 2.3. we obtain:

$$\det\,[L[D_2(C_n)] - \lambda I] = (4-\lambda)^n \times 2^n 2\left[T_n\left(\frac{2-\frac{\lambda}{2}}{2}\right) - 1\right]$$

Using Eq. (2.12) and Eq. (2.8) then

$$\det[L[D_2(C_n)] - \lambda I] = (4-\lambda)^n \times 2^{n-1}(-\lambda)\prod_{i=1}^{n-1}\left(\frac{4-\lambda}{2} - 2\cos\frac{2i\pi}{n}\right)$$

When $\det[L[D_2(C_n)] - \lambda I] = 0$, then the eigenvalues λ_i are :

$$0, 4^{(n)}, 2^{n-1}\left(2 - 2\cos\frac{2i\pi}{n}\right),\ i = 1,2,\cdots,n-1$$

Using Eq. (2.10) and From Eq. (1.1). Then $\tau[D_2(C_n)] = \frac{1}{2n}\prod_{i=1}^{2n-1}\lambda_i = n2^{3n-2}$

5.3 Complexity of the Shadow of the complete graph

Theorem 5.3. For $n \geq 3$, the number of the spanning trees of the shadow graph $D_2(K_n)$ of the complete graph K_n is given by: $\tau[D_2(K_n)] = 2^{2n-2}(n-1)^n(n)^{n-2}$

Proof. The shadow graph $D_2(K_n)$ of the complete graph K_n is obtained by taking two copies of K_n, say K_n', K_n'' and joining each vertex u' of K_n' to the neighbors of the corresponding vertex u'' of K_n''. According to the construction, the number of total vertices $|V(D_2(K_n)|= 2n, n = 3,4,\dots$, see Fig. 8 .

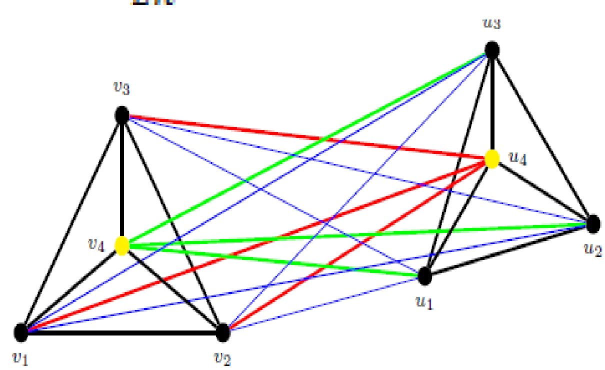

Fig. 8: The shadow graph $D_2(K_4)$

Let $L[D_2(K_n)] = D[D_2(K_n)] - A[D_2(K_n)]$ be the Laplacian matrix of the shadow of the complete graph K_n, then

$$\det\,[L[D_2(K_n)] - \lambda I] = \det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \det\,(A-B) \times \det\,(A+B)$$

$A =$

$$A = \begin{pmatrix} 2n-2-\lambda & -1 & -1 & \dots & -1 & -1 \\ -1 & 2n-2-\lambda & -1 & \dots & -1 & -1 \\ -1 & -1 & 2n-2-\lambda & \dots & -1 & -1 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & 2n-2-\lambda & -1 \\ -1 & -1 & -1 & \dots & -1 & 2n-2-\lambda \end{pmatrix}$$

and $$B = \begin{pmatrix} 0 & -1 & -1 & \dots & -1 & -1 \\ -1 & 0 & -1 & \dots & -1 & -1 \\ -1 & -1 & 0 & \dots & -1 & -1 \\ \vdots & \dots & \dots & \ddots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 0 & -1 \\ -1 & -1 & -1 & \dots & -1 & 0 \end{pmatrix}$$

$$\det[L[D_2(K_n)] - \lambda I] = \det \begin{pmatrix} 2n-2-\lambda & 0 & 0 & \dots & 0 & & 0 \\ 0 & 2n-2-\lambda & 0 & \dots & 0 & & 0 \\ 0 & 0 & 2n-2 & -\lambda & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \dots & \vdots & & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 2n-2-\lambda & & 0 \\ 0 & 0 & 0 & \dots & 0 & & 2n-2-\lambda \end{pmatrix}$$

$$\times \det \begin{pmatrix} 2n-2-\lambda & -2 & -2 & \dots & -2 & -2 \\ -2 & 2n-2-\lambda & -2 & \dots & -2 & -2 \\ -2 & -2 & 2n-2-\lambda & \dots & -2 & -2 \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ \vdots & \vdots & \dots & \dots & \vdots & \vdots \\ -2 & -2 & -2 & \dots & 2n-2-\lambda & -2 \\ -2 & -2 & -2 & \dots & -2 & 2n-2-\lambda \end{pmatrix}$$

straightforward induction using the properties of determinants and Lemma. 2.4. we obtain:

$$\det\,[L[D_2(K_n)] - \lambda I] = (2n-2-\lambda)^n \times (-2)^n \left(\frac{\lambda}{2}\right)\left(\frac{\lambda}{2} - n\right)^{n-1}$$

When $\det[L[D_2(K_n)] - \lambda I] = 0$, then the eigenvalues λ_i are : $0, [2(n-1)]^n, [2(n)]^{n-1}$
From Eq. (1.1). Then

$$\tau[D_2(K_n)] = \frac{1}{2n}\prod_{i=1}^{2n-1}\lambda_i = 2^{2n-2}(n-1)^n(n)^{n-2}$$

Conclusions

Enumerating the number of spanning trees (complexity) in networks is a significant invariant, not only helpful from a combinatorial standpoint, but it is also an important measurement of the reliability of a network and electrical circuit design. In this paper, we found clear formulas for the number of spanning trees for some duplicating edges by vertex and vertex by edge graphs. Also, we found the Laplacian spectrum for the shadow of the path, cycle and complete graphs, then we used it in calculating the complexity of these graphs.

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