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Numerical Solution of Gardner Equation Via Composite Finite Difference Scheme

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Abstract:

In this paper, a new numerical solution of Gardner equation via composite finite difference scheme is introduced. Numerical experiments compare the approximate solution and exact solution.

Keywords: *Evolution equations; Gardner equation; numerical solution of partial differential equations; composite finite difference scheme.*

1 Introduction

Partial differential equations play a very important role in various scientific and engineering fields. Such as fluid mechanics, plasma physics, optical fibers, solid state physics and geochemistry. In recent years a variety of powerful and efficient methods have been proposed. For example, Sine-Cosine method [1], Exp-function method [2], the Darboux transformation [3], the Lie group analysis [4], the modified homogeneous balance method [5], and the extended tanh method [6]. New methods for determining solutions of partial differential equations can be given in [7–15].

The Gardner equation belongs to the category of integrable non-linear partial differential equations. The introduction of this equation is attributed to the famous mathematician Clifford Gardner in 1968 [16]. It is an important model to understand the propagation of negative ion acoustic plasma waves [17]. We can derive Gardner equation from the system of plasma motion equations in one dimension with arbitrarily charged cold. The Gardner equation is a good model for describing internal waves with large amplitudes [18]. In [19] The tanh method is applied for generating interacting solutions for this equation. Some interacting of two wave solutions were presented in [20]. These solutions have various terms including trigonometric or hyperbolic functions in rational forms. G'/G gave Some solitary wave, periodic, exponential, rational and complex-type traveling wave solutions [21]. Numerical solutions of Gardner equations are presented in various papers. The conservative finite difference schemes are developed to determine propagation of one soliton and collusion of two soliton solutions numerically [22]. Restrictive Taylor's technique has been implemented to simulate the propagation of some solutions numerically [23].

Composite finite difference scheme (CFDS) has been introduced and applied to some nonlinear equations, such as Burger equation, KdV equation, KdVB equation and reaction diffusion equations [24-25]. In this article, a new numerical solution of Gardner equation via composite finite different scheme is obtained.

2 Composite Finite Difference Scheme

Consider the general Gardner equation [26] of the form

$$\mathcal{U}_t + (\alpha\mathcal{U} + \beta\mathcal{U}^2)\mathcal{U}_x + \gamma\mathcal{U}_{xxx} = 0, \qquad (x,t) \in Z_T \tag{1}$$

Where $Z_T = \sigma \times \zeta$, $\sigma = (a,b)$, $\zeta = (0,\mathrm{T})$, a and b are real positive constants, α, β and γ are parameters. We consider equation (1) associated with initial condition $\mathcal{U}(x,0) = \mathcal{U}_0(x)$. In finite difference method [24] the domain is discretized forming a grid of a finite number of intersected points with horizontal step $h = \frac{b-a}{N}$, where N is the number of horizontal intervals, $0 < \iota < N$ and κ is the time step such that $\mathrm{T} = \kappa j$, $0 < j < M$.

Equation (1) known as the mixed KdV- mKdV equation which have very widely applications in physics, plasma, Quantum Field theory. We can rewrite it in the form

$$\mathcal{U}_t = -(\alpha\mathcal{U} + \beta\mathcal{U}^2)\mathcal{U}_x - \gamma\mathcal{U}_{xxx} \tag{2}$$

Multiply both sides of (2) by $\frac{dF}{d\mathcal{U}}$, where $F(\mathcal{U})$ is a continuous and differentiable function, we obtain

$$\frac{dF}{d\mathcal{U}}\frac{\partial\mathcal{U}}{\partial t} = -F'(\mathcal{U})\big((\alpha\mathcal{U} + \beta\mathcal{U}^2)\mathcal{U}_x + \gamma\mathcal{U}_{xxx}\big) \tag{3}$$

In Equation (3) reset $\mathcal{U}\mathcal{U}_x = \frac{1}{2}\mathcal{U}^2{}_x$,we have

$$\frac{\partial F}{\partial t} = -F'(\mathcal{U})\left((\alpha + \beta\mathcal{U})\frac{\mathcal{U}_x{}^2}{2} + \gamma\mathcal{U}_{xxx}\right). \tag{4}$$

FDM based on replacing derivatives by difference formulas [30, 31] as follows, for $1 \le \iota \le 2$ we apply the forward difference formulas,

$$\begin{aligned}
(\mathcal{U}_x)_\iota^j &= \frac{-3\mathcal{U}_\iota^j + 4\mathcal{U}_{\iota+1}^j - \mathcal{U}_{\iota+2}^j}{2h},\\
(\mathcal{U}_{xxx})_\iota^j &= \frac{-5\mathcal{U}_\iota^j + 18\mathcal{U}_{\iota+1}^j - 24\mathcal{U}_{\iota+2}^j + 14\mathcal{U}_{\iota+3}^j - 3\mathcal{U}_{\iota+4}^j}{2h^3},\\
(\mathcal{U}_t)_\iota^j &= \frac{\mathcal{U}_\iota^{j+1} - \mathcal{U}_\iota^j}{\kappa}, \quad (F_t)_\iota^j = \frac{F(\mathcal{U}_\iota^{j+1}) - F(\mathcal{U}_\iota^j)}{\kappa}.
\end{aligned} \tag{5}$$

Substitute from (5) into (4) we get,

$$F(\mathcal{U}_i^{j+1}) = F(\mathcal{U}_\imath^j) - \kappa\, F'(\mathcal{U}_\imath^j)\left(\left(\alpha+\beta\mathcal{U}_\imath^j\right)\ \frac{-3(\mathcal{U}_\imath^j)^2+4(\mathcal{U}_{\imath+1}^j)^2-(\mathcal{U}_{\imath+2}^j)^2}{4h} + \frac{-5\mathcal{U}_\imath^j+18\mathcal{U}_{\imath+1}^j-24\mathcal{U}_{\imath+2}^j+14\mathcal{U}_{\imath+3}^j-3\mathcal{U}_{\imath+4}^j}{2h^3}\right). \tag{6}$$

When $3 \le \imath \le \mathrm{N}-2$ we apply the central difference formulas,

$$(\mathcal{U}_x)_\imath^j = \frac{\mathcal{U}_{\imath+1}^j-\mathcal{U}_{\imath-1}^j}{2h},$$
$$(\mathcal{U}_{xxx})_\imath^j = \frac{\mathcal{U}_{\imath+2}^j-2\mathcal{U}_{\imath+1}^j+2\mathcal{U}_{\imath-1}^j-\mathcal{U}_{\imath-2}^j}{2h^3} \tag{7}$$

From (7) substitute in (4), we obtain

$$F(\mathcal{U}_i^{j+1}) = F(\mathcal{U}_\imath^j) - \kappa\, F'(\mathcal{U}_\imath^j)\left(\left(\alpha+\beta\mathcal{U}_\imath^j\right)\frac{(\mathcal{U}_{\imath+1}^j)^2-(\mathcal{U}_\imath^j)^2}{4h} + \frac{\mathcal{U}_{\imath+2}^j-2\mathcal{U}_{\imath+1}^j+2\mathcal{U}_{\imath-1}^j-\mathcal{U}_{\imath-2}^j}{2h^3}\right). \tag{8}$$

When $N-1 \le \imath \le N$ we apply the backward difference formulas,

$$(\mathcal{U}_x)_\imath^j = \frac{3\mathcal{U}_\imath^j-4\mathcal{U}_{\imath-1}^j+\mathcal{U}_{\imath-2}^j}{2h},$$
$$(u_{xxx})_\imath^j = \frac{5\mathcal{U}_\imath^j-18\mathcal{U}_{\imath-1}^j+24\mathcal{U}_{\imath-2}^j-14\mathcal{U}_{\imath-3}^j+3\mathcal{U}_{\imath-4}^j}{2h^3}, \tag{9}$$

Substitute from (9) into (4) we get

$$F(\mathcal{U}_i^{j+1}) = F(\mathcal{U}_\imath^j) - \kappa\, F'(\mathcal{U}_\imath^j)\left(\left(\alpha+\beta\mathcal{U}_\imath^j\right)\ \frac{3(\mathcal{U}_\imath^j)^2-4(\mathcal{U}_{\imath-1}^j)^2+(\mathcal{U}_{\imath-2}^j)^2}{4h} + \frac{5\mathcal{U}_\imath^j-18\mathcal{U}_{\imath-1}^j+24\mathcal{U}_{\imath-2}^j-14\mathcal{U}_{\imath-3}^j+3\mathcal{U}_{\imath-4}^j}{2h^3}\right). \tag{10}$$

2.1 Exponential Finite Difference Method

In this sub section we apply the exponential finite difference method (Exp FDM) which was developed by Bhattachary [27, 28]. He used Exp FDM for solving the one-dimensional heat conduction in a solid slab. In [29] aA. R. Bahadir applied Exp FDM to find the numerical solution of Korteweg de Vries equation. In [24] M. S. EL-AZAB et al studied the stability of CFDS when applied to linearized version of KdVB equation.

In Exp FDM, we set $F(\mathcal{U}) = \mathrm{Ln}\,\mathcal{U}$, then $F'(\mathcal{U}) = \frac{1}{\mathcal{U}}$ substitute in equations (6), (8) and (10) we obtain, for $1 \le \imath \le 2$

$$\mathrm{Ln}\,(\mathcal{U}_i^{j+1}) = \mathrm{Ln}\,(\mathcal{U}_\imath^j) - \frac{\kappa}{\mathcal{U}_\imath^j}\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{-3(\mathcal{U}_\imath^j)^2 + 4(\mathcal{U}_{\imath+1}^j)^2 - (\mathcal{U}_{\imath+2}^j)^2}{4h} + \frac{-5\mathcal{U}_\imath^j + 18\mathcal{U}_{\imath+1}^j - 24\mathcal{U}_{\imath+2}^j + 14\mathcal{U}_{\imath+3}^j - 3\mathcal{U}_{\imath+4}^j}{2h^3}\right), \qquad (11)$$

When $3 \le \imath \le N - 2$, equation (8) becomes,

$$\mathrm{Ln}\,(\mathcal{U}_i^{j+1}) = \mathrm{Ln}\,(\mathcal{U}_\imath^j) - \frac{\kappa}{\mathcal{U}_\imath^j}\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{(\mathcal{U}_{\imath+1}^j)^2 - (\mathcal{U}_\imath^j)^2}{4h} + \frac{\mathcal{U}_{\imath+2}^j - 2\mathcal{U}_{\imath+1}^j + 2\mathcal{U}_{\imath-1}^j - \mathcal{U}_{\imath-2}^j}{2h^3}\right), \qquad (12)$$

$$\left(\mathrm{Ln}\,(\mathcal{U}_i^{j+1}\right) = \mathrm{Ln}\,(\mathcal{U}_\imath^j) - \frac{\kappa}{\mathcal{U}_\imath^j}\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{3(\mathcal{U}_\imath^j)^2 - 4(\mathcal{U}_{\imath-1}^j)^2 + (\mathcal{U}_{\imath-2}^j)^2}{4h} + \frac{5\mathcal{U}_\imath^j - 18\mathcal{U}_{\imath-1}^j + 24\mathcal{U}_{\imath-2}^j - 14\mathcal{U}_{\imath-3}^j + 3\mathcal{U}_{\imath-4}^j}{2h^3}\right),$$

$$N - 1 \le \imath \le N \qquad (13)$$

Simplify Equations (11-13) we obtain the following equations,

$$\mathcal{U}_\mathrm{i}^{\mathrm{j}+1} = \mathcal{U}_\imath^\mathrm{j} \exp\left(\frac{-\kappa}{\mathcal{U}_\imath^\mathrm{j}}\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{-3(\mathcal{U}_\imath^\mathrm{j})^2 + 4(\mathcal{U}_{\imath+1}^\mathrm{j})^2 - (\mathcal{U}_{\imath+2}^\mathrm{j})^2}{4\mathrm{h}} + \frac{-5\mathcal{U}_\imath^\mathrm{j} + 18\mathcal{U}_{\imath+1}^\mathrm{j} - 24\mathcal{U}_{\imath+2}^\mathrm{j} + 14\mathcal{U}_{\imath+3}^\mathrm{j} - 3\mathcal{U}_{\imath+4}^\mathrm{j}}{2\mathrm{h}^3}\right)\right), 1 \le \imath \le 2, \qquad (14)$$

$$\mathcal{U}_{\mathrm{i}}^{\mathrm{j+1}} = \mathcal{U}_{\imath}^{\mathrm{j}} \exp\left(\frac{-\kappa}{\mathcal{U}_{\imath}^{\mathrm{j}}} \left(\left(\alpha + \beta \mathcal{U}_{\imath}^{j}\right) \frac{(\mathcal{U}_{\imath+1}^{\mathrm{j}})^2 - (\mathcal{U}_{\imath}^{\mathrm{j}})^2}{2\mathrm{h}} + \frac{\mathcal{U}_{\imath+2}^{\mathrm{j}} - 2\mathcal{U}_{\imath+1}^{\mathrm{j}} + 2\mathcal{U}_{\imath-1}^{\mathrm{j}} - \mathcal{U}_{\imath-2}^{\mathrm{j}}}{2\mathrm{h}^3} \right)\right), \qquad (15)$$

$$\mathcal{U}_{\mathrm{i}}^{\mathrm{j+1}} = \mathcal{U}_{\imath}^{\mathrm{j}} \exp\left(-\frac{\kappa}{\mathcal{U}_{\imath}^{\mathrm{j}}} \left(\left(\alpha + \beta \mathcal{U}_{\imath}^{j}\right) \frac{3(\mathcal{U}_{\imath}^{\mathrm{j}})^2 - 4(\mathcal{U}_{\imath-1}^{\mathrm{j}})^2 + (\mathcal{U}_{\imath-2}^{\mathrm{j}})^2}{4\mathrm{h}} + \frac{5\mathcal{U}_{\imath}^{\mathrm{j}} - 18\mathcal{U}_{\imath-1}^{\mathrm{j}} + 24\mathcal{U}_{\imath-2}^{\mathrm{j}} - 14\mathcal{U}_{\imath-3}^{\mathrm{j}} + 3\mathcal{U}_{\imath-4}^{\mathrm{j}}}{2\mathrm{h}^3} \right)\right),$$

$$\mathrm{N} - 1 \leq \imath \leq \mathrm{N}. \qquad (16)$$

Logarithmic finite difference method was introduced by M. S. El-Azab and S. A. El Morsy for solving the nonlinear evolution equations. They applied the method to a class of solitary waves equations, such as KdV and KdVB equations [30]. In [25] M. S. El-Azab et al applied Logarithmic finite difference method to nonlinear reaction diffusion equation. The method is simple and effective for arbitrarily large values of parameters.

Consider $F(\mathcal{U}) = \exp(\mathcal{U})$, then $\mathrm{F}'(\mathcal{U}) = \exp(\mathcal{U})$, substitute in equations (6), (8) and (10) to obtain,

we obtain,

$$\exp(\mathcal{U}_i^{j+1}) = \exp\left(\mathcal{U}_\imath^j\right) - \kappa \exp\left(\mathcal{U}_\imath^j\right) \left(\left(\alpha + \beta \mathcal{U}_\imath^j\right) \frac{-3(\mathcal{U}_\imath^j)^2 + 4(\mathcal{U}_{\imath+1}^j)^2 - (\mathcal{U}_{\imath+2}^j)^2}{4h} + \frac{-5\mathcal{U}_\imath^j + 18\mathcal{U}_{\imath+1}^j - 24\mathcal{U}_{\imath+2}^j + 14\mathcal{U}_{\imath+3}^j - 3\mathcal{U}_{\imath+4}^j}{2h^3} \right), 1 \leq \imath \leq 2 \qquad (17)$$

When $3 \leq \imath \leq \mathrm{N} - 2$, equation (8) becomes,

$$\exp(\mathcal{U}_i^{j+1}) = \exp\left(\mathcal{U}_\imath^j\right) - \kappa \exp\left(\mathcal{U}_\imath^j\right) \left(\left(\alpha + \beta \mathcal{U}_\imath^j\right) \frac{-3(\mathcal{U}_\imath^j)^2 + 4(\mathcal{U}_{\imath+1}^j)^2 - (\mathcal{U}_{\imath+2}^j)^2}{4h} + \frac{-5\mathcal{U}_\imath^j + 18\mathcal{U}_{\imath+1}^j - 24\mathcal{U}_{\imath+2}^j + 14\mathcal{U}_{\imath+3}^j - 3\mathcal{U}_{\imath+4}^j}{2h^3} \right) \qquad (18)$$

$$\exp\left(\mathcal{U}_i^{j+1}\right) = \exp\left(\mathcal{U}_\imath^j\right) - \kappa \exp\left(\mathcal{U}_\imath^j\right)\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right) \frac{3(\mathcal{U}_\imath^j)^2 - 4(\mathcal{U}_{\imath-1}^j)^2 + (\mathcal{U}_{\imath-2}^j)^2}{2h} + \frac{5\mathcal{U}_\imath^j - 18\mathcal{U}_{\imath-1}^j + 24\mathcal{U}_{\imath-2}^j - 14\mathcal{U}_{\imath-3}^j + 3\mathcal{U}_{\imath-4}^j}{2h^3}\right), \quad \mathrm{N}-1 \le \imath \le \mathrm{N} \tag{19}$$

Simplify Equations (17-19) we can obtain the following algebraic system of equations,

$$\mathcal{U}_i^{j+1} = \mathcal{U}_\imath^j + \mathrm{Ln}\left(-\kappa\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{-3(\mathcal{U}_\imath^j)^2 + 4(\mathcal{U}_{\imath+1}^j)^2 - (\mathcal{U}_{\imath+2}^j)^2}{4h} + \frac{-5\mathcal{U}_\imath^j + 18\mathcal{U}_{\imath+1}^j - 24\mathcal{U}_{\imath+2}^j + 14\mathcal{U}_{\imath+3}^j - 3\mathcal{U}_{\imath+4}^j}{2h^3}\right) + 1\right), 1 \le \imath \le 2 \tag{20}$$

When $3 \le \imath \le \mathrm{N} - 2$, equation (8) becomes,

$$\mathcal{U}_i^{j+1} = \mathcal{U}_\imath^j + \mathrm{Ln}\left(-\kappa\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{-3(\mathcal{U}_\imath^j)^2 + 4(\mathcal{U}_{\imath+1}^j)^2 - (\mathcal{U}_{\imath+2}^j)^2}{4h} + \frac{-5\mathcal{U}_\imath^j + 18\mathcal{U}_{\imath+1}^j - 24\mathcal{U}_{\imath+2}^j + 14\mathcal{U}_{\imath+3}^j - 3\mathcal{U}_{\imath+4}^j}{2h^3}\right) + 1\right), \tag{21}$$

$$\mathcal{U}_i^{j+1} = \mathcal{U}_\imath^j + \mathrm{Ln}\left(-\kappa\left(\left(\alpha + \beta\mathcal{U}_\imath^j\right)\frac{3(\mathcal{U}_\imath^j)^2 - 4(\mathcal{U}_{\imath-1}^j)^2 + (\mathcal{U}_{\imath-2}^j)^2}{4h} + \frac{5\mathcal{U}_\imath^j - 18\mathcal{U}_{\imath-1}^j + 24\mathcal{U}_{\imath-2}^j - 14\mathcal{U}_{\imath-3}^j + 3\mathcal{U}_{\imath-4}^j}{2h^3}\right) + 1\right),$$
$$\mathrm{N}-1 \le \imath \le \mathrm{N} \tag{22}$$

3 Numerical Experiments

Case study 1

Consider Gardner equation [31] when $\alpha = 4$, $\beta = -3$ and $\gamma = 1$, with the initial condition $\mathcal{U}(x,0) = \frac{2}{12+3\sqrt{14}\,\mathrm{Cosh}\left(-\frac{x}{3}+\frac{5}{3}\right)}$ and the exact solution $(x,t) = \frac{2}{12+3\sqrt{14}\,\mathrm{Cosh}\left(-\frac{x}{3}+\frac{5}{3}+\frac{t}{27}\right)}$.

Table 1. Absolute errors for Exp. FDM and Log.FDM when h =0.1

$\kappa = 0.0001, h = 0.1$ and $t = 0.001$		
x	**Exp. FDM**	**Log. FDM**
0.0	1.9331×10^{-8}	3.79303×10^{-7}
0.1	7.2528×10^{-9}	2.80823×10^{-6}
0.2	2.9034×10^{-9}	3.46015×10^{-6}
0.3	9.4034×10^{-10}	4.24513×10^{-6}
0.4	1.8390×10^{-10}	4.96094×10^{-6}
0.5	2.3134×10^{-10}	5.68259×10^{-6}
0.6	6.7394×10^{-11}	6.10547×10^{-6}
0.7	2.0092×10^{-11}	4.4415×10^{-6}
0.8	4.7687×10^{-11}	4.42334×10^{-6}
0.9	4.6039×10^{-10}	7.04767×10^{-6}
1.0	1.4084×10^{-9}	6.42047×10^{-7}

Table 2. Absolute errors for Exp. FDM and Log. FDM when h =1.0

$\kappa = 0.0001, h = 1$ and $t = 0.001$		
x	**Exp. FDM**	**Log. FDM**
0.0	3.60921×10^{-7}	1.97198×10^{-6}
1.0	5.83107×10^{-7}	1.82722×10^{-6}
2.0	5.431×10^{-7}	1.73770×10^{-6}
3.0	1.00206×10^{-8}	5.80394×10^{-6}
4.0	8.86987×10^{-9}	3.45538×10^{-6}
5.0	4.63718×10^{-12}	1.30074×10^{-6}
6.0	8.99999×10^{-9}	3.45796×10^{-6}
7.0	9.94002×10^{-9}	5.80334×10^{-6}
8.0	2.15564×10^{-9}	6.52653×10^{-6}
9.0	8.06268×10^{-9}	1.83816×10^{-6}
10.0	1.41102×10^{-8}	1.97448×10^{-6}

Case study 2

Consider Gardner equation [32] when $\alpha = 1,\ \beta = -5$ and γ =1, with the initial condition $\mathcal{U}(x,0) = 0.1(\,1 - \tanh(\,\frac{x}{2\sqrt{30}}))$ with exact solution $\mathcal{U}(x,t) = \,0.1\left(1 - \tanh(\,\frac{x-0.1t}{2\sqrt{30}})\right)$

.

Table 3. Absolute errors for Exp. FDM and Log. FDM when h =0.1

$\kappa = 0.0001, h = 0.1$and $t = 0.001$		
x	**Exp. FDM**	**Log. FDM**
0.0	1.53809×10^{-6}	1.63261×10^{-7}
0.1	1.51488×10^{-6}	4.08396×10^{-7}
0.2	1.52308×10^{-6}	4.87864×10^{-7}
0.3	1.51889×10^{-6}	6.30548×10^{-7}
0.4	1.51844×10^{-6}	7.35689×10^{-7}
0.5	1.51628×10^{-6}	8.03524×10^{-7}
0.6	1.51415×10^{-6}	8.35254×10^{-7}
0.7	1.51148×10^{-6}	6.04582×10^{-7}
0.8	1.50851×10^{-6}	5.97121×10^{-7}
0.9	1.50529×10^{-6}	9.42587×10^{-7}
1.0	1.5017×10^{-6}	1.59377×10^{-7}

Table 4. Absolute errors for Exp. FDM and Log. FDM when h =0.1

$\kappa = 0.00001, h = 1.0$ and $t = 0.0001$		
x	**Exp. FDM**	**Log. FDM**
0.0	1.53816×10^{-7}	2.8845×10^{-7}
1.0	1.51316×10^{-7}	2.96338×10^{-7}
2.0	1.44889×10^{-7}	3.17768×10^{-7}
3.0	1.35135×10^{-7}	6.73696×10^{-7}
4.0	1.23766×10^{-7}	6.16856×10^{-7}
5.0	1.10903×10^{-7}	5.53123×10^{-7}
6.0	9.74436×10^{-8}	4.86214×10^{-7}
7.0	8.41595×10^{-8}	4.20061×10^{-7}
8.0	7.16318×10^{-8}	3.57718×10^{-7}
9.0	5.88746×10^{-8}	4.02963×10^{-7}
10.0	4.88345×10^{-8}	3.84064×10^{-7}

4 Conclusion

Exponential and Logarithmic Finite Difference methods are applied effectively to Gardner equation. Numerical experiments show that the absolute error between numerical solution and exact solution is neglectable for different times and for different exact solutions.

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