Parameters and Reliability Estimation of Extended Exponential Distribution under Type-II Progressive Hybrid Censoring

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Abstract

The estimating problems of the model parameters, reliability and hazard functions of extended exponential distribution used Type-II progressive hybrid censoring scheme (Type-II PHCS) will be considered. The maximum likelihood estimation (MLE) has been obtained for any function of the model parameters. Based on the normality property of the classical estimators, approximate confidence intervals (ACIs) for the unknown parameters and any function of them are constructed. Further, construct the asymptotic confidence interval of the reliability and hazard rate function. Using independent gamma priors, the Bayes estimators of the unknown parameters are derived based on both the symmetric (squared error (SE)) and asymmetric (LINEX) loss functions. Since the Bayes estimators are obtained in a complex form therefore, Markov Chain Monte Carlo (MCMC) using Metropolis-Hastings (MH) algorithm has been used to carry out the Bayes estimates and also to construct the associate highest posterior density credible intervals. To evaluate the performance of the proposed methods, a Monte Carlo simulation study is carried out. Finally, we consider engineering data to illustrate the applicability of the methods covered in the paper.

Keywords: Extended exponential distribution; Reliability and hazard rate functions; Bayesian and non-Bayesian estimation; MCMC; Type-II progressive hybrid censoring.

1. Introduction

A new generalization of the exponential distribution as an alternative to gamma, Weibull and generalized-exponential lifetime models has been introduced by Nadarajah and Haghighi (2011). The extension of the exponential distribution was named NHD by Lemonte (2013) as an abbreviation for the name authors Nadarajah and Haghighi. Also, many properties of extended exponential distribution are discussed by Nadarajah and Haghighi (2011). Suppose that the lifetime X of a testing unit follows two-parameter extended exponential distribution (α, λ). The probability density function $f(\cdot)$, cumulative distribution function $F(\cdot)$, reliability function $S(\cdot)$ and hazard rate function $H(\cdot)$, for given mission time t, are given by

$$f(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}\exp(1-(1+\lambda x)^{\alpha}) \qquad ;x > 0,\ \ \alpha, \lambda > 0, \tag{1}$$

$$F(x) = 1-\exp(1-(1+\lambda x)^{\alpha}) \qquad ;x > 0,\ \ \alpha, \lambda > 0, \tag{2}$$

$$S(t;\alpha,\lambda) = \exp(1-(1+\lambda t)^{\alpha}) \qquad ;t > 0, \tag{3}$$

and

$$H(t;\alpha,\lambda) = \alpha\lambda(1+\lambda t)^{\alpha-1} \qquad ;t > 0 \tag{4}$$

respectively, where α and λ are the shape and scale parameters, respectively.

Recently, many studies on estimating the unknown parameters of extended exponential distribution based on different life-testing experiments have been carried out by many authors. Singh et al. (2015a) obtained the MLE and Bayes estimators of the extended exponential distribution under Type-II progressive censoring scheme (Type-II PCS).

Singh et al. (2015b) discussed the MLEs and Bayes estimators of the unknown parameters and reliability characteristics of the extended exponential distribution based on complete sampling.. Sanku et al. (2017) introduced a comparisons between several methods for estimating the unknown parameters of extended exponential distribution. Sana and Faizan (2019) discussed MLE and Bayes estimation of the two unknown parameters of extended exponential distribution based on record values. Ashour et al. (2020) obtained The MLE and Bayes inferential approaches for estimating the unknown two parameters and some lifetime parameters such as reliability and hazard rate functions of extended exponential distribution in presence of progressive first-failure censored sampling. Wu, M. and Gui, W. (2021) obtained estimation and prediction for extended exponential distribution under progressive Type-II censoring.

In conventional Type-I and Type-II censoring, a life test is terminated at a prescribed time span or at a predefined number of failures. The main drawback of these censoring schemes is, the units cannot be removed from the test at any time point except the final closure point. However, the Type-II PCS gives the flexibility of eliminating the test units before the final termination. On other hand, the major drawback of the Type-II PCS is that, it can take a lot of time to reach the final termination point (Childs et al. (2008)). They introduced Type-II progressive hybrid censoring scheme (Type-II PHCS). Type-II PHCS involves the termination of the life test at time $T^* = \max(x_{(r)}, T)$. Let D denote the number of failures that occur before time T, if $x_{(r)} > T$, the experiment would terminate at the r^{th} failure, with the withdrawal of units occurring after each failure according to the pre-fixed progressive censoring scheme $R_1, R_2, ..., R_r$ .However, if $x_{(r)} < T$, then instead of terminating the experiment by removing all remaining R_r units after the r^{th} failure, the experiment would continue to observe failures without any further withdrawals up to time T . Thus, in this case $R_r = R_{r+1} = ...R_D = 0$.

Based on the above Type-II PHCS ,the observed date will be one of the following two form;

$$Case \begin{cases} I : X_{(1)} < X_{(2)} < ...X_{(r)} & if \quad X_{(r)} \geq T , \\ \\ II : X_{(1)} < ... < X_{(r)} < X_{(r+1)} < ... < X_{(D)} & if \quad X_{(r)} < T . \end{cases}$$

The likelihood function of the observed data (without constant term) is given by

$$L(\underline{x};\theta) \propto \begin{cases} (\prod_{i=1}^{r} f(x_{(i)})(1-F(x_{(i)}))^{R_i}), & for\ case\ I \\ \\ (\prod_{i=1}^{r} f(x_{(i)})(1-F(x_{(i)}))^{R_i})(\prod_{i=r+1}^{D} f(x_{(i)}))(1-F(T))^{R_D^*}. & for\ case\ II \end{cases} \quad (5)$$

where R_D^* is the number of remaining units left at the time point T for case II. This procedure is guarantees that the life test would yield at last r complete failure times.

For more details and some recent references on progressive hybrid censoring schemes, see Kundu and Joarder (2006), Lin et al. (2009), Joarder et al. (2009), Bayat Mokhtari et al. (2011), Hemmati and Khorram (2013), Gurunlu Alma and Arabi Belaghi. (2016) and Kayal et al. (2017).

The aim of this paper is the estimation of the unknown parameters, hazard rate and reliability functions of extended exponential distribution under Type-II PHCS .In section 2, The MLEs and the information matrix will be discussed to obtain asymptotic confidence intervals for the parameters and estimate reliability and hazard rate functions. Further, Bayesian estimation under the assumption of independent gamma priors using SE and LINEX loss functions will be discussed in section 3. Numerically proposed methods using Monte Carlo simulations and a real data set is compared in Section 4. Finally a conclusion is given in Section 5.

2. Maximum Likelihood Estimation

In this section, maximum likelihood estimation and its information matrix for the unknown parameters of the extended exponential distribution (1) will be obtained using Type II progressive hybrid censoring (5).

The likelihood function is given by

$$L(\underline{x};\alpha,\lambda)=\begin{cases}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} & \text{for case I}\\ \prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} & \text{for case II}\\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*} & \end{cases} \quad (6)$$

Taking natural logarithm, we get

$$\ln L(\underline{x};\alpha,\lambda)=\begin{cases} r\ln\alpha+r\ln\lambda+(\alpha-1)\sum_{i=1}^{r}\ln(1+\lambda x_{(i)})+\\ \sum_{i=1}^{r}(1-(1+\lambda x_{(i)})^{\alpha})+R_i\ln(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))\ \text{for case I}\\ r\ln\alpha+r\ln\lambda+(\alpha-1)\sum_{i=1}^{r}\ln(1+\lambda x_{(i)})+\\ \sum_{i=1}^{r}(1-(1+\lambda x_{(i)}))^{\alpha}+R_i((1-(1+\lambda x_{(i)})^{\alpha})\\ +D\ln\alpha+D\ln\lambda+(\alpha-1)\ln\sum_{i=r+1}^{D}(1+\lambda x_{(i)})\\ +\sum_{i=r+1}^{D}(1-(1+\lambda x_{(i)}))^{\alpha}+R_D^*((1-(1+\lambda T)^{\alpha}))\ \ \text{for case II}\end{cases} \quad (7)$$

Differentiating $\ln L(\underline{x};\alpha,\lambda)$ partially with respect to α and λ, we get the following two equations.

$$\frac{\partial \ln L(\underline{x};\alpha,\lambda)}{\partial \alpha}=\begin{cases}\frac{r}{\alpha}+\sum_{i=1}^{r}\ln(1+\lambda x_{(i)})-\sum_{i=1}^{r}(1+\lambda x_{(i)})^{\alpha}\ln(1+\lambda x_{(i)}) \\ \qquad (1+R_i) \qquad \textit{for case I} \\ \frac{r}{\alpha}+\sum_{i=1}^{r}\ln(1+\lambda x_{(i)})-\sum_{i=1}^{r}(1+\lambda x_{(i)})^{\alpha}\ln(1+\lambda x_{(i)})[1+R_i] \\ +\frac{D}{\alpha}+\sum_{i=r+1}^{D}\ln(1+\lambda x_{(i)})-\sum_{i=r+1}^{D}(1+\lambda x_{(i)})^{\alpha}\ln(1+\lambda x_{(i)}) \\ -R_i^{*}(1+\lambda T)^{\alpha}\ln(1+\lambda T) \qquad \textit{for case II}\end{cases} \tag{8}$$

and

$$\frac{\partial \ln L(\underline{x};\alpha,\lambda)}{\partial \lambda}=\begin{cases}\frac{r}{\lambda}+(\alpha-1)\sum_{i=1}^{r}\frac{x_{(i)}}{1+\lambda x_{(i)}}-\alpha\sum_{i=1}^{r}x_{(i)}(1+\lambda x_{(i)})^{\alpha-1} \\ \qquad [1+R_i] \qquad \textit{for case I} \\ \frac{r}{\lambda}+(\alpha-1)\sum_{i=1}^{r}\frac{x_{(i)}}{1+\lambda x_{(i)}}-\alpha\sum_{i=1}^{r}x_{(i)}(1+\lambda x_{(i)})^{\alpha-1}[1+R_i] \\ +\frac{D}{\lambda}+(\alpha-1)\sum_{i=r+1}^{D}\frac{x_{(i)}}{1+\lambda x_{(i)}}-\alpha\sum_{i=r+1}^{D}x_{(i)}(1+\lambda x_{(i)})^{\alpha-1} \\ -R_i^{*}\alpha T(1+\lambda T)^{\alpha-1} \qquad \textit{for case II}\end{cases} \tag{9}$$

Since these equations after equating them to zero are clearly transcendental equations in $\hat{\alpha}$ and $\hat{\lambda}$ that is, no closed form solutions are known they must be solved by iterative numerical techniques to provide solutions (estimates), $\hat{\alpha}$ and $\hat{\lambda}$, in the desired degree of accuracy.

If $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of the parameters then by using the invariance properties, the MLEs of hazard rate function and survival function are given by, respectively.

$$\hat{H}(x)=\hat{\alpha}\hat{\lambda}(1+\hat{\lambda}t)^{\hat{\alpha}-1} \tag{10}$$

and

$$\hat{S}(x)=e^{(1-(1+\hat{\lambda}t)^{\hat{\alpha}})} \tag{11}$$

To study the variation of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, the asymptotic variance of these estimators are obtained. The asymptotic variance covariance matrix of , $\hat{\alpha}$ and $\hat{\lambda}$, is obtained by inverting the information matrix with elements that are negative expected values of the second order derivatives of natural logarithms of the likelihood function, for sufficiently large samples, a reasonable approximation to the asymptotic variance covariance matrix of the estimators can be obtained as

$$I^{-1}(\hat{\alpha},\hat{\lambda}) \cong \begin{bmatrix} -\dfrac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial \alpha^2} & -\dfrac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial \lambda \partial \alpha} \\ -\dfrac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial \lambda \partial \alpha} & -\dfrac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\lambda})} \cong \begin{bmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\lambda}) \\ Cov(\hat{\alpha},\hat{\lambda}) & Var(\hat{\lambda}) \end{bmatrix} \qquad (12)$$

The elements of the previous sample information matrix can be obtained such that

$$\frac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial \alpha^2} = \begin{cases} \dfrac{-r}{\alpha^2} - \sum\limits_{i=1}^{r} (\ln(1+\lambda x_{(i)}))^2 (1+\lambda x_{(i)})^{\alpha} [1+R_i] & \textit{for case I} \\ \dfrac{-r}{\alpha^2} - \sum\limits_{i=1}^{r} (\ln(1+\lambda x_{(i)}))^2 (1+\lambda x_{(i)})^{\alpha} [1+R_i] & \\ -\dfrac{D}{\alpha^2} - \sum\limits_{i=r+1}^{D} (\ln(1+\lambda x_{(i)}))^2 (1+\lambda x_{(i)})^{\alpha} & \\ -R_D^{*} (1+\lambda T)^{\alpha} [\ln(1+\lambda T)]^2 & \textit{for case II} \end{cases},$$

$$\frac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial\lambda^2}=\begin{cases}\frac{-r}{\lambda^2}-(\alpha-1)\sum_{i=1}^{r}\frac{x_{(i)}^2}{(1+\lambda x_{(i)})^2}-\\ \alpha(\alpha-1)\sum_{i=1}^{r}x_{(i)}^2(1+\lambda x_{(i)})^{\alpha-2}[1+R_i] & for\ case\ I\\ \frac{-r}{\lambda^2}-(\alpha-1)\sum_{i=1}^{r}\frac{x_{(i)}^2}{(1+\lambda x_{(i)})^2}-\alpha(\alpha-1)\sum_{i=1}^{r}x_{(i)}^2(1+\lambda x_{(i)})^{\alpha-2}[1+R_i]\\ -\frac{D}{\lambda^2}-(\alpha-1)\sum_{i=r+1}^{D}\frac{x_{(i)}^2}{(1+\lambda x_{(i)})^2}-\alpha(\alpha-1)\sum_{i=r+1}^{D}x_{(i)}^2(1+\lambda x_{(i)})^{\alpha-2}\\ -R_D^*[\alpha(\alpha-1)T^2(1+\lambda T)^{\alpha-2}] & for\ case\ II\end{cases}$$

and

$$\frac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial\lambda\partial\alpha}=\frac{\partial^2 \ln L(\underline{x};\alpha,\lambda)}{\partial\alpha\partial\lambda}=\begin{cases}\sum_{i=1}^{r}\frac{x_{(i)}}{(1+\lambda x_{(i)})}-\sum_{i=1}^{r}x_{(i)}(1+\lambda x_{(i)})^{\alpha-1}\\ (\alpha\ln(1+\lambda x_{(i)})+1)[1+R_i] & for\ case\ I\\ \sum_{i=1}^{r}\frac{x_{(i)}}{(1+\lambda x_{(i)})}-\sum_{i=1}^{r}x_{(i)}(1+\lambda x_{(i)})^{\alpha-1}\\ (\alpha\ln(1+\lambda x_{(i)})+1)[1+R_i]+\sum_{i=r+1}^{D}\frac{x_{(i)}}{(1+\lambda x_{(i)})}\\ -\sum_{i=r+1}^{D}x_{(i)}(1+\lambda x_{(i)})^{\alpha-1}(\alpha\ln(1+\lambda x_{(i)})+1)-\\ R_D^*[T(1+\lambda T)^{\alpha-1}][\alpha\ln(1+\lambda T)+1] & for\ case\ II\end{cases}$$

Diagonal elements of $I^{-1}(\hat{\alpha},\hat{\lambda})$ provides the asymptotic variance of α and λ respectively. Then using large sample theory a two sided $100(1-\beta)\%$ approximate confidence interval for α can be constructed as $\hat{\alpha}\pm z_{1-\beta/2}\sqrt{\mathrm{var}(\hat{\alpha})}$ and similarly, for λ the two sided $100(1-\beta)\%$ approximate confidence interval can be obtained as $\hat{\lambda}\pm z_{1-\beta/2}\sqrt{\mathrm{var}(\hat{\lambda})}$.

To construct the ACIs of $S(t)$ and $H(t)$, The variances of them is needed Therefore, the delta method is considered to obtain the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{H}(t)$. Delta method is a general approach for computing ACIs for any function of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, (See Greene (2012)). According to this method, the variance of $\hat{S}(t)$ and $\hat{H}(t)$, can be approximated, by

$$\hat{\sigma}^2_{\hat{S}(t)} = [\nabla \hat{S}(t)]^T I_0^{-1} [\nabla \hat{S}(t)] \quad \text{and} \quad \hat{\sigma}^2_{\hat{H}(t)} = [\nabla \hat{H}(t)]^T I_0^{-1} [\nabla \hat{H}(t)]$$

respectively, where the gradient vector of first partial derivatives of $S(t)$ and $H(t)$ with respect to α and λ obtained at $\hat{\alpha}$ and $\hat{\lambda}$ are given by

$$[\nabla \hat{S}(t)]^T = \left[\frac{\partial \nabla \hat{S}(t)}{\partial(\alpha)}, \frac{\partial \nabla \hat{S}(t)}{\partial(\lambda)}\right]_{(\hat{\alpha},\hat{\lambda})} \quad \text{and} \quad [\nabla \hat{H}(t)]^T = \left[\frac{\partial \nabla \hat{H}(t)}{\partial(\alpha)}, \frac{\partial \nabla \hat{H}(t)}{\partial(\lambda)}\right]_{(\hat{\alpha},\hat{\lambda})}$$

Hence, the $100(1-\beta)\%$ ACIs of $S(t)$ and $H(t)$, are given by

$$\hat{S}(t) \pm z_{1-\beta/2}\sqrt{\hat{\sigma}^2_{\hat{S}(t)}} \quad \text{and} \quad \hat{H}(t) \pm z_{1-\beta/2}\sqrt{\hat{\sigma}^2_{\hat{H}(t)}}$$

respectively.

3. Bayesian Estimation

In this section, Bayesian method is used to obtain the estimators for the unknown parameters of extended exponential distribution using squared error and LINEX loss functions

We consider independent gamma priors for the parameters α and λ as

$$\pi(\alpha) \propto \alpha^{a-1} e^{-b\alpha} \quad , \alpha > 0, a, b > 0 \qquad \text{and} \qquad \pi(\lambda) \propto \lambda^{c-1} e^{-d\lambda} \quad , \lambda > 0, c, d > 0$$

then the joint priors distribution is

$$\pi(\alpha,\lambda) \propto \alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)} \qquad \lambda,\alpha > 0, a,b,c,d > 0, \tag{13}$$

Combining equation (13) with equation (6) and using Bayes theorem, the joint posterior distribution can be obtained as

$$\pi(\alpha,\lambda\,|\,\underline{x})=\frac{\pi(\alpha)\pi(\lambda)L(\underline{x};\alpha,\lambda)}{\int\limits_{\alpha}\int\limits_{\lambda}\pi(\alpha)\pi(\lambda)L(\underline{x};\alpha,\lambda)\quad d\lambda d\alpha}$$

$$=\begin{cases}\dfrac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod\limits_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} & \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} & for\, case\, I \\ \dfrac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod\limits_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} & \\ \prod\limits_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*} & for\, case\, II\end{cases} \qquad (14)$$

where

$$\psi_3=\int\limits_0^{\infty}\int\limits_0^{\infty}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}d\lambda d\alpha$$

and

$$\psi_4=\int\limits_{\alpha}\int\limits_{\lambda}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}$$
$$\prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}d\lambda d\lambda$$

The Bayesian estimators of α and λ of extended exponential distribution under the squared error loss function is the mean of the posterior density function, given by

$$\tilde{\alpha}_{SE}=\int\limits_{\alpha}\alpha\pi(\alpha,\lambda\,|\,\underline{x})\quad d\alpha$$ and

$$\tilde{\lambda}_{SE}=\int\limits_{\lambda}\lambda\pi(\alpha,\lambda\,|\,\underline{x})\quad d\lambda$$

respectively. These estimators can be expressed a

$$\tilde{\alpha}_{SE}=\begin{cases}\int_{\alpha}\frac{1}{\psi_3}\alpha^{a}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\,d\alpha \qquad \text{for case I} \\ \int_{\alpha}\frac{1}{\psi_4}\alpha^{a}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} \\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\,d\alpha \qquad \text{for case II}\end{cases} \tag{15}$$

and

$$\tilde{\lambda}_{SE}=\begin{cases}\int_{\lambda}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\,d\lambda \qquad \text{for case I} \\ \int_{\lambda}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} \\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\,d\lambda \quad \text{for case II}\end{cases} \tag{16}$$

and the form of reliability function and hazard function are given as the following equation,

$$\tilde{S}(t)_{SE}=\begin{cases}\int_{\alpha}\int_{\lambda}e^{(1-(1+\lambda x)^{\alpha})}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\,d\lambda d\alpha \ \text{for case I} \\ \int_{\alpha}\int_{\lambda}e^{(1-(1+\lambda x)^{\alpha})}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} \\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\,d\lambda d\alpha \ \text{for case II}\end{cases} \tag{17}$$

and

$$\tilde{H}(t)_{SE}=\begin{cases}\int_{\alpha}\int_{\lambda}\alpha\lambda(1+\lambda x)^{\alpha-1}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}\\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\;d\lambda d\alpha \qquad for\,case\,I\\ \int_{\alpha}\int_{\lambda}\alpha\lambda(1+\lambda x)^{\alpha-1}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}\\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\;d\lambda d\alpha\;for\,case\,II\end{cases}\tag{18}$$

respectively.

Following Zellner (1986), the Bayes estimators under LINEX loss function are

$$\tilde{\alpha}_{LINEX}=\frac{1}{c^*}\ln(E(\mathrm{e}^{-c^*\alpha}))$$ and

$$\tilde{\lambda}_{LINEX}=\frac{1}{c^*}\ln(E(\mathrm{e}^{-c^*\lambda}))$$

respectively, where $E(\cdot)$ denotes the posterior expectation. These estimators can be expressed as

$$\tilde{\alpha}_{LINEX}=\begin{cases}\frac{1}{c^*}\ln\int_{\alpha}\mathrm{e}^{-c^*\alpha}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}\\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\quad d\alpha\;\;for\,case\,I\\ \frac{1}{c^*}\ln\int_{\alpha}\mathrm{e}^{-c^*\alpha}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}\\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\,d\alpha\;for\,case\,II\end{cases}\tag{19}$$

and

$$\tilde{\lambda}_{LINEX}=\begin{cases}\frac{1}{c^*}\ln\int_{\lambda} e^{-c^*\lambda}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\; d\lambda \quad for\ case\ I \\ \frac{1}{c^*}\ln\int_{\lambda} e^{-c^*\lambda}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} \\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\; d\lambda \quad for\ case\ II\end{cases} \tag{20}$$

respectively, and the form of reliability function and hazard function are given as,

$$\tilde{S}(t)_{LINEX}=\begin{cases}\frac{1}{c^*}\ln\int_{\alpha}\int_{\lambda} e^{-c^* e^{(1-(1+\lambda x)^{\alpha})}}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\, d\lambda d\alpha \qquad for\ case\ I \\ \frac{1}{c^*}\ln\int_{\alpha}\int_{\lambda} e^{-c^* e^{(1-(1+\lambda x)^{\alpha})}}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} \\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\, d\lambda d\alpha \quad for\ case\ II\end{cases} \tag{21}$$

and

$$\tilde{H}(t)_{LINEX}=\begin{cases}\frac{1}{c^*}\ln\int_{\alpha}\int_{\lambda} e^{-c^*\alpha\lambda(1+\lambda x)^{\alpha-1}}\frac{1}{\psi_3}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i}\; d\lambda d\alpha \qquad for\ case\ I \\ \frac{1}{c^*}\ln\int_{\alpha}\int_{\lambda} e^{-c^*\alpha\lambda(1+\lambda x)^{\alpha-1}}\frac{1}{\psi_4}\alpha^{a-1}\lambda^{c-1}e^{-(b\alpha+d\lambda)}\prod_{i=1}^{r}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})} \\ \qquad\times(1-(1-e^{(1-(1+\lambda x_{(i)})^{\alpha})}))^{R_i} \\ \prod_{i=r+1}^{D}\alpha\lambda(1+\lambda x_{(i)})^{\alpha-1}e^{(1-(1+\lambda x_{(i)})^{\alpha})}(1-(1-e^{(1-(1+\lambda T)^{\alpha})}))^{R_D^*}\, d\lambda d\alpha \quad for\ case\ II\end{cases} \tag{22}$$

respectively.

Equations (15), (16), (17), (18), (19), (20), (21) and (22) in general cannot be obtained in a closed form, so the approximate methods is employed. MCMC using MH algorithm has been used to carry out the Bayes estimates and also to construct the associate HPD credible intervals.

4. Simulated Results and Real Data Analysis

The aim of this section is to compare the performance of the different methods of estimation discussed in the previous sections. A Monte Carlo study is employed to check the behavior of the proposed methods as well as to assess the statistical performances of the estimators under Type-II progressive hybrid. Also, a real data set is analyzed for illustrative purpose. *R*-statistical programming language will be used for calculation.

4.1 Simulated Study

In this section, we perform a Monte Carlo simulated study 1000 times to compare the performance of different estimators of unknown parameters of the extended exponential distribution. We also assess the behavior of predictors of censored observations under the considered censoring scheme. The performance of different estimators is compared in terms of corresponding average estimates and mean square error (MSE) values. For this purpose, we generate Type-II progressive hybrid censored samples using various sampling schemes by considering different combinations of (n,r) and assuming that T is either (0.63, 1.79). We used the R-statistical software for all computations. The MLEs of α and λ are computed and the information matrix will be discussed to obtain asymptotic confidence intervals for the parameters and estimate reliability and hazard rate functions. Bayes estimates of parameters are computed with respect to a gamma prior distribution under squared error and LINEX loss functions. Both MLEs and Bayes estimates of parameters are obtained for arbitrarily taken unknown parameters $\alpha = 1.5$ and $\lambda = 0.5$.

For the MLEs, one may generate 1000 data from the extended exponential distribution with the following assumptions:

1. Assume the following selected cases of parameters of the extended exponential distribution: $(\alpha, \lambda) = (1.5, 0.5)$.
2. Sample sizes, are $n = 50, 100, 200$ and number of observed failures $r = 20, 40, 80$, respectively.
3. Censoring times Type-II PHCS are assumed T_q corresponding to the selected quantiles q^{th} quantiles, where $q = (40\%, 80\%)$. The q^{th} quantiles of lifetimes distribution is given by :

$$P(X \le T_q) = q \quad \Rightarrow \quad T_q = Q(q)$$

where $Q(.)$ is the inverse of the cdf (quantile) of the given distribution.

4. Removed items R_i are assumed to as follows:

Scheme I: $R_1 = n - r$ **and** $R_2, ..., R_r = 0$.

Scheme II: $R_1 ... R_{\frac{r}{2}} = 1$ **and** $R_{(\frac{r}{2}+1)} ... R_r = 2$.

Scheme III: $R_1, ..., R_{r-1} = 0$ **and** $R_r = n - r$.

Table 1. Removal patterns of units in various censoring schemes

(n, r)	Censoring Schemes		
	I	II	III
(50,20)	(30, 0^{*19})	(1^{*10}, 2^{*10})	($0*^{19}$, 30)
(100,40)	(60, $0*^{39}$)	(1^{*20}, 2^{*20})	($0*^{39}$, 60)
(200,80)	(120, $0*^{79}$)	(1^{*40}, 2^{*40})	($0*^{79}$, 120)

Here, (1^{*5}, 0), for example, means that the censoring scheme employed is (1, 1, 1, 1, 1, 0).

The values of hyper-parameters are chosen to satisfy the prior mean become the expected value of the corresponding parameter, one can assume the hyper-parameters as: $a = 1.6, b = 1, c = 1$ and $d = 1.5$. These values, hyper parameters, are then plugged-in to calculate the desired estimates. While utilizing MH algorithm, the MLEs are taken into account as initial guess values, and the associated variance-covariance matrix $(\theta^{(0)}) = (\ln(\hat{\alpha}), \ln(\hat{\lambda}))$. At the end, 2000 burn-in samples are discarded among the overall 10000 samples generated from the posterior density, and subsequently obtained Bayes estimates and highest posterior density credible interval estimates.

Further, we have also obtained the MLEs and Bayesian estimates of the reliability function and hazard function where the true values of $S(t)$ and $H(t)$ are taken form the specified time censoring, termination point of the test $T^* = \max(T, x_{(r)})$, of Type-II progressive hybrid scheme. The true values of hazard function are $h(t = 0.63, \alpha, \lambda) = 0.8606$ and $h(t = 1.79, \alpha, \lambda) = 1.0325$ and the true values of reliability function are $S(t = 0.63, \alpha, \lambda) = 0.6000$ and $S(t = 1.79, \alpha, \lambda) = 0.2000$. All the average estimates and associated MSEs for both methods are reported in Table (1.a) and Table (1.b). Further, the corresponding average interval lengths (AILs) and coverage probabilities (CPs) are reported in Table (2.a) and Table (2.b) for all the proposed confidence intervals, namely; asymptotic confidence interval (Asy-CI), HPD interval, and ACI for $\hat{S}(t)$ and $\hat{H}(t)$.

Table (1.a): Average estimates values and MSEs of the ML and Bayes estimates based on Type-II progressive hybrid censoring schemes at different time censoring and different values of (n, r) for $\alpha = 1.5, \lambda = 0.5$

(n,r)	Method	$q_i = 40\%$			$q_i = 80\%$		
		I	II	III	I	II	III
(50,20)	MLE_α	2.7525 12.2799	2.4499 12.9524	2.2997 14.2271	2.5906 9.6574	2.5978 15.3345	0.1386 2.1217
	MLE_λ	0.6772 0.5743	0.9701 1.3841	1.1280 2.0412	0.6790 0.5920	0.8828 1.1739	1.7169 2.9320
	Bayes SE_α	2.0261 1.3482	1.6667 1.3087	1.5555 1.7610	2.0208 1.1514	1.7138 1.5766	0.1164 1.9548
	Bayes SE_λ	0.5251 0.1535	0.6894 0.3583	0.7974 0.6537	0.5099 0.1426	0.6257 0.3144	1.5380 1.9878
	Bayes $LINEX_\alpha$	1.9015 1.3720	1.5036 0.7944	1.4072 1.4972	1.8946 1.2122	1.5894 1.4045	0.1130 1.9579
	Bayes $LINEX_\lambda$	0.4617 0.1003	0.5905 0.2210	0.6781 0.3939	0.4499 0.0942	0.5434 0.2016	1.4354 1.5810
(100,40)	MLE_α	2.4166 7.0952	2.8080 14.5896	2.5945 19.0184	2.5289 9.2707	2.2224 10.6330	1.1463 2.2712
	MLE_λ	0.5366 0.1725	0.6835 0.4288	1.0585 1.6447	0.5436 0.1774	0.7516 0.5236	0.8966 0.4364
	Bayes SE_α	2.0339 1.3979	1.9069 1.2692	1.8067 3.9895	2.0794 1.5459	1.8236 1.1268	1.2241 0.5448
	Bayes SE_λ	0.4988 0.1036	0.5685 0.2067	0.8053 0.6762	0.4817 0.0932	0.6006 0.2217	0.7502 0.2731
	Bayes $LINEX_\alpha$	1.9735 1.4760	1.8679 1.7048	1.6729 2.9964	1.9984 1.6548	1.6765 1.1132	1.1365 0.5479
	Bayes $LINEX_\lambda$	0.4616 0.0780	0.5101 0.1482	0.7130 0.4829	0.4450 0.0704	0.5351 0.1540	0.6867 0.2028
(200,80)	MLE_α	2.0809 4.1238	2.4660 10.6670	2.5847 14.6691	2.1042 3.3813	2.3040 7.9377	0.9412 0.4840
	MLE_λ	0.5183 0.0915	0.5960 0.1822	0.7427 0.5887	0.4964 0.0812	0.6205 0.2331	0.8313 0.2334
	Bayes SE_α	1.9831 1.3706	1.9811 1.6678	1.9895 2.8323	2.0124 1.4481	2.0030 2.1565	1.0807 0.3716

	Bayes SE_{λ}	0.4920 0.0744	0.5467 0.1276	0.6513 0.4072	0.4850 0.0764	0.5595 0.1561	0.7423 0.1717
	Bayes $LINEX_{\alpha}$	1.9240 1.3358	1.8836 1.7406	1.9193 3.0643	1.9650 1.4645	1.9295 2.0577	1.0266 0.3958
	Bayes $LINEX_{\lambda}$	0.4692 0.0597	0.5073 0.0964	0.5952 0.3164	0.4643 0.0618	0.5181 0.1204	0.6992 0.1373

Note that:

$T_{q=40\%} = Q(40\%, \alpha = 1.5, \lambda = 0.5) = 0.6334$ & $T_{q=80\%} = Q(80\%, \alpha = 1.5, \lambda = 0.5) = 1.7908$.

True value of $h(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.8606$ & $h(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 1.0325$.

True value of

$S(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.6000$ & $S(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 0.2000$.

Table (1.b): Average estimates values and MSEs of S(t) and H(t) for the MLE and Bayes estimates based Type-II progressive hybrid censoring schemes at different time censoring and different values of (n, r) for $\alpha = 1.5, \lambda = 0.5$

(n,r)	Method	$q_i = 40\%$			$q_i = 80\%$		
		I	II	III	I	II	III
(50,20)	$MLE_{h(t)}$	0.9133 0.0530	0.8768 0.0613	0.6979 0.0563	1.1734 0.2853	0.8584 0.2943	0.0333 0.9985
	$MLE_{S(t)}$	0.5830 0.0076	0.5851 0.0050	0.6381 0.0042	0.1909 0.0057	0.2807 0.0188	0.9078 0.5012
	Bayes $SE_{h(t)}$	1.0458 2.1195	1.0356 0.4697	0.8486 0.7496	1.2920 0.3654	1.0053 0.3486	0.0335 0.9982
	Bayes $SE_{S(t)}$	0.5673 0.0115	0.5559 0.0106	0.6115 0.0092	0.1775 0.0083	0.2475 0.0169	0.9098 0.5041
	Bayes $LINEX_{h(t)}$	0.7109 0.0555	0.6374 0.0899	0.5297 0.1319	0.8520 0.1230	0.6071 0.2381	0.0326 1.0000
	Bayes $LINEX_{S(t)}$	0.6556 0.0087	0.6772 0.0128	0.7148 0.0174	0.2855 0.0215	0.3791 0.0512	0.9129 0.5084
(100,40)	$MLE_{h(t)}$	0.8762 0.0228	0.8847 0.0264	0.7101 0.0450	1.1000 0.0741	0.9471 0.1895	0.5882 0.2062
	$MLE_{S(t)}$	0.5970 0.0036	0.5894 0.0024	0.6352 0.0033	0.1970 0.0027	0.2399 0.0090	0.3311 0.0173
	Bayes $SE_{h(t)}$	0.9343	1.0798	0.8539	1.2008	1.1498	0.7007

		0.1412	2.8310	0.3685	0.1492	0.2435	0.1383
	Bayes $SE_{S(t)}$	0.5846 0.0058	0.5597 0.0077	0.6045 0.0077	0.1824 0.0046	0.1984 0.0089	0.2969 0.0114
	Bayes $LINEX_{h(t)}$	0.7666 0.0308	0.7118 0.0557	0.5850 0.0950	0.9315 0.0580	0.7670 0.1412	0.5689 0.2221
	Bayes $LINEX_{S(t)}$	0.6354 0.0050	0.6536 0.0085	0.6912 0.0118	0.2487 0.0089	0.3046 0.0221	0.3588 0.0258
(200,80)	$MLE_{h(t)}$	0.8671 0.0114	0.8656 0.0132	0.7359 0.0372	1.0888 0.0358	1.0482 0.1180	0.5863 0.2035
	$MLE_{S(t)}$	0.5990 0.0020	0.5967 0.0013	0.6355 0.0042	0.1960 0.0014	0.2089 0.0034	0.3315 0.0174
	Bayes $SE_{h(t)}$	0.8911 0.0206	0.9434 0.0376	0.8292 0.0466	1.1283 0.0476	1.1658 0.1304	0.6681 0.1475
	Bayes $SE_{S(t)}$	0.5931 0.0033	0.5764 0.0034	0.6091 0.0050	0.1891 0.0021	0.1832 0.0041	0.3044 0.0120
	Bayes $LINEX_{h(t)}$	0.8077 0.0130	0.7773 0.0223	0.6444 0.0704	1.0085 0.0268	0.9136 0.0817	0.5800 0.2094
	Bayes $LINEX_{S(t)}$	0.6202 0.0022	0.6291 0.0028	0.6739 0.0097	0.2185 0.0021	0.2508 0.0093	0.3474 0.0220

Note that:

$T_{q=40\%} = Q(40\%, \alpha = 1.5, \lambda = 0.5) = 0.6334$ & $T_{q=80\%} = Q(80\%, \alpha = 1.5, \lambda = 0.5) = 1.7908$.

True value of $h(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.8606$ & $h(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 1.0325$.

True value of

$S(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.6000$ & $S(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 0.2000$.

Table (2.a): The AILs and CPs (%) for the MLE and Bayes estimates based on Type-II progressive hybrid censoring schemes at different time censoring and different values of (n, r) for $\alpha = 1.5, \lambda = 0.5$.

(n,r)	Method	$q_i = 40\%$			$q_i = 80\%$		
		I	II	III	I	II	III
(50,20)	MLE_{α}	9.1712 94.8	9.2595 94.8	9.5357 93.4	8.2979 94.9	9.9734 93.5	1.1542 98.9
	MLE_{λ}	2.1222 95.6	3.0856 94.5	3.6467 94.2	2.1467 95.0	2.8715 94.5	4.0792 94.5
	Bayes SE_{α}	3.7734	2.9700	2.9965	3.4764	3.1222	0.2779

		95.2	95.1	95.5	95.6	95.3	95.1
	Bayes SE_{λ}	1.1789 95.2	1.9162 95.1	2.4626 95.1	1.1578 95.0	1.7455 95.1	3.4487 95.1
	Bayes $LINEX_{\alpha}$	3.6509 95.3	3.0679 95.3	3.1637 95.5	3.5704 95.1	3.4174 95.0	0.2620 95.1
	Bayes $LINEX_{\lambda}$	1.0022 95.2	1.6061 95.1	2.0336 95.1	0.9686 95.0	1.4973 95.1	3.0657 95.3
(100,40)	MLE_{α}	7.3212 95.8	9.8467 94.3	10.8802 93.4	8.1486 95.3	8.4605 95.7	4.0190 97.9
	MLE_{λ}	1.3480 95.8	1.9165 95.2	3.3242 95.1	1.3651 95.1	2.0824 95.3	1.9326 95.8
	Bayes SE_{α}	3.6870 95.2	3.5416 95.2	3.7552 95.0	3.9321 95.7	3.3773 95.1	1.9021 95.1
	Bayes SE_{λ}	1.0343 95.0	1.3655 95.2	2.4571 95.3	0.9843 95.0	1.4126 95.1	1.5944 95.6
	Bayes $LINEX_{\alpha}$	3.7545 95.0	3.8874 95.1	4.1038 95.0	3.9482 95.1	3.3376 95.3	1.7681 95.1
	Bayes $LINEX_{\lambda}$	0.9022 95.1	1.2017 95.2	2.1934 95.3	0.8690 95.1	1.2331 95.6	1.4097 95.6
(200,80)	MLE_{α}	5.8966 96.7	8.5846 95.2	9.7924 93.8	5.5099 96.1	7.5999 94.9	1.6257 96.2
	MLE_{λ}	1.1104 95.9	1.4118 95.3	2.1707 95.6	1.0552 96.0	1.5373 95.7	1.3786 96.5
	Bayes SE_{α}	3.4812 95.3	3.9154 95.8	4.5478 95.4	3.4315 95.2	4.2517 95.5	1.3138 95.5
	Bayes SE_{λ}	0.9352 95.2	1.1593 95.5	1.9380 95.2	0.9192 95.3	1.1958 95.3	1.2235 96.6
	Bayes $LINEX_{\alpha}$	3.4846 95.2	3.8216 95.6	5.1189 95.1	3.5248 95.3	4.2573 95.0	1.2102 95.2
	Bayes $LINEX_{\lambda}$	0.8409 95.2	1.0486 96.2	1.7600 95.2	0.8295 95.5	1.1154 95.2	1.1366 96.5

Note that: $T_{q=40\%} = Q(40\%, \alpha = 1.5, \lambda = 0.5) = 0.6334$ &

$T_{q=80\%} = Q(80\%, \alpha = 1.5, \lambda = 0.5) = 1.7908$.

True value of $h(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.8606$ & $h(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 1.0325$.

True value of

$S(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.6000$ & $S(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 0.2000$.

Table (2.b): The AILs and CPs (%) of S(t) and H(t) for the MLE and Bayes estimates based on hybrid progressive Type-II censoring schemes at different time censoring and different values of (n, r) for $\alpha = 1.5, \lambda = 0.5$.

(n, r)	Method	$q_i = 40\%$			$q_i = 80\%$		
		I	II	III	I	II	III
(50,20)	$MLE_{h(t)}$	1.5381 99.9	2.4760 98.5	4.2954 98.5	2.8372 98.8	4.5402 98.5	0.0330 95.6
	$MLE_{S(t)}$	0.5521 98.5	0.6646 98.5	1.3145 98.5	0.4871 98.5	1.7674 98.5	0.0495 98.5
	Bayes $SE_{h(t)}$	1.2356 95.9	1.2299 96.2	1.0353 96.1	1.7932 95.6	1.8436 95.1	0.0393 95.7
	Bayes $SE_{S(t)}$	0.3661 98.5	0.3263 98.4	0.3415 98.2	0.3175 96.3	0.4722 95.6	0.0522 99.0
	Bayes $LINEX_{h(t)}$	0.7253 98.3	0.8868 96.7	0.5704 99.4	1.2873 95.9	0.9390 96.9	0.0376 96.2
	Bayes $LINEX_{S(t)}$	0.2971 96.0	0.3686 96.2	0.2401 95.5	0.4206 95.3	0.4765 95.5	0.0487 98.9
(100,40)	$MLE_{h(t)}$	0.8236 99.7	1.6193 98.5	2.7615 98.5	1.6640 99.9	4.1534 98.5	0.6568 98.5
	$MLE_{S(t)}$	0.3200 98.5	0.4574 98.5	0.8260 98.5	0.3286 98.5	1.2748 98.5	0.2109 98.5
	Bayes $SE_{h(t)}$	0.7638 95.8	0.9671 95.6	0.9340 97.0	1.1168 96.0	1.5966 95.1	0.6488 96.3
	Bayes $SE_{S(t)}$	0.2680 98.2	0.2735 99.7	0.3003 98.3	0.2481 97.6	0.3503 95.8	0.1649 99.4
	Bayes $LINEX_{h(t)}$	0.5495 98.3	0.8683 96.2	0.5171 99.7	0.8330 97.2	1.0464 96.3	0.3262 98.6
	Bayes $LINEX_{S(t)}$	0.2168 97.1	0.3505 96.4	0.2175 95.6	0.2375 96.0	0.4193 95.8	0.0901 96.8
(200,80)	$MLE_{h(t)}$	0.5008 99.6	0.7601 99.8	2.4558 98.5	0.7671 97.7	2.6863 98.5	0.3771 99.5
	$MLE_{S(t)}$	0.2034 99.6	0.2266 98.5	0.8162 98.5	0.1601 99.3	0.5366 98.5	0.1191 98.5
	Bayes $SE_{h(t)}$	0.5360 96.5	0.6172 96.1	0.9278 96.9	0.6959 96.4	1.2375 96.9	0.4261 95.8
	Bayes $SE_{S(t)}$	0.2050	0.1985	0.3136	0.1739	0.2229	0.1146

		98.3	99.8	95.6	98.5	96.8	98.7
	Bayes LINEX$_{h(t)}$	0.3889 96.8	0.4414 97.0	0.6060 99.4	0.5826 96.9	1.0210 96.6	0.2604 96.7
	Bayes LINEX$_{S(t)}$	0.1612 97.5	0.1466 96.5	0.2498 95.8	0.1493 97.4	0.2329 95.5	0.0634 97.1

Note that:

$T_{q=40\%} = Q(40\%, \alpha = 1.5, \lambda = 0.5) = 0.6334$ & $T_{q=80\%} = Q(80\%, \alpha = 1.5, \lambda = 0.5) = 1.7908$.

True value of

$h(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.8606$ & $h(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 1.0325$.

True value of

$S(t = 0.6334, \alpha = 1.5, \lambda = 0.5) = 0.6000$ & $S(t = 1.7908, \alpha = 1.5, \lambda = 0.5) = 0.2000$.

3.2 Real Data Set.

A real data set is analyzed for illustrative purpose as well as to assess the statistical performances of the MLEs and Bayes estimators for the extended exponential distribution under Type-II Progressive Hybrid censoring schemes.

A real-life data set is analyzed to illustrate how the proposed methodology can be applied in real life phenomenon. We shall use the real-life data set originally presented by Linhart and Zucchini (1986), which represents the failure times of the air conditioning system of an air-plane. The ordered data with n = 30 are as follows: 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71,71, 87, 90, 95, 120, 120, 225, 246 and 261. Recently, this real data set was analyzed by Singh et al. (2015a,b).

We first check whether the extended exponential distribution is suitable for analyzing this data set or not. The value of Kolmogorov–Smirnov (K–S) test statistic is calculated to judge the goodness of fit. The calculated Kolmogorov-Smirnov (K-S) distance between the empirical and the fitted extended exponential distribution is 0.1992 and its p-value is

0.1847, which indicate that this distribution can be considered as an adequate model for the given data set. The MLEs of the parameters are obtained where $\hat{\alpha} = 0.5339$ and $\hat{\lambda} = 0.0808$.

From the original data, one can generate, three Type-II progressive hybrid censoring samples with number of stages $r = 15$ at time censoring $T = 50$ and removed items R_i are assumed to as follows:

Scheme I: $R_1 = n - r$ **and** $R_2, ..., R_r = 0$ $(15, 0^{*14})$.

Scheme II: $R_1, ..., R_{\frac{r}{2}} = 0, R(\frac{r}{2} + 1) = n - r$ **and** $R(\frac{r}{2} + 2), ..., R_r = 0$. $(0^{*7}, 15, 0^{*7})$.

Scheme III: $R_1, ..., R_{r-1} = 0$ **and** $R_r = n - r$. $(0^{*14}, 15)$.

Table (3.a) and Table (3.b) give the MLEs of the parameters α and λ and calculated their associated asymptotic confidence interval at proposed schemes for Type II progressive hybrid censoring samples in the given real data set. Also, Bayes estimates under two loss functions; namely: squared error loss function and LINEX loss function, were computed by utilizing the MH algorithm under the Non-informative prior, i.e. $a = b = c = d = 0$. It is indicated that, while generating samples from the posterior distribution utilizing the MH algorithm, initial values of (α, λ) are considered as

$(\alpha^{(0)}, \lambda^{(0)}) = (\hat{\alpha}, \hat{\lambda})$ where $\hat{\alpha}, \hat{\lambda}$ are the MLEs of the parameters (α, λ) respectively. Finally, discarded 2000 burn-in samples among the total 10000 samples created from the posterior density, and subsequently obtained Bayes estimates and HPD interval. Further, the estimates of the of $S(t)$ and $H(t)$ are obtained in case of MLEs and Bayesian estimates at a specified time censoring $T = 50$.

Table (3.a): ML, Bayesian, and standard errors for real data set based on Type- II progressive hybrid censoring under various censoring <u>schemes</u>

Scheme	Parameter	MLE		SE		LINEX	
		Estimate	St.E*	Estimate	St.E	Estimate	St.E
I	α	0.5549	0.2362	0.4806	0.0131	0.4678	0.0133
	λ	0.0381	0.0346	0.0539	0.0005	0.0534	0.0004
	$h(t)$	0.0005	---	0.0131	---	0.0125	---
	$s(t)$	0.9592	---	0.4170	---	0.4324	---
II	α	0.5172	0.2108	1.0179	0.0967	0.9316	0.1041
	λ	0.0466	0.0362	0.0137	3.64 e-5	0.0137	3.62 e-5
	$h(t)$	0.0006	---	0.0141	---	0.0123	---
	$s(t)$	0.9439	---	0.4940	---	0.5336	---
III	α	2.3726	4.8814	1.8933	0.07445	1.8181	0.0801
	λ	0.0092	0.0217	0.0124	1.32e-05	0.0124	1.31e-05
	$h(t)$	5.88e-05	---	0.0362	---	0.0335	---
	$s(t)$	0.9964	---	0.2234	---	0.2447	---

Scheme	Parameter	MLE		SE		LINEX	
		Estimate	St.E*	Estimate	St.E	Estimate	St.E
I	α	0.5549	0.2362	0.4806	0.0131	0.4678	0.0133
	λ	0.0381	0.0346	0.0539	0.0005	0.0534	0.0004
	$h(t)$	0.0005	---	0.0131	---	0.0125	---
	$s(t)$	0.9592	---	0.4170	---	0.4324	---
II	α	0.5172	0.2108	1.0179	0.0967	0.9316	0.1041
	λ	0.0466	0.0362	0.0137	3.64 e-5	0.0137	3.62 e-5
	$h(t)$	0.0006	---	0.0141	---	0.0123	---
	$s(t)$	0.9439	---	0.4940	---	0.5336	---
III	α	2.3726	4.8814	1.8933	0.07445	1.8181	0.0801
	λ	0.0092	0.0217	0.0124	1.32e-05	0.0124	1.31e-05
	$h(t)$	5.88e-05	---	0.0362	---	0.0335	---
	$s(t)$	0.9964	---	0.2234	---	0.2447	---

* St.E – Standard error .

Table (3.b): Associated interval estimates for ML and Bayesian for real data set based on Type II progressive hybrid censoring under various censoring schemes

Scheme	Parameter	Asy-CI MLE*	HPD Bayes SE	HPD Bayes LINEX
I	α λ	(0.2701, 2.3603) (0.0037, 0.2061)	(0.2732, 0.7144) (0.0236, 0.0963)	(0.2702, 0.7145) (0.0233, 0.0971)
	$h(t)$ $s(t)$	(0.0000, 0.0067) (0.5061, 1.4123)	(0.0014, 0.0248) (0.2169, 0.6170)	(0.0010, 0.0240) (0.2274, 0.6375)
II	α λ	(0.2397, 1.4941) (0.0076, 0.2138)	(0.4960, 1.6629) (0.0054, 0.0266)	(0.4958, 1.6631) (0.0050, 0.0268)
	$h(t)$ $s(t)$	(0.0000, 0.0068) (0.4920, 1.3958)	(0.0000, 0.0797) (0.0000, 2.1319)	(0.0000, 0.0670) (0.0000, 2.0676)
III	α λ	(0.0000, 6.5421) (0.0000, 0.0304)	(1.3995, 2.3769) (0.0062, 0.0201)	(1.3980, 2.3762) (0.0064, 0.0203)
	$h(t)$ $s(t)$	(0.0000, 0.0608) (0.000, 4.6228)	(0.0000, 0.2218) (0.0000, 1.4292)	(0.0000, 0.2151) (0.0000, 1.5718)

* Asy CI- Asymptotic confidence interval.

The convergence of MCMC estimation in case of scheme I of Type-II progressive hybrid censoring can be showed for α and λ in Figure (1)

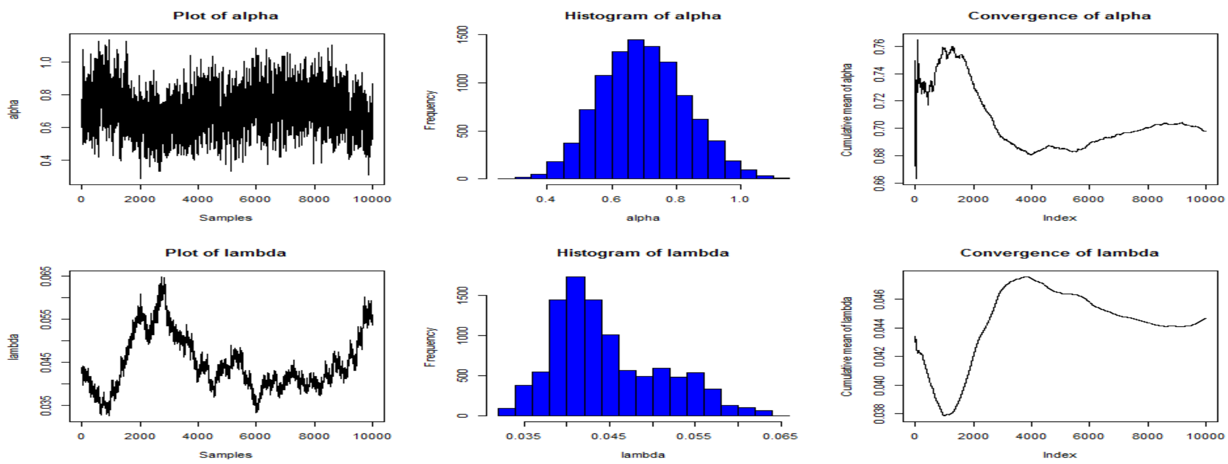

Figure (1) : Convergence of MCMC estimators for α and λ using MH algorithm

5. Concluding Remarks

In this article, The estimation of the unknown parameters and reliability and hazard functions of an extended exponential distribution under Type-II PHCS is considered. Different estimates for the unknown parameters using ML and Bayesian approaches are computed. The asymptotic confidence intervals are also constructed. Bayes estimates of unknown parameters are developed using MH algorithm with respect to gamma prior distributions under SE and LINEX loss functions. HPD intervals based on MH procedure are considered. A real data set and simulation study was conducted to examine and compare the performance of the proposed methods for different; sample sizes, censoring times and censoring schemes.

From the results of simulation study we reported some comments observed from numerical results.

- When n is increasing: the bias and MSE of the MLE estimate of α is decreasing at Scheme I but increasing at Scheme II and Scheme III but the bias of the MLE estimate of λ is decreasing at all schemes of removing item. Also, the bias and MSE of the Bayes estimate of α under the loss functions SE and LINEX is decreasing at all schemes of removing and the bias and MSE of the Bayes estimate of λ under the loss functions SE and LINEX is decreasing at all schemes of removing item.
- When T is increasing: the bias and MSE of the MLE estimate of α and λ is decreasing at all schemes of removing item. But, the bias and MSE of the Bayes estimate of α and λ under the loss functions SE and LINEX is increasing at all schemes of removing.
- The average interval lengths and associated coverage probabilities of highest posterior density credible intervals are better than those of SE loss function and the MLEs.

- For the estimates of S (t) and H (t), it is notice that the MLEs is better than the Bayes estimates under two error loss functions.
- The performance of the estimates in Scheme II is better than other two schemes.

From the results of real data we reported some comments observed from numerical results.

- The performance of Bayes estimates for the parameters α and λ obtained under squared error loss function is better than the performance of Bayes estimates obtained under LINEX loss function and the MLEs.
- For the estimates of S(t) and H(t), it is notice that the MLEs is better than the Bayes estimates under two error loss functions.
- Furthermore, the performance of the estimates in scheme I is better than other two schemes (III and II).

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