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Fixed point results on complex-valued metric spaces for fuzzy mappings

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ABSTRACT

Banach Contraction Principle (BCP) is a fundamental result in metric fixed point theory and it is a very powerful tool in solving the existence problems in pure and applied sciences. Also, the fuzzy set theory has many applications in various branches of engineering, mathematical sciences including artificial intelligence, control engineering, computer science, management science etc., see [1]. The aim of this paper is to study a common fixed point results for fuzzy mappings under implicit relation in a complex-valued metric space. we introduces a new class of an implicit relation to prove a common fixed point theorems for fuzzy mappings in this paper and constructed some examples to illustrate the main theorem. Also, we gave the consequences of our main result. The results have been reached in our current research work that consider applying for an integral type contractive condition, these results are extention of many results in this field.

KeyWords:
Fuzzy mapping, Fuzzy fixed point, Common fuzzy fixed point , Complex-valued metric space.

1. PRELIMINARIES

In 2011, Azam *et al.* [2] introduced complex valued metric spaces and established fixed point theorems for a pair of mappings satisfying contractive type conditions. In 1965, Zadeh [3] introduced the concepts of fuzzy sets. Motivated by the work of Popa [4], Azam *et al.*[2] and by the ongoing research in this direction, we study a common fixed point results for fuzzy mappings under implicit relation in a complex-valued metric spaces.

Now, we present some basic definitions and lemmas that help us in our sequel.

Definition 1.1: [2] Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as below:

$$z_1 \precsim z_2 \quad \text{iff} \qquad \text{Re}\{z_1\} \leq \text{Re}\{z_2\}, \qquad \text{Im}\{z_1\} \leq \text{Im}\{z_2\}.$$

So, $z_1 \precsim z_2$ if one of the following conditions holds:

(i) $\text{Re}\{z_1\} = \text{Re}\{z_2\}$ and $\text{Im}\{z_1\} = \text{Im}\{z_2\}$,

(ii) $\text{Re}\{z_1\} < \text{Re}\{z_2\}$ and $\text{Im}\{z_1\} = \text{Im}\{z_2\}$,

(iii) $\text{Re}\{z_1\} = \text{Re}\{z_2\}$ and $\text{Im}\{z_1\} < \text{Im}\{z_2\}$,

(iv) $\mathrm{Re}\{z_1\} < \mathrm{Re}\{z_2\}$ and $\mathrm{Im}\{z_1\} < \mathrm{Im}\{z_2\}$.

$z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of **(ii)**, **(iii)** and **(iv)** is satisfied. Also, $z_1 \prec z_2$ if only **(iv)** is satisfied. The symbol $\prec$ means that only **(iv)** is satisfied.

Definition 1.2: [2] A mapping $d: Z \times Z \to \mathbb{C}$, where Z is a nonempty set, is said to be *complex-valued metric* on Z if it satisfies the following conditions:

(i) $\theta \precsim d(x,y)$;

(ii) $d(x,y) = \theta$ if and only if $x = y$;

(iii) $d(x,y) = d(y,x)$;

(iv) $d(x,y) \precsim d(x,z) + d(z,y)$.

$\forall x,y,z \in Z$, θ is zero vector and a complex-valued metric space is denoted by (Z,d).

Let X be a nonempty set, a fuzzy set A in X is characterized by a function $\mu_A: X \to [0,1]$ is called ***membership function*** of A, " such that $x \in X$ a real number in the interval [0, 1] and the value of μ_A at x representing the grade of membership of x in A". Clearly, any crisp subset A of X is fuzzy set if $\mu_A(x) = 1$, when $x \in A$ and $\mu_A(x) = 0$ otherwise.

Let Y be a nonempty subset of a vector space V and D be a nonempty set, a mapping $F: D \to \mathfrak{F}(Y)$, where $\mathfrak{F}(Y)$ be the collection of all fuzzy sets of Y, is called ***a fuzzy mapping***, and $F(x),\ x \in D$ is a ***fuzzy set*** in $\mathfrak{F}(Y)$, denoted by F_x and $F_x(y),\ y \in Y$ is the ***grade of membership*** of y in F_x, see for details .

Let $B \in \mathfrak{F}(X)$ and $\alpha \in [0,1]$, then the set

$$B_\alpha = \{u \in X: B(u) \geq \alpha\}$$

is called an α-*cut* or (α-*level*) set of B.

Definition 1.3: **[5]** Fuzzy set A in X is an ***approximate quantity*** if and only if its α-level set is a nonempty compact subset of X for each $\alpha \in [0,1]$. The set of all approximate quantities is denoted by $W^*(X)$, is a sub collection of $\mathfrak{F}(X)$.

Definition 1.4: [6] Assume that X represents an arbitrary set and Y represents a metric space. If $G: X \to W(Y)$, then G is said to be a fuzzy mapping. A fuzzy mapping G is a fuzzy subset on $X \times Y$ with a membership function $G(z)(x)$. The function $G(z)(x)$ is the grade of membership of $x \in G(z)$.

Definition 1.5: Assume that (Z,d) represents a complex-valued metric space. Azam, etc in [6, 7] represented

$$\mathfrak{s}(q) = \{w \in \mathbb{C}: q \precsim w\} \quad \text{for } q \in \mathbb{C},$$

$$\mathfrak{s}(z,B) = \bigcup_{s\in B} \mathfrak{s}(d(z,s)) = \bigcup_{s\in B} \{w \in \mathbb{C}: d(z,s) \precsim w\} \ \text{ for} B \in CB(Z) \text{ and } s \in B,$$

$$\mathfrak{s}(A,B) = \Big(\bigcap_{r\in A} \mathfrak{s}(r,B)\Big) \cap \Big(\bigcap_{s\in B} \mathfrak{s}(s,A)\Big) \ \text{ for } A,B \in CB(Z),$$

where $CB(Z)$ is the family of all nonempty closed bounded subsets of Z.

***Remark*: [7]** If $\mathbb{C}$ is replaced by $\mathbb{R}$ in definition 1.2 and $\leq$ instead of $\precsim$, then (Z, d) is metric space. Also Hausdorff distance induced by d is

$$H(A,B) = \inf\ \mathfrak{s}(A,B) \quad \text{such that } A,B \in CB(Z).$$

Lemma 1.1: [9] Let (Z, d) be a complex-valued metric space, $A \subseteq Z$, then

$$\overline{A} = \{z \in Z: d(z, A) = \theta\},$$

such that $d(z, A) = \inf_{x\in A} d(x, z)$**.**

Also, A is closed set if and only if $z \in \overline{A} = A$.

Definition 1.6: [5] Assume that (Z, d) is a complex-valued metric space and $G, F: Z \to W^*(Z)$ are fuzzy mappings. A point $z \in Z$ is said to be a **fuzzy fixed point** of G if $z \in \{Gz\}_\alpha$ for some $\alpha \in [0,1]$ and z is said to be a **common fuzzy fixed point** of G, F if $z \in \{Gz\}_\alpha \cap \{Fz\}_\alpha$.

Lemma 1.2: [11] Suppose that $\{y_n\}$ is a sequence in Z and $h \in [0,1)$. If $z_n = d(y_n, y_{n+1})$ satisfies $z_n \precsim hz_{n-1}$ $\forall n \in \mathbb{N}$, then $\{y_n\}$ is a Cauchy sequence.

Definition 1.7: [12] The max. function for complex numbers with partial order relation $\precsim$ is defined as

$$\max\{z_2, z_3\} = z_3 \text{ iff } |z_2| \leq |z_3| \qquad \forall z_2, z_3 \in \mathbb{C}$$

2. IMPLICIT RELATION

Following Popa [4], we introduce a new class of an implicit relation to prove common fixed point theorems for fuzzy mappings in the next section.
Let φ be the family of all complex continuous mappings $\varrho: \mathbb{C}_+^6 \to \mathbb{C}_+$ satisfy the following properties as below:

($\boldsymbol{\varrho_1}$) ϱ is non-decreasing in the 1^{st} variable and non-increasing in the 2^{nd}, 3^{rd}, 4^{th}, 5^{rd} and 6^{th} coordinate variables,

($\boldsymbol{\varrho_2}$) there exists $h \in [0,1)$ such that for every $u, v \succsim \theta$, $k \in (1,2]$ with
($\boldsymbol{\varrho_{21}}$) $\varrho(u, v, v, u, u+v, \theta) \precsim \theta$ or
($\boldsymbol{\varrho_{22}}$) $\varrho(u, v, u, v, \theta, u+v) \precsim \theta$ implies $u \precsim hv$.

($\boldsymbol{\varrho_3}$) $\varrho(u, u, \theta, \theta, u, u) \succ \theta \quad \forall u \succ \theta$.

($\boldsymbol{\varrho_4}$) $\varrho(u, u, ku, \theta, u, u) \succ \theta \quad \forall u \succ \theta$.

($\boldsymbol{\varrho_5}$) $\varrho(v, v, \theta, kv, v, v) \succ \theta \;\; \forall v \succ \theta$.

Example 2.1: A function $\varrho: \mathbb{C}_+^6 \to \mathbb{C}_+$ defined as

$$\varrho(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{3}{2}z_1 - \frac{1}{2}(z_5 + z_6)$$

(ϱ_1) is obvious.

(ϱ_2) $\varrho(u, v, v, u, u+v, \theta) = \frac{3}{2}u - \frac{1}{2}(u+v) = \frac{3}{2}u - \frac{1}{2}u - \frac{1}{2}v \precsim \theta \Rightarrow u \precsim \frac{1}{2}v.$

(ϱ_3) $\varrho(u, u, \theta, \theta, u, u) = \frac{3}{2}u - \frac{1}{2}u - \frac{1}{2}u = \frac{1}{2}(u) \succ \theta. \;\; \forall u \succ \theta$

(ϱ_4) $\varrho(u,u,ku,\theta,u,u)=\frac{3}{2}u-\frac{1}{2}u-\frac{1}{2}u=\frac{1}{2}(u)\succ\theta\ \ \forall u\succ\theta.$

(ϱ_5) $\varrho(v,v,\theta,kv,v,v)=\frac{3}{2}v-\frac{1}{2}v-\frac{1}{2}v=\frac{1}{2}v\succ\theta\ \forall v\succ\theta\ \ k\in(1,2].$

Definition 2.1: Assume that a fuzzy mapping G and a self mapping g from a complex-valued metric space into itself are known as weakly g-biased iff $Gx=gx$ implies $d(gGx,gx)\precsim d(Ggx,Gx)$, $\forall x\in C(G,g)$.

3. MAIN RESULTS

In the following, we will introduce our main results.

Theorem 3.1: Assume that g,f represent two self mappings from a complex-valued metric space (Z,d) into itself and G,F are fuzzy mappings from Z into $W^*(Z)$ such that

(i) $\{GZ\}_\alpha\subseteq f(Z)$, $\{FZ\}_\alpha\subseteq g(Z)$

(ii) the pairs (G,g) and (F,f) are weakly g-biased and weakly f-biased mappings respectively,

(iii) $g(Z)$ is z_0 joint orbitally complete for some $z_0\in Z$, $\forall a,b\in Z$. If there is $\varrho\in\varphi$ such that

$$\varrho\begin{pmatrix}s(\{Fa\}_\alpha,\{Gb\}_\alpha),d(fa,gb),d(fa,\{Fa\}_\alpha),\\ d(gb,\{Gb\}_\alpha),d(fa,\{Gb\}_\alpha),d(gb,\{Fa\}_\alpha)\end{pmatrix}\precsim\theta. \tag{1}$$

Then $\boldsymbol{G},\boldsymbol{F},\boldsymbol{g}$ **and** $\boldsymbol{f}$ have a common fixed point.

Proof: We build an orbit $O(G,F,g,f,z_0)$ with two sequences $\{y_n\}$ and $\{x_n\}$ in Z, where $z_0\in Z$,

$$y_{2n-1}=gz_{2n-1}\subseteq Fz_{2n-2}\quad\text{and}\quad y_{2n}=fz_{2n}\subseteq Gz_{2n-1}\,. \tag{2}$$

Now, we prove that $\{y_n\}$ is Cauchy sequence. As

$$y_1=gz_1\subseteq Fz_0\quad\text{and}\quad\{Gz_1\}_\alpha,\{Fz_0\}_\alpha\in CP(Z)\ \exists\ y_2=fz_2\subseteq Gz_1. \tag{3}$$

Taking $a=z_0$ and $b=z_1$ in (1) and by using (2), (3) and triangle inequality. Also, we suppose that $\mathfrak{s}(\{Fz_0\}_\alpha,\{Gz_1\}_\alpha)\succsim d(fz_1,gz_2)=d(y_1,y_2)$, then we have

$$\varrho(d(y_1,y_2),d(y_0,y_1),d(y_0,y_1),d(y_1,y_2),d(y_0,y_1)+d(y_1,y_2),\theta)\precsim$$

,

$$\varrho\begin{pmatrix}\mathfrak{s}(\{Fz_0\}_\alpha,\{Gz_1\}_\alpha),d(fz_0,gz_1),d(fz_0,\{Fz_0\}_\alpha),\\ d(gz_1,\{Gz_1\}_\alpha),d(fz_0,\{Gz_1\}_\alpha),d(gz_1,\{Fz_0\}_\alpha)\end{pmatrix}\precsim\theta. \tag{4}$$

From $(\varrho_{21})\Rightarrow$

$$\exists\ h\in[0,1)\ :\ d(y_1,y_2)\precsim hd(y_0,y_1).$$

Similarity, $d(y_2, y_3) \precsim hd(y_1, y_2)$. By using induction, we get $d(y_n, y_{n+1}) \precsim h^n d(y_0, y_1)$. From lemma 1.2, then $\{y_n\}$ is a Cauchy sequence. As $\{y_{2n-1}\}$ is a Cauchy sequence in $g(Z)$ and $g(Z)$ is z_0 joint orbitally complete, then

$$\exists\, z \in Z \;:\; y_{2n-1} \to z = ft \;:\; z \in g(Z),\; t \in Z \text{ and } y_{2n} \to z \text{ as } n \to \infty$$

Now, we must prove that $z \in \{Ft\}_\alpha$. Since

$$\varrho(d(y_{2n-1}, Ft), d(z, y_{2n-2}), d(z, Ft), d(y_{2n-2}, y_{2n-1}), d(z, y_{2n-1}), d(y_{2n-2}, Ft)) \precsim$$

$$\varrho\begin{pmatrix} \mathfrak{s}(Gz_{2n-2}, Ft), d(ft, gz_{2n-2}), d(ft, Ft), \\ d(gz_{2n-2}, Gz_{2n-2}), d(ft, Gz_{2n-2}), d(gz_{2n-2}, Ft) \end{pmatrix} \precsim \theta. \tag{5}$$

at $n \to \infty$

$$\varrho(d(z, Ft), d(z, z), d(z, Ft), d(z, z), d(z, z), d(z, Ft)) =$$

$$\varrho(d(z, Ft), \theta, d(z, Ft), \theta, \theta, d(z, Ft)) \precsim \theta \tag{6}$$

From (ϱ_{22}) and lemma 1.1, then we get $d(z, Ft) \precsim h\theta = \theta \Rightarrow z \in Ft$, where

$$z = ft \in Ft \subseteq g(Z), \text{ Also } \exists\, w \in Z : z = gw \in Gw \subseteq f(Z). \tag{7}$$

Let
$$ft = Ft \text{ and } gw = Gw \text{ for some } w, t \in Z. \tag{8}$$

From (7) and (8), we can deduce $z = ft = Ft = gw = Gw$. Now we see that z is fixed point of f. Suppose that $fz \neq z$, put $a = z$ and $b = w$ in (1)

$$\begin{gathered} \varrho(d(Fz, z), d(fz, z), d(fz, z) + d(z, Fz), d(z, z), d(fz, z), d(z, Fz)) \precsim \\ \varrho(\mathfrak{s}(Fz, z), d(fz, z), d(fz, Fz), d(z, z), d(fz, z), d(z, Fz)) = \\ \varrho(\mathfrak{s}(Fz, Gw), d(fz, gw), d(fz, Fz), d(gw, Gw), d(fz, Gw), d(gw, Fz)) \precsim \theta. \end{gathered} \tag{9}$$

Since (f, F) is f-weakly biased and from (8), we get

$$d(fFt, ft) \precsim d(Fft, Ft) \Longrightarrow d(fz, z) \precsim d(Fz, z) \tag{10}$$

From (9) and (10), we get the following:

$$\begin{gathered} \varrho(d(Fz, z), d(Fz, z), 2d(Fz, z), \theta, d(Fz, z), d(Fz, z)) \precsim \\ \varrho(d(Fz, z), d(fz, z), 2d(Fz, z), \theta, d(fz, z), d(Fz, z)) \precsim \\ \varrho(\mathfrak{s}(Fz, z), d(fz, z), d(Fz, z) + d(Fz, z), \theta, d(fz, z), d(Fz, z)) \precsim \\ \varrho(\mathfrak{s}(Fz, Gw), d(fz, gw), d(fz, z) + d(Fz, z), \theta, d(fz, Gw), d(Fz, gw)) \precsim \theta. \end{gathered} \tag{11}$$

From (ϱ_4), it contradicts with assumption, then $fz = z, z \in \{Fz\}_\alpha$. (i.e., z is a common fixed point of F, f).
Next, we prove that z is common fixed point of G, g. Suppose that $gz \neq z$. As (G, g) is g-weakly biased and from (8), we get

$$d(gGw, gw) \precsim d(Ggw, Gw) \Longrightarrow d(gz, z) \precsim d(Gz, z) \tag{12}$$

From (1), (12) and by using triangle inequality, we obtain

$$\begin{aligned}
&\varrho(d(z, Gz), d(z, Gz), \theta, 2d(z, Gz), d(z, Gz), d(Gz, z)) = \\
&\varrho(d(z, Gz), d(z, Gz), d(z, z), d(Gz, z) + d(z, Gz), d(z, Gz), d(Gz, z)) \precsim \\
&\varrho(\mathfrak{s}(z, Gz), d(z, gz), d(z, z), d(gz, z) + d(z, Gz), d(z, Gz), d(gz, z)) = \\
&\varrho(\mathfrak{s}(Ft, Gz), d(ft, gz), d(ft, Ft), d(gz, z) + d(z, Gz), d(ft, Gz), d(gz, Ft)) \precsim \\
&\varrho(\mathfrak{s}(Ft, Gz), d(ft, gz), d(ft, Ft), d(gz, Gz), d(ft, Gz), d(gz, Ft)) \precsim \theta.
\end{aligned} \tag{13}$$

From (ϱ_5), it contradicts with assumption, then $gz = z, z \in \{Gz\}_\alpha$. (i.e., z is a common fixed point of G, g). So G, F, g and f have a common fixed point.

Remark: The content of the theorem 3.1 remains correct if the completeness of $g(Z)$ is replaced by the completeness of $f(Z)$.

Example 3.1: Suppose that (Z, d) is a complex-valued metric space defined as $d(z_1, z_2) = i|z_1 - z_2|$ such that $z_1, z_2 \in Z$, $Z = [0,1]$ and f, g are two mappings from a complex-valued metric space into itself defined by:

$$gz_1 = \frac{2}{3}z_1 \quad , \qquad fz_1 = \frac{z_1}{5}$$

and G, F are two fuzzy mappings as mentioned below:

$$(F\theta)(z_1) = \begin{cases} 1 & \text{if } z_1 = \theta \\ \frac{1}{5} & \text{if } \theta \prec z_1 \precsim \frac{z_2}{5} \\ \frac{1}{10} & \text{if } \frac{z_2}{5} \prec z_1 \precsim 1 \end{cases} \quad , \qquad (G\theta)(z_1) = \begin{cases} 1 & \text{if } z_1 = \theta \\ \frac{1}{3} & \text{if } \theta \prec z_1 \precsim \frac{z_2}{10} \\ \frac{1}{9} & \text{if } \frac{z_2}{10} \prec z_1 \precsim 1 \end{cases}$$

Now, for $\alpha = 1$, then $\{F\theta\}_1 = \{G\theta\}_1 = f(\theta) = g(\theta) = \theta$. The pairs (f, F) and (g, G) are weakly f biased and weakly g-biased mappings respectively. Also $\varrho(z_1, z_2, z_3, z_4, z_5, z_6) = \theta$, then $\theta = g\theta = f\theta = \{G\theta\}_\alpha = \{F\theta\}_\alpha$, θ is a common fixed point for mappings G, F, g, f.

Example 3.2: Suppose that (Z, d) is a complex-valued metric space, f, g are two mappings from a complex-valued metric space into itself is defined by: $gz_2 = fz_2 = z_2$

$$d(\{Fz_2\}_\alpha, z_1) = \begin{cases} \theta & \text{if } z_1 \in \{Fz_2\}_\alpha \text{or } \{z_1\} = \{Fz_2\}_\alpha \\ \\ i|z_1 - z_2| & \text{if } otherwise \end{cases}$$

and G, F are two fuzzy mappings defined as:

$$(Fz_2)(z_1) = \begin{cases} \alpha & \text{if } \theta \precsim z_1 \precsim z_2 \\ \\ \frac{\alpha}{4} & otherwise \end{cases} \quad , \quad (Gz_2)(z_1) = \begin{cases} \alpha & \text{if } \theta \precsim z_1 \precsim z_2 \\ \\ \frac{\alpha}{3} & \text{if } z_2 \prec z_1 \precsim 1 \end{cases}$$

$\{Fz_2\}_\alpha = [\theta, z_2]$, $\{Gz_2\} = [\theta, z_2]$. If $z_2 = \theta$, the pairs (G, g) and (F, f) are weakly g-biased and weakly f-biased mappings respectively. Also, $\theta = f\theta = g\theta \in \{G\theta\}_\alpha \cap \{F\theta\}_\alpha$ (i.e., θ is common fixed point for mappings G, F, g, f).

Theorem 3.2: Let $Y \subseteq Z$ (complex-valued metric space) and $g, f: Y \to Z$ and $\{G_n\}_\alpha: Y \to W^*(Z)$ such that

(i) $\{G_i Z\}_\alpha \subset f(Z), \{G_j Z\}_\alpha \subset g(Z)$,

(ii) the pairs (G_i, g) and (G_j, f) are weakly g-biased and weakly f-biased mappings respectively,

(iii) $g(Z)$ is Z_0 joint orbitally complete for some $z_0 \in Z$.

If there is $\varrho \in \varphi$ where

$$\varrho\begin{pmatrix} d(\{G_i a\}_\alpha, \{G_j b\}_\alpha), d(fa, gb), d(fa, \{G_i a\}_\alpha), \\ \\ d(gb, \{G_j b\}_\alpha), d(fa, \{G_j b\}_\alpha), d(gb, \{G_i a\}_\alpha) \end{pmatrix} \precsim \theta. \tag{14}$$

$\forall\, n \in \mathbb{N}, \forall a, b \in Z, j = 2n + 1, i = 2n + 2$. Then (G_i, g) and (G_j, f) have a common fixed point.

Proof: From theorem 3.1, let $G_i = F$ and $G_j = G$, then the proof is complete.

4. INTEGRAL TYPE RESULTS

Let $\hat{\varphi}$ be the family of all complex continuous mappings $\hat{\varrho}: \mathbb{C}_+^6 \to \mathbb{C}_+$ satisfying the following properties:

($\hat{\varrho}_1$) $\hat{\varrho}$ is non-decreasing in the 1^{st} variable and non-increasing in the 2^{nd}, 3^{rd}, 4^{th} 5^{th} and 6^{th} coordinate variables,

($\hat{\varrho}_2$) $\exists\, h \in [0,1)$ such that for every $u, v \succsim \theta$ with

$$(\hat{\varrho}_{21}) \int_\theta^{\hat{\varrho}\left(\int_\theta^u \vartheta(r)dr, \int_\theta^v \vartheta(r)dr, \int_\theta^v \vartheta(r)dr, \int_\theta^u \vartheta(r)dr, \int_\theta^{u+v} \vartheta(r)dr, \theta\right)} \psi(s)ds \precsim \theta$$

Or

$$\left(\hat{\varrho}_{22}\right) \int_\theta^{\hat{\varrho}\left(\int_\theta^u \vartheta(r)dr, \int_\theta^v \vartheta(r)dr, \int_\theta^u \vartheta(r)dr, \int_\theta^v \vartheta(r)dr, \theta, \int_\theta^{u+v} \vartheta(r)dr\right)} \psi(s)ds \precsim \theta \text{ implies}$$

$$\int_\theta^u \vartheta(r)dr \precsim \int_\theta^v \vartheta(r)dr$$

$$(\hat{\varrho}_3)\int_0^{\hat{\varrho}\left(\int_\theta^u \vartheta(r)dr,\int_\theta^u \vartheta(r)dr,\theta,\theta,\int_\theta^u \vartheta(r)dr,\int_\theta^u \vartheta(r)dr\right)}\psi(s)ds \succ \theta \quad \forall u \succ \theta.$$

$$(\hat{\varrho}_4)\int_\theta^{\hat{\varrho}\left(\int_\theta^u \vartheta(r)dr,\int_\theta^u \vartheta(r)dr,\int_\theta^{ku} \vartheta(r)dr,\theta,\int_\theta^u \vartheta(r)dr,\int_\theta^u \vartheta(r)dr\right)}\psi(s)ds \succ \theta \quad \forall u \succ \theta.$$

$$(\hat{\varrho}_5)\int_\theta^{\hat{\varrho}\left(\int_\theta^v \vartheta(r)dr,\int_\theta^v \vartheta(r)dr,\theta,\int_\theta^{kv} \vartheta(r)dr,\int_\theta^v \vartheta(r)dr,\int_\theta^v \vartheta(r)dr\right)}\psi(s)ds \succ \theta \quad \forall v \succ \theta.$$

Where $\psi,\vartheta:\mathbb{C}_+\to\mathbb{C}_+$ is a summable non negative Lebesgue integrable function such that for each $\epsilon \succsim \theta$, $\int_\theta^\epsilon \psi(s)ds \succsim \theta$ and $\int_\theta^\epsilon \vartheta(r)dr \succsim \theta$. Note that if $\psi(s)ds = 1, \vartheta(r)dr = 1$, then $\hat{\varrho} \Rightarrow \varrho$.

Theorem 4.1: Assume that g, f are two self mappings from a complex-valued metric space (Z,d) into itself and G, F are fuzzy mappings from Z into $W^*(Z)$ such that

(i) $\{GZ\}_\alpha \subseteq f(Z)$, $\{FZ\}_\alpha \subseteq g(Z)$,

(ii) the pairs (G,g) and (F,f) are weakly g-biased and weakly f-biased mappings respectively,

(iii) $g(Z)$ is z_0 joint orbitally complete for some $z_0 \in Z$. If there is $\hat{\varrho} \in \hat{\varphi}$ such that $\forall a,b \in Z$,

$$\int_\theta^{\hat{\varrho}\begin{pmatrix}\int_\theta^{d(\{Fa\}_\alpha,\{Gb\}_\alpha)}\vartheta(r)dr,\int_\theta^{d(fa,gb)}\vartheta(r)dr,\int_\theta^{d(fa,\{Fa\}_\alpha)}\vartheta(r)dr,\\ \int_\theta^{d(gb,\{Gb\}_\alpha)}\vartheta(r)dr,\int_\theta^{d(fa,\{Gb\}_\alpha)}\vartheta(r)dr,\int_\theta^{d(gb,\{Fa\}_\alpha)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim \theta. \tag{15}$$

Then G, F, g and f have a common fixed point.

Proof : We build an orbit $O(G,F,g,f,z_0)$ with two sequences $\{y_n\}$ and $\{x_n\}$ in Z, where $z_0 \in Z$,

$$y_{2n-1} = gz_{2n-1} \subseteq Fz_{2n-2} \quad \text{and} \quad y_{2n} = fz_{2n} \subseteq Gz_{2n-1}. \tag{16}$$

Now, we prove that $\{y_n\}$ is Cauchy sequence. As

$$y_1 = gz_1 \subseteq Fz_0 \quad \text{and} \quad \{Gz_1\}_\alpha, \{Fz_0\}_\alpha \in CP(Z) \exists\, y_2 = fz_2 \subseteq Gz_1. \tag{17}$$

Put $a = z_0$ and $b = z_1$ in (15) and by using (16), (17) and triangle inequality. Also, we suppose that $d(\{Fz_0\}_\alpha,\{Gz_1\}_\alpha) \succsim d(fz_1, gz_2) = d(y_1,y_2)$, then we have

$$\int_\theta^{\hat{\varrho}\begin{pmatrix}\int_\theta^{d(y_1,y_2)}\vartheta(r)dr,\int_\theta^{d(y_0,y_1)}\vartheta(r)dr,\int_\theta^{d(y_0,y_1)}\vartheta(r)dr,\\ \int_\theta^{d(y_1,y_2)}\vartheta(r)dr,\int_\theta^{d(y_0,y_1)+d(y_1,y_2)}\vartheta(r)dr,\theta\end{pmatrix}}\psi(s)ds \precsim$$

$$\int_\theta^{\hat{\varrho}\begin{pmatrix}\int_\theta^{s(\{Fz_0\}_\alpha,\{Gz_1\}_\alpha)}\vartheta(r)dr,\int_\theta^{d(fz_0,gz_1)}\vartheta(r)dr,\int_\theta^{d(fz_0,\{Fz_0\}_\alpha)}\vartheta(r)dr,\\ \int_\theta^{d(gz_1,\{Gz_1\}_\alpha)}\vartheta(r)dr,\int_\theta^{d(fz_0,\{Gz_1\}_\alpha)}\vartheta(r)dr,\int_\theta^{d(gz_1,\{Fz_0\}_\alpha)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim \theta. \tag{18}$$

From $(\varrho_2) \Rightarrow$

$$\exists h \in [0,1) \ :\ \int_\theta^{d(y_1,y_2)}\vartheta(r)dr \precsim h\int_\theta^{d(y_0,y_1)}\vartheta(r)dr \tag{19}$$

Similarity, we get that $\int_{\theta}^{d(y_2,y_3)} \vartheta\,(r)dr \precsim h \int_{\theta}^{d(y_1,y_2)} \vartheta\,(r)dr$ and by using induction, we obtain $\int_{\theta}^{d(y_n,y_{n+1})} \vartheta\,(r)dr \precsim h^n \int_{\theta}^{d(y_0,y_1)} \vartheta\,(r)dr$. Thus

$$\begin{aligned}\int_{\theta}^{d(y_n,y_m)} \vartheta\,(r)dr &\precsim \sum_{i=n}^{m-1} \int_{\theta}^{d(y_i,y_{i+1})} \vartheta\,(r)dr \\ &\precsim \frac{h^n}{1-h} \int_{\theta}^{d(y_0,y_1)} \vartheta\,(r)dr\end{aligned} \tag{20}$$

Therefore $\lim_{m,n\to\infty} d(y_n, y_m) = \theta$, and then $\{y_n\}$ is a Cauchy sequence.
As $\{y_{2n-1}\}$ is a Cauchy sequence in $g(Z)$ and $g(Z)$ is z_0 joint orbitally complete, then

$$\exists\, z \in Z \,:\, y_{2n-1} \to z = ft \,:\, z \in g(Z),\ t \in Z \text{ and } y_{2n} \to z \text{ as } n \to \infty$$

Now, we prove that $z \in \{Ft\}_{\alpha}$. Since

$$\int_{\theta}^{\hat{\varrho}\left(\substack{\int_{\theta}^{d(y_{2n-1},Ft)} \vartheta(r)dr,\int_{\theta}^{d(z,y_{2n-2})} \vartheta(r)dr,\int_{\theta}^{d(z,Ft)} \vartheta(r)dr,\\ \int_{\theta}^{d(y_{2n-2},y_{2n-1})} \vartheta(r)dr,\int_{\theta}^{d(z,y_{2n-1})} \vartheta(r)dr,\int_{\theta}^{d(y_{2n-2},Ft)} \vartheta(r)dr}\right)} \psi\,(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\left(\substack{\int_{\theta}^{s(Gz_{2n-2},Ft)} \vartheta(r)dr,\int_{\theta}^{d(ft,gz_{2n-2})} \vartheta(r)dr,\int_{\theta}^{d(ft,Ft)} \vartheta(r)dr,\\ \int_{\theta}^{d(gz_{2n-2},Gz_{2n-2})} \vartheta(r)dr,\int_{\theta}^{d(ft,Gz_{2n-2})} \vartheta(r)dr,\int_{\theta}^{d(gz_{2n-2},Ft)} \vartheta(r)dr}\right)} \precsim \theta \tag{21}$$

at $n \to \infty$

$$\int_{\theta}^{\hat{\varrho}\left(\int_{\theta}^{d(z,Ft)} \vartheta(r)dr,\int_{\theta}^{d(z,z)} \vartheta(r)dr,\int_{\theta}^{d(z,Ft)} \vartheta(r)dr,\int_{\theta}^{d(z,z)} \vartheta(r)dr,\int_{\theta}^{d(z,z)} \vartheta(r)dr,\int_{\theta}^{d(z,Ft)} \vartheta(r)dr\right)} \psi\,(s)ds =$$

$$\int_{\theta}^{\hat{\varrho}\left(\int_{\theta}^{d(z,Ft)} \vartheta(r)dr,\theta,\int_{\theta}^{d(z,Ft)} \vartheta(r)dr,\theta,\theta,\int_{\theta}^{d(z,Ft)} \vartheta(r)dr\right)} \psi\,(s)ds \precsim \theta \tag{22}$$

From $(\hat{\varrho}_{22})$, we get $\int_{\theta}^{d(z,Ft)} \vartheta\,(r)dr \precsim h\theta = \theta \Rightarrow z \in F$. Such that

$$z = ft \in Ft \subseteq g(Z), \text{ Also there exists } w \in z : z = gw \in Gw \subseteq f(Z). \tag{23}$$

Take

$$ft = Ft \text{ and } gw = Gw \text{ for some } w,t \in Z. \tag{24}$$

From (23) and (24), we can deduce $z = ft = Ft = gw = Gw$.

Now, we prove that z is a fixed point of f, let $fz \neq z$. Put $a = z$ and $b = w$ in (15)

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{d(Fz,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)+d(Fz,z)}\vartheta(r)dr,\\ \int_{\theta}^{d(z,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(z,Fz)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{s(Fz,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,Fz)}\vartheta(r)dr,\\ \int_{\theta}^{d(z,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(z,Fz)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds = \tag{25}$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{s(Fz,Gw)}\vartheta(r)dr,\int_{\theta}^{d(fz,gw)}\vartheta(r)dr,\int_{\theta}^{d(fz,Fz)}\vartheta(r)dr,\\ \int_{\theta}^{d(gw,Gw)}\vartheta(r)dr,\int_{\theta}^{d(fz,Gw)}\vartheta(r)dr,\int_{\theta}^{d(gw,Fz)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim \theta.$$

Since (f,F) is f-weakly biased and from (24), then

$$d(fFt,ft) \precsim d(Fft,Ft) \Longrightarrow d(fz,z) \precsim d(Fz,z) \tag{26}$$

From (25) and (26), we get

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{d(Fz,z)}\vartheta(r)dr,\int_{\theta}^{d(Fz,z)}\vartheta(r)dr,\int_{\theta}^{2d(Fz,z)}\vartheta(r)dr,\\ \theta,\int_{\theta}^{d(Fz,z)}\vartheta(r)dr,\int_{\theta}^{d(Fz,z)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{d(Fz,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{2d(Fz,z)}\vartheta(r)dr,\\ \theta,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(Fz,z)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{s(Fz,z)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(Fz,z)+d(Fz,z)}\vartheta(r)dr,\\ \theta,\int_{\theta}^{d(fz,z)}\vartheta(r)dr,\int_{\theta}^{d(Fz,z)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix}\int_{\theta}^{s(Fz,Gw)}\vartheta(r)dr,\int_{\theta}^{d(fz,gw)}\vartheta(r)dr,\int_{\theta}^{d(fz,z)+d(Fz,z)}\vartheta(r)dr,\\ \theta,\int_{\theta}^{d(fz,Gw)}\vartheta(r)dr,\int_{\theta}^{d(Fz,gw)}\vartheta(r)dr\end{pmatrix}}\psi(s)ds \precsim \theta. \tag{27}$$

Which contradicts with $(\hat{\varrho}_4)$, then $fz = z$, $z \in \{Fz\}_{\alpha}$. (i.e., z is common fixed point of F, f). Finally, we prove that z is a common fixed point of G, g. Assume that $gz \neq z$, as (G,g) is g-weakly biased, from (24), then

$$d(gGw,gw) \precsim d(Ggw,Gw) \Longrightarrow d(gz,z) \precsim d(Gz,z) \tag{28}$$

From (28) and triangle inequality, we get

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,z)}\vartheta(r)dr, \\ \int_{\theta}^{2d(Gz,z)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(Gz,z)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds =$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,z)}\vartheta(r)dr, \\ \int_{\theta}^{d(Gz,z)+d(Gz,z)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(Gz,z)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{s(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,gz)}\vartheta(r)dr, \int_{\theta}^{d(z,z)}\vartheta(r)dr, \\ \int_{\theta}^{d(gz,z)+d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(gz,z)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds =$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{s(Ft,Gz)}\vartheta(r)dr, \int_{\theta}^{d(ft,gz)}\vartheta(r)dr, \int_{\theta}^{d(ft,Ft)}\vartheta(r)dr, \\ \int_{\theta}^{d(gz,z)+d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(ft,Gz)}\vartheta(r)dr, \int_{\theta}^{d(gz,Ft)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds \precsim$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{s(Ft,Gz)}\vartheta(r)dr, \int_{\theta}^{d(ft,gz)}\vartheta(r)dr, \int_{\theta}^{d(ft,Ft)}\vartheta(r)dr, \\ \int_{\theta}^{d(gz,Gz)}\vartheta(r)dr, \int_{\theta}^{d(ft,Gz)}\vartheta(r)dr, \int_{\theta}^{d(gz,Ft)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds \precsim \theta. \tag{29}$$

Then

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,gz)}\vartheta(r)dr, \int_{\theta}^{d(z,z)}\vartheta(r)dr, \\ \int_{\theta}^{d(Gz,z)+d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(gz,z)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds =$$

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(z,gz)}\vartheta(r)dr, \theta, \\ \int_{\theta}^{2d(Gz,z)}\vartheta(r)dr, \int_{\theta}^{d(z,Gz)}\vartheta(r)dr, \int_{\theta}^{d(gz,z)}\vartheta(r)dr \end{pmatrix}} \psi(s)ds \precsim \theta \tag{30}$$

Which contradicts with ($\hat{\varrho}_5$), then $gz = z$, $z \in \{G_\alpha\}$. (i.e., z is common fixed point of G, g), so G, F, g and f have a common fixed point.

When we put $\vartheta(t) = 1$ in theorem 4.1, then we get the following result:

Corollary 4.1: The conclusion of theorem 4.1 remains true if we replace the condition (15) by:

If there is a $\hat{\varrho} \in \hat{\varphi}$ such that $\forall a, b \in Z$,

$$\int_{\theta}^{\hat{\varrho}\begin{pmatrix} d(\{Fa\}_\alpha,\{Gb\}_\alpha), d(fa,gb), d(fa,\{Fa\}_\alpha), \\ d(gb,\{Gb\}_\alpha), d(fa,\{Gb\}_\alpha), d(gb,\{Fa\}_\alpha) \end{pmatrix}} \psi(s)ds \precsim \theta. \tag{31}$$

5. CONCLUSION

In this paper, we studied common fixed point findings for fuzzy mapping under a new class of an implicit relation in the complex-valued metric spaces. We hope that our presented idea herein will be source of motivation for other researches to extend and improve these findings for their applications.

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