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HEAT TRANSFER IN ENCLOSURES FILLED WITH NANOFLUIDS IN CASE OF THE FRACTIONAL DERIVATIVES

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In this article, influences of the fractional derivatives on the heat transfer enhancement by free convective flow of nanofluids in a square enclosure that includes a heat source at the bottom are examined. All the cavity walls are considered to be cold except the bottom wall that is considered thermally insulated and contains a heat source. The nanofluids consist of water as a base fluid and copper as nanoparticles. The fractional partial differential equations are transformed to non-dimensional form and then solved numerically using the finite differences method. The obtained numerical data are presented in terms of streamlines and isotherms contours as well as local and average Nusselt numbers. The results revealed that both of the local and average Nusselt numbers are supported as the order of the fractional derivatives decreases.

Key words: Fractional derivatives; heat transfer; heat source; nanofluids; finite difference method.

1. INTRODUCTION

Study of natural convection heat transfer was and remains an area of interest of researchers from the standpoint of basic and applied research. Natural convection has various applications in many engineering fields such as cooling electronic system, building insulation. It also was applied in solar energy collection, cooling of heat-generating, components in the electrical and nuclear industries [1-5]**.** Kandaswamy et al. [6] studied the effect of natural convection in a square cavity for different values of Grashof number and different aspect ratios and position of heated plat, the study found that with increase of Gr heat transfer rate increased in both vertical and horizontal position of the plate and aspect ratio of heated thin plate is decreased when the heat transfer also decreases. The heat transfer performance and entropy generation of natural convection in a nanofluid-filled U-shaped cavity studied by Cho et al. [7], the results show that the mean Nusselt number and the total entropy generation are both increased as the Rayleigh number increase. The effect of increasing the size of the heater on natural heat transfer convection in square enclosure studied by Ragui et al. [8]. They found with higher Rayleigh number, the cavity heat transfer increases with the width of the heater until it reaches a critical value, where the heat transfer reaches its maximum. An et al. [9] could obtain hybrid numerical analytical solution of natural convection in a cavity with volumetric heat generation by the generalized integral transform technique (GITT). Mansour and Ahmed [10] discusses the natural convection heat transfer in an inclines triangular enclosure filled with Cu-water nanofluid saturated porous medium in the presence of heat generation effect. Mansour et al. [11] studied that natural convection fluid flow and heat transfer between two enclosures filled with a water- based nanofluid and has been investigated numerically using finite difference method. Mansour et al. [12] studied natural convection fluid flow and heat transfer inside C-shaped enclosures filled with Cu-water nanofluid numerically using the finite difference method.

Fractional derivative is as old as calculus. Many researchers tried to put a definition of a fractional derivative. Most of them used an integral form for the fractional derivative. The most popular definitions are Riemann-Liouville, Caputo, Riesz and Gr¨unwald-Letnikov see [16-20]. There are many applications of the fractional derivatives used in many fields such as control theory of dynamical systems, nanotechnology, viscoelasticity [21-25]. The governing equations for fractional fluids are obtained from those of ordinary fluids through substituting derivatives of an integer order with fractional derivatives of order α. For example, α=1 corresponds to the classical diffusion, whereas for 0< α<1 the transport phenomena exhibits sub diffusion and for α >1 it exhibits super diffusion. Many of the familiar properties of standard (integer) derivatives such as product, quotient and chain rules are not provided for the fractional derivatives. Since these basic rules cannot be used, algebraic operations in non-integer calculus have many difficulties. For these reasons it was appeared a new definition well-behaved simple fractional derivative called “the conformable fractional derivative” depending just on the basic limit definition, see [26-29]. Khalil et al. [26] introduced the conformable fractional derivative by using the limits in the form

$$D^{\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\beta}) - f(t)}{\varepsilon} \quad \forall t > 0,\ \beta \in (0,1], \qquad f^{(\beta)}(0) = \lim_{t \to 0^{+}} f^{(\beta)}(t).$$

The conformable fractional derivative has the following properties:

$$D^{\beta} t^{p} = p\, t^{p-\beta}, \quad p \in N, \quad D^{\beta} c = 0, \quad \forall f(t) = c,$$

$$D^{\beta}(af + b\,g) = a\,D^{\beta} f + b\,D^{\beta} g, \ \forall\, a,b \in N$$

$$D^{\beta}(f\,g) = f\,D^{\beta} g + f\,D^{\beta} g,$$

$$D^{\beta} f(g) = \frac{df}{dg} D^{\beta} g, \qquad D^{\beta} f(g) = t^{1-\beta} \frac{df}{dg},$$

Abdeljawad [30] developed the definitions of conformable fractional and set the basic concepts in this new simple fractional calculus. Iyiola and Nwaeze [31] proved some new results on the recently proposed conformable fractional derivatives and fractional integral. They also apply the D'Alambert approach to the conformable fractional differential equation as application.

The main objective of this paper is to study the natural convention inside an enclosure filled with nanofluid using the conformable fractional derivative. The conformable fractional definition used to convert the governing equation from ordinary to fractional to study the behavior fractional Newtonian fluid and discuss differences about classical Newtonian fluid in previous studies and studies effects fractional parameter α on properties of the fluid. This paper provides a detailed discussion as well as a graphical representation of the obtained results.

2. PROBLEM DESCRIBTION

Let us consider a steady two-dimensional natural convection flow inside a square cavity of length L filled with nanofluid, as shown in Fig 1. A heat source is located on the lower wall with length B. The nanofluids used are assumed to be incompressible and laminar, the base fluid (water) and the solid spherical nanoparticles (Cu) are in thermal equilibrium. The thermo-physical properties of the nanofluid are assumed constant except for the density variation, which is determined based on the Boussinesq approximation.

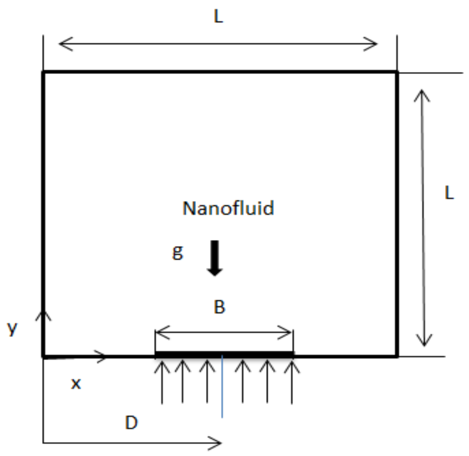


Figure1. Physical model of the problem

3. MATHEMATICAL FORMULATION

The continuity, momentum and energy equations for the laminar and steady state natural convection in the two-dimensional enclosure can be written in dimensional form as follow, see [10].

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho_{nf}}\left(-\frac{\partial P}{\partial x}+\mu_{nf}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right), \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{\rho_{nf}}\left(-\frac{\partial P}{\partial y}+\mu_{nf}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+g(\rho\beta)_{nf}(T-T_c)\right), \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha_{nf}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right). \tag{4}$$

The conformable fractional derivative of the integer system can be written as:

$$D_x^{\alpha}u+D_y^{\alpha}v=0, \tag{5}$$

$$uD_x^{\alpha}u+vD_y^{\alpha}u=\frac{1}{\rho_{nf}}(-D_x^{\alpha}p+\mu_{nf}(D_x^{\alpha}(D_x^{\alpha}u)+D_y^{\alpha}(D_y^{\alpha}u)), \tag{6}$$

$$uD_x^{\alpha}v+vD_y^{\alpha}v=\frac{1}{\rho_{nf}}(-D_y^{\alpha}p+\mu_{nf}(D_x^{\alpha}(D_x^{\alpha}v)+D_y^{\alpha}(D_y^{\alpha}v)+(\rho\beta)_{nf}g(T-T_c)), \tag{7}$$

$$uD_x^{\alpha}T+vD_y^{\alpha}T=\alpha_{nf}((D_x^{\alpha}(D_x^{\alpha}T)+D_y^{\alpha}(D_y^{\alpha}T)), \tag{8}$$

where D^{α} is conformable fractional derivative symbol
The boundary conditions are:

for
$$y=0,\ u=v=0,$$
$$\frac{\partial T}{\partial y}=\frac{q''}{k_{nf}}\quad,\ (D-0.5B)\le\frac{x}{L}\le(D+0.5B),$$
$$\frac{\partial T}{\partial y}=0\ \ otherwise$$
$$y=L\ \text{ and}\ 0\le x\le 1$$
$$u=v=0, T=T_C$$
$$\mathbf{x=L\ ,0\le y\le 1}$$
$$\boldsymbol{u=v=0,T=T_C}$$

$$\text{x=0 and } 0\le \text{y}\le 1$$
$$u=v=0,T=T_C \tag{9}$$

We use the formulations for the thermo-physical properties of nanofluids according to the nanoparticles volume fraction only and which were proven and used in many previous studies K. Khanafer [32] as follows:

The effective density of the nanofluid is given as:

$$\rho_{nf}=(1-\phi)\rho_f+\phi\rho_p, \tag{10}$$

where ϕ is the solid volume fraction of the nanofluid, ρ_f and ρ_p are the densities of the fluid and of the solid fractions respectively, and the heat capacitance of the nanofluid given is by

$$(\rho c_p)_{nf}=(1-\phi)(\rho c_p)_f+\phi\,(\rho c_p)_p. \tag{11}$$

The thermal expansion coefficient of the nanofluid can be determined by

$$(\rho_\beta)_{nf}=(1-\phi)\big(\rho_\beta\big)_f+\phi\big(\rho_\beta\big)_p\ , \tag{12}$$

where β_f and β_p are the coefficients of thermal expansion of the fluid and of the solid fractions respectively. Thermal diffusivity, α_{nf} of the nanofluid:

$$\alpha_{nf}=\frac{k_{nf}}{(\rho c_p)_{nf}}, \tag{13}$$

where k_{nf} is the thermal conductivity of the nanofluid which for spherical nanoparticles, according to the Maxwell-Garnetts [33] model is:

$$\frac{k_{nf}}{k_f}=\frac{(k_p+2k_f)-2\phi(k_f-k_p)}{(k_p+2k_f)+\phi(k_f-k_p)}, \tag{14}$$

The effective dynamic viscosity of the nanofluid based on the Brinkman [34] model is given by

$$\mu_{nf}=\frac{\mu_f}{(1-\phi)^{2.5}}, \tag{15}$$

where μ_f is the viscosity of the fluid fraction

Introducing the following dimensionless variables:

$$X=\frac{x}{L},Y=\frac{y}{L},U=\frac{uL}{\alpha_f},V=\frac{vL}{\alpha_f},P=\frac{pL^2}{\rho_{nf}\ \alpha_f^2},\theta=\frac{T-T_c}{\Delta T},\Delta T=\frac{q''L}{k_f} \tag{16}$$

Substituting Eq. (16) into Eqs. (5)- (8), the dimensionless form of the governing equations are:

$$D_X^\alpha U+D_Y^\alpha V=0 \tag{17}$$

$$UD_X^\alpha U+VD_Y^\alpha U=-D_X^\alpha p+\frac{\mu_{nf}}{\rho_{nf}\ \alpha_f}(D_X^\alpha(D_X^\alpha U)+D_Y^\alpha(D_Y^\alpha U)) \tag{18}$$

$$UD_X^\alpha V+VD_Y^\alpha V=-D_Y^\alpha p+\frac{\mu_{nf}}{\rho_{nf}\ \alpha_f}(D_X^\alpha(D_X^\alpha V)+D_Y^\alpha(D_Y^\alpha V))+\frac{(\rho\beta)_{nf}}{\rho_{nf}\ \beta_f}Ra\Pr\theta \tag{19}$$

$$UD_X^\alpha\theta+VD_Y^\alpha\theta=\frac{\alpha_{nf}}{\alpha_f}((D_X^\alpha(D_X^\alpha\theta)+D_Y^\alpha(D_Y^\alpha\theta)) \tag{20}$$

Where

$\Pr=\frac{\nu_f}{\alpha_f},Ra=\frac{g\beta_f L^3\Delta T}{\nu_f\ \alpha_f}$,are the Prandtl number and Rayleigh number equations.

The dimensionless boundary condition for Eqs. (17-20). are as follows:

$$Y=0, U=V=0,$$

$$\frac{\partial\theta}{\partial Y}=\frac{q''}{k_{nf}} \quad , (D-0.5B)\le X\le(D+0.5B),$$

$$\frac{\partial\theta}{\partial Y}=0\ \ otherwise$$

$Y = L$ and $0 \le X \le 1$
$U = V = 0, \theta = 0$

$X = L, 0 \le Y \le 1$
$U = V = 0, \theta = 0$

$X = 0$ and $0 \le Y \le 1$
$U = V = 0, \theta = 0$ (21)

The local Nusselt number is defined as:

$$Nu_s = \frac{1}{(\theta)_{heat\ source}} , \tag{22}$$

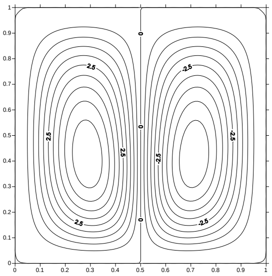
and the average Nusselt number is defined as:

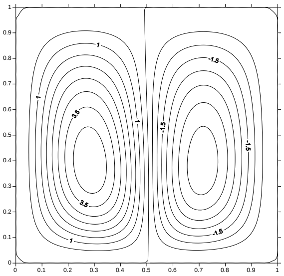
$$Nu_m = \left(\frac{1}{B} \int_{D-0.5B}^{D+0.5B} Nu_s \, dX \right)_{Y=0} , \tag{23}$$

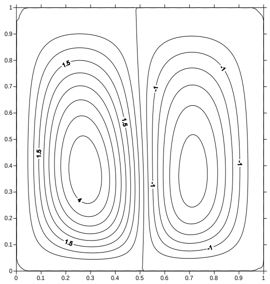
4. RESULTS AND DISCUSSION

The aim of the study was to identify the resulting changes within the fluid from the use of fractional derivatives. The governing equations for fractional fluids are obtained from those of ordinary fluids through substituting derivatives of an integer order with fractional derivatives of order α. Equations solved numerically by finite difference method. We used FORTRAN program for several value of the fractional parameter α and ($Ra=10^5$, D =0.5, B =0.5, phi=0.05)

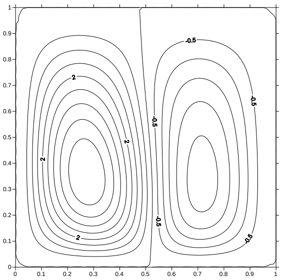
$a)$ $\alpha = 1$ $\alpha = 0.95$ $\alpha = 0.9$

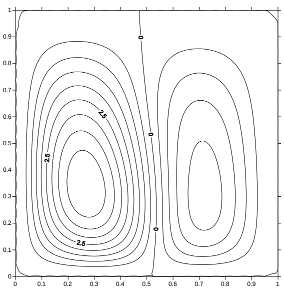




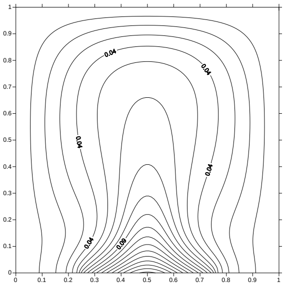


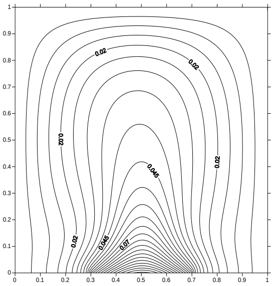
$a)$ $\alpha = 0.85$ $\alpha = 0.8$

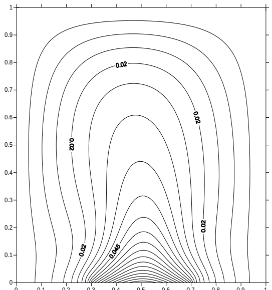




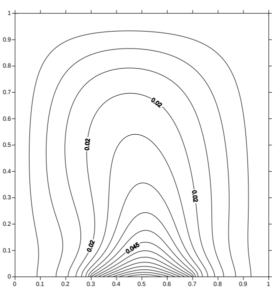
$b)$ $\alpha = 1$ $\alpha = 0.95$ $\alpha = 0.9$







$b)$ $\alpha = 0.85$ $\alpha = 0.8$



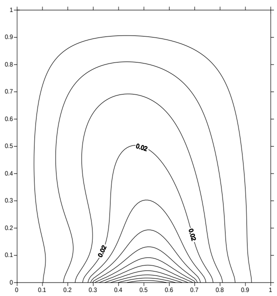


Figure 2, a- Streamlines and b- Isothermal for Cu-water at
$$\varphi = 0.05, D = 0.5, B = 0.5, Ra = 10^5$$

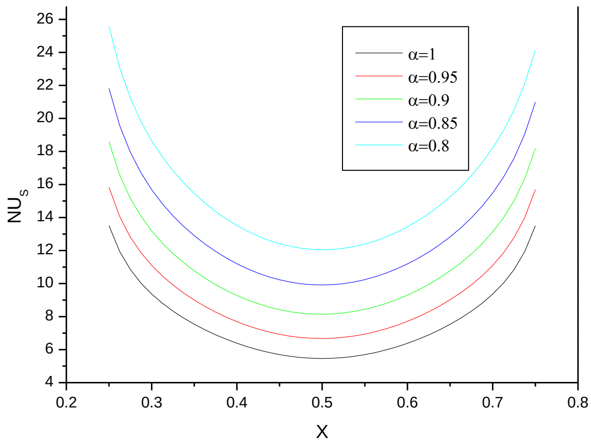


Figure 3, Profiles of the local Nusselt number for Cu-water at
$$\varphi = 0.05, D = 0.5, Ra = 10^5, B = 0.5$$

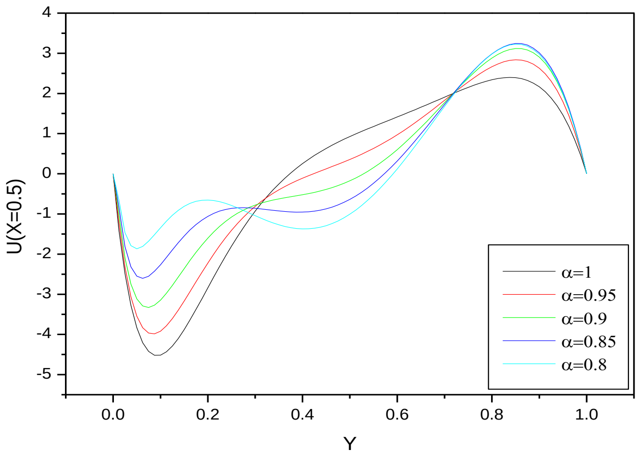


Figure 4, Horizontal velocity along the mid-section of the enclosure for Cu-water at
$$\varphi = 0.05, D = 0.5, Ra = 10^5, B = 0.5$$

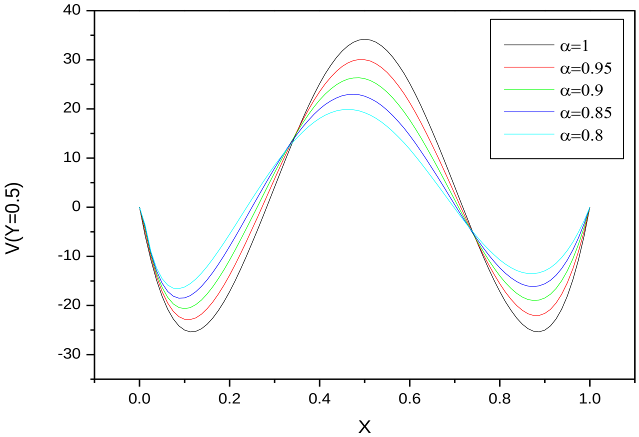


Figure 5, Vertical velocity along the mid-section of the enclosure for Cu-water at

$$\varphi = 0.05, D = 0.5, Ra = 10^5, B = 0.5$$

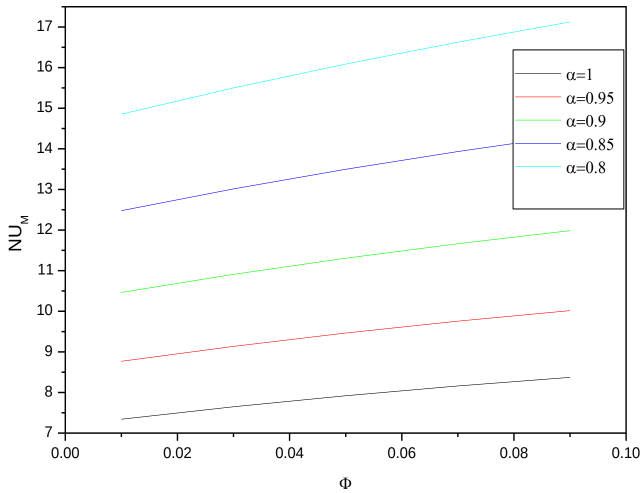


Figure 6, Variation of the average Nusselt number for **Cu**-water at

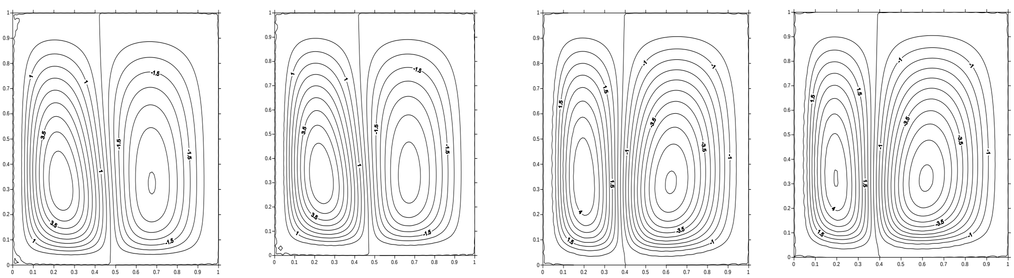
$$B = 0.5, D = 0.5, Ra = 10^5$$

In Fig. 2 we noted that the different values of α affects the on flow of fluid and distribution of temperature**.** When α=1 this gave the same result to ordinary fluids. When 0.7<α<1 the Streamlines concentrated on the left side of the cavity and Isothermal approaching to the bottom of the cavity. Fig. 3

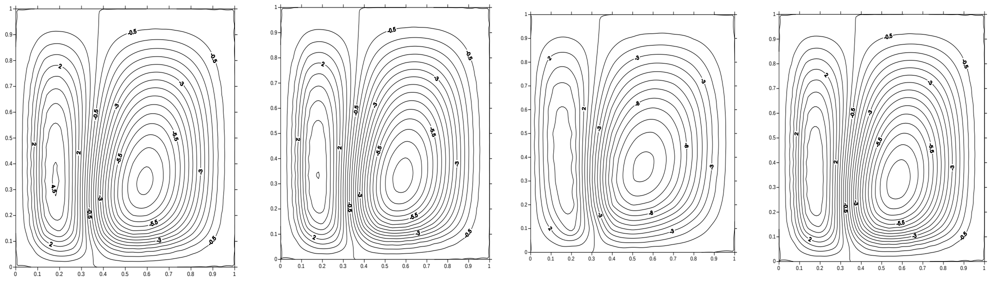
displays the profiles of the local Nusselt number for different values of the fractional parameter α from 0.8 to 1, we noted that the figure Parabolic curves, the minimum values of local Nusselt number approximately in the mid x-axis, curves don't intersect with x-axis for any value of α and the curves don't intersect each other, the local Nusselt number increases when α reduces. This can be attributed to the convection became more active at the small value of

the fractional parameter α. It is observed from Fig. 4 that the horizontal velocity curves along the mid-section of the enclosure intersect at two points, from the left and right sides in the figure we found that α is inversely proportional to velocity, and in the middle of figure α is proportional to velocity, but the horizontal velocity in the right side more than the left side this explain that the fluid moves from right to left of enclosure. Also, in Fig. 5 the vertical velocity carves along the mid-section of the enclosure intersect at two points. From the left and right sides in the figure we found that by increasing α velocity decrease, in the middle of figure occurred the convers when the α increase the velocity also increase. According to the Nusselt number Fig. 6 showed that when α increases the average Nusselt number decrease

$c)$ $\alpha = 0.7$ $\alpha = 0.69$ $\alpha = 0.67$ $\alpha = 0.65$



$c)$ $\alpha = 0.64$ $\alpha = 0.63$ $\alpha = 0.62$ $\alpha = 0.6$



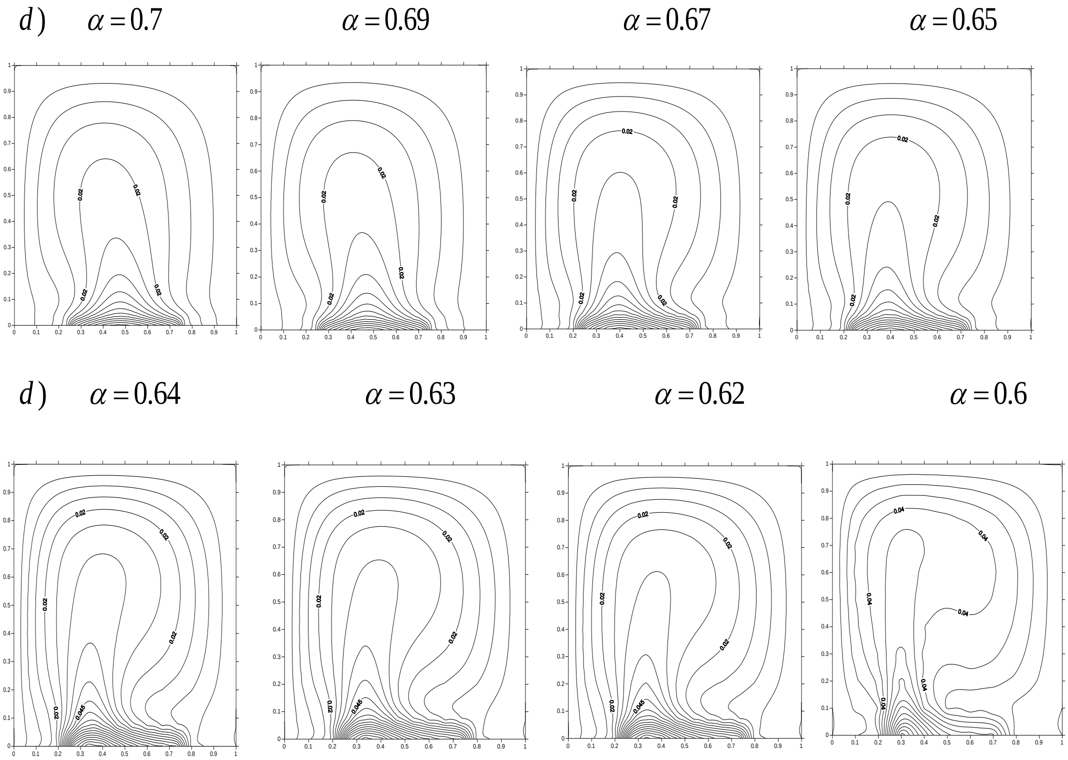


Figure 7, a- Streamlines and b- Isothermal for Cu-water at

$\varphi = 0.05, D = 0.5, B = 0.5, Ra = 10^5$

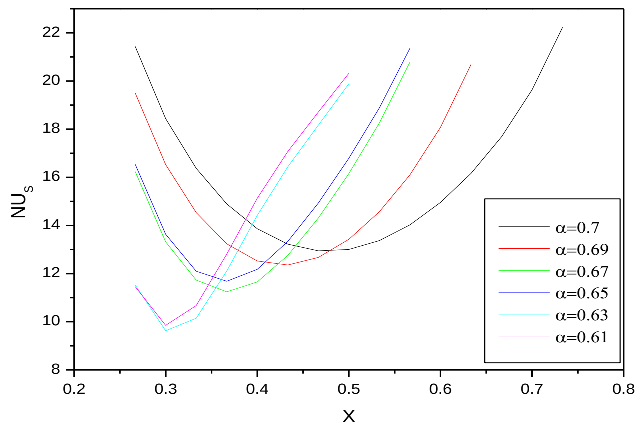


Figure 8, Profiles of the local Nusselt number for Cu-water at

$$\varphi = 0.05, D = 0.5, Ra = 10^5, B = 0.5$$

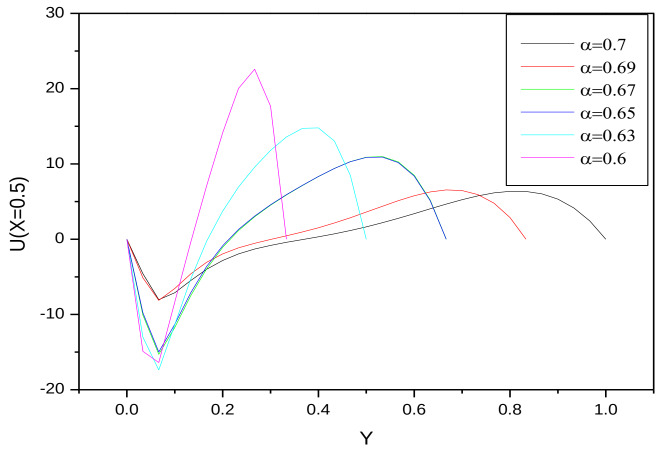


Figure 9, Horizontal velocity along the mid-section of the enclosure for Cu-water at

$$\varphi = 0.05, D = 0.5, Ra = 10^5, B = 0.5$$

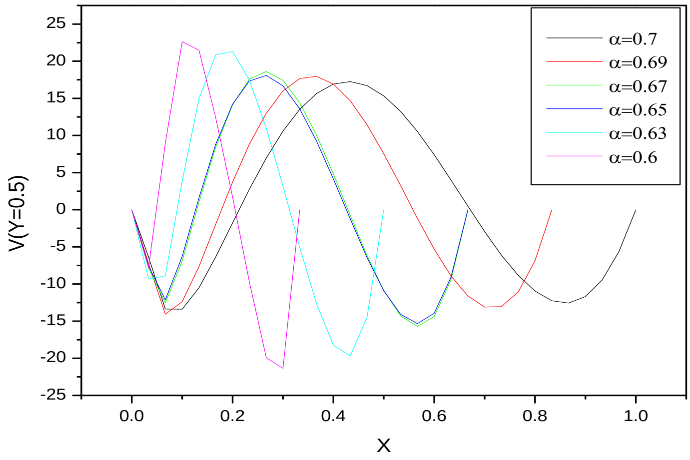


Figure 10, vertical velocity along the mid-section of the enclosure for Cu-water at

$$\varphi = 0.05, D = 0.5, Ra = 10^5, B = 0.5$$

When we used the values for α less than 0.8 we noted as show in Fig.7 that the motion of fluid return to the right side of cavity and there is critical value of α =0.69 after this value the streamlines increased on the right side of the cavity and became more density and isothermal

Spread in cavity toward the top. This mean that the conduction became more active in right side of cavity. Fig. 8 displays the profiles of the local Nusselt number α. We noted that the minimum value for local Nusselt number was in the middle of cavity but after the critical value of fractional parameterα the minimum value became in the left of cavity. This ensures that the convection more active in right side. In Fig.9 we noted that when α reduce the horizontal velocity decreases and increases significantly, explaining that the fluid movement becomes more on the left side as it begins to move towards the right side. Also, in Fig. 10 When α reduces, the change in the vertical velocity shifts gradually to the left side of the enclosure, because the fluid moves from left to right and the fluid in the right side becomes more stable

5. **CONCLUSION**

In this paper we studied the natural convection in the formula of fractional derivatives by using conformable definition in enclosure filled with a Cu-water nanofluid. Dimensionless governing equations are solved by finite difference method. The effect of the fractional parameter α on the isotherms and streamlines are given in the following:

Firstly, when α lies between 1and 0.7

- The streamlines gradually concentrated on the left side of cavity and isotherms increasing near to bottom.
- Nusselt number increased when fractional parameter α reduced this mean that the convection became more active.
- The horizontal velocity increase with the reduced of fractional parameter α.
- The horizontal velocity in the right side of enclosure more than it is in the left side this explained that the movement of fluid in enclosure from right to left.
- The change in vertical velocity was symmetric along the enclosure.
- The average Nusselt number increased when α reduced.
- Secondly, when α=0.69
- There was critical value for the fractional parameter α =0.69.At this value the distribution of streamlines and isotherms are very similar to the case when α=1.
- Finally, when α less than 0.69
- The streamlines gradually concentrated on the right side of cavity and isotherms increasing far of bottom.
- Nusselt number decreased in the left side of enclosure when fractional parameter α reduced.
- The change in the vertical velocity shifts gradually to the left side of the enclosure.
- When α reduces the horizontal velocity decreases and increases significantly.
- The fluid became more stable in left side.

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في هذه المقالة ، ندرس تأثيرات المشتقات الجزيئية على تحسين نقل الحرارة في حالة التدفق الحراري الحر لسوائل نانونية في حاوية مربعة فى وجود مصدر للطاقة فى منصف الجدار السفلى الذى يعتبر معزولا حراريا مع حفظ الجدران الثلاثة باردة.اما المادة النانونية فتتكون من الماء كسائل أساسي ونحاس كجسيمات نانوية. يتم تحويل المعادلات التفاضلية الجزئية الجزئية إلى معادلات لابعديه يتم حلها عدديًا باستخدام طريقة الفروق المحددة. يتم عرض النتائج التي تم الحصول عليه لكلا من streamlines and isotherms ولقد لوحظ ان نتائج كلا من

average Nusselt numbers و Nusselt numbers

تكون افضل كلماقلت قيمة الالفا(بارامتر المشتقة الجزيئية)