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POSITION VECTORS OF A PARIALLY NULL AND PSEUDO NULL W-CURVES IN MINKOWSKI SPACE-TIME

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In this paper, we introduced a position vector of a space-like curve with space-like principal normal vector and null first binormal vector. Also, we introduced a position vector of a space-like curve with space-like first binormal vector and null principal normal vector in the Minkowski space E_1^4 **.**

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1. Introduction

In Euclidean space E^3, a regular smooth curve α is called a helix if the tangent vector makes a constant angle with a fixed straight line (the axis of the helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. De Saint Venant in 1845 [1, 2]. The necessary and sufficient condition for a curve to be a helix is that the ratio of curvature to torsion be constant. If both of κ and τ of a curve α are non-zero constant, then it is a general helix. Also we call it a circular helix or W-curve. A helix in E_1^3 is a regular curve such that $\langle T(s), v\rangle$ is a constant function for some fixed vector $v \neq 0$. Any line parallels this direction v is called the axis of the helix [3].

All W-curves in the Minkowski 4-space are completely classified by Walrave [4]. For example, circles and hyperbolas are the only planar spacelike W-curves. In the Minkowski space-time E_1^4, all space-like W-curves are studied in [5]. General helices (W-curve) in the Lorentz-Minkowski spaces are studied in [6-9]. K. Ilarslan and O. Boyacioglu [10] obtained the position vectors of a spacelike W-curve with space-like, time-like and null principal normal in the Minkowski 3-space E_1^3.

In this paper, we study the position vectors of a partially null and pseudo null W-curves in Minkowski 4-space E_1^4.

2. Preliminaries

The Minkowski 4-space E_1^4 is the Pseudo Euclidean 4-space E^4 provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of E_1^4.

We say that a vector $v \in E_1^4$ can have one of three Lorentzian causal characters: it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null (light-like) if $g(v,v) = 0$ and $v \neq 0$. Also, a curve α in E_1^4 can have one of the following causal characters: $\alpha(s)$ is space-like, null or time-like, which means that $g(\alpha', \alpha') > 0$, $g(\alpha', \alpha') = 0$, $g(\alpha', \alpha') < 0$ respectively.

The Frenet frame along α is the orthonormal frame $\{T, N, B_1, B_2\}$ which is determined as follows: T is the velocity or the unit tangent vector fields of α, N is the principal normal vector fields of α, B_1 and B_2 are the first binormal and the second binormal vector fields of α respectively.

Denote by $\{T, N, B_1, B_2\}$ the moving Frenet frame along the space-like curve α, where s is a pseudo arc-length parameter. Then T is a space-like tangent vector, so depending on the causal character of the principal normal vector N and the binormal vector B_1, we have the following cases [4].

Case 1: N is space-like and B_1 is light-like;
The Frenet frame are:

$$\begin{pmatrix} T' \\ N' \\ B_1' \\ B_2' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 & 0 \\ 0 & -\kappa_2 & 0 & -\kappa_3 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}$$

(2.1)

where T, N, B_1, B_2 are mutually orthogonal vectors with the following properties

$$g(T,T)=g(N,N)=1\ ,\ g(B_1,B_1)=g(B_2,B_2)=0\ ,\ g(B_1,B_2)=1,$$
$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(N,B_2)=0\,.$$

Such a curve α is known as a *partially null* curve.

Case 2: N is light-like and B_1 is space-like;

The Frenet frame are:

$$\begin{pmatrix} T' \\ N' \\ B_1' \\ B_2' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ 0 & 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 0 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B_1 \\ B_2 \end{pmatrix}$$

(2.2)

where The functions κ_1, κ_2 and κ_3 are called the first, second and third curvatures of α . the " curvature " κ_1 can only take two values; 0 , when α is a null straight line, or 1 in all other cases.

where T, N, B_1, B_2 are mutually orthogonal vectors with the following properties

$$g(T,T)=g(B_1,B_1)=1\ ,\ g(N,N)=g(B_2,B_2)=0\ ,\ g(N,B_2)=1,$$
$$g(T,N)=g(T,B_1)=g(T,B_2)=g(N,B_1)=g(B_1,B_2)=0\,.$$

Such a curve α is known as a *pseudo null* curve.

3. Position vectors of a partially null W-curve in E_1^4

In this part, we get a position vector of a partially null W-curve in Minkowski 4-space.

If $\alpha(s)$ is a space-like curve with space-like principal normal and null first binormal in E_1^4 .

Then we can write its position vector as follows:

$$\alpha(s)=\lambda(s)T(s)+\mu(s)N(s)+\nu(s)B_1(s)+\gamma(s)B_2(s), \qquad (3.1)$$

for some differentiable functions λ , μ , ν and γ of $s \in I \subset \mathbb{R}$.

Differentiating (3.1) with respect to s and using the corresponding Frenet equation (2.1), we get the system of linear differential equations as follows:

$$\lambda'(s) - \mu(s)\kappa_1(s) = 1, \tag{3.2}$$

$$\lambda(s)\kappa_1(s) + \mu'(s) - \gamma(s)\kappa_2(s) = 0, \tag{3.3}$$

$$\gamma'(s) - \gamma(s)\kappa_3(s) = 0, \tag{3.4}$$

$$\mu(s)\kappa_2(s) + \nu'(s) + \nu(s)\kappa_3(s) = 0. \tag{3.5}$$

From (3.4) we get,

$$\frac{d\gamma(s)}{\gamma(s)} = \kappa_3 ds, \tag{3.6}$$

Integration (3.6) we get,

$$\gamma(s) = c_3 e^{\kappa_3 s}. \tag{3.7}$$

where c_3 is an arbitrary constant.

Differentiation (3.2) we obtained,

$$\mu'(s) = \frac{\lambda''(s)}{\kappa_1}, \qquad \kappa_1 \neq 0 \tag{3.8}$$

Substituting from (3.8) in (3.3) we get,

$$\lambda''(s) + \lambda(s)\kappa_1^2 = c_3\kappa_1\kappa_2 e^{\kappa_3 s}. \tag{3.9}$$

Solving equation (3.9) we get,

$$\lambda(s) = c_1\cos(\kappa_1 s) + c_2\sin(\kappa_1 s) + \frac{c_3\kappa_1\kappa_2 e^{\kappa_3 s}}{\kappa_1^2 + \kappa_3^2}, \qquad \kappa_1^2 \neq -\kappa_3^2 \tag{3.10}$$

where c_1 and c_2 are an arbitrary constants.

Differentiation (3.10) twice and substituting in (3.8) we get,

$$\mu''(s) = -c_1\kappa_1\cos(\kappa_1 s) - c_2\kappa_1\sin(\kappa_1 s) + \frac{c_3\kappa_2\kappa_3^2 e^{\kappa_3 s}}{\kappa_1^2 + \kappa_3^2}, \tag{3.11}$$

Integration (3.11) twice we get,

$$\mu(s) = -c_1\sin(\kappa_1 s) + c_2\cos(\kappa_1 s) + \frac{c_3\kappa_2\kappa_3 e^{\kappa_3 s}}{\kappa_1^2 + \kappa_3^2}. \tag{3.12}$$

From (3.5) we get,

$$\nu'(s)+\nu(s)\kappa_3(s)=-\mu(s)\kappa_2(s), \tag{3.13}$$

Substituting from (3.12) in (3.13) we get the differential equation in ν as

$$\nu'(s)+\nu(s)\kappa_3(s)=c_1\kappa_2\sin(\kappa_1 s)-c_2\kappa_2\cos(\kappa_1 s)-\frac{c_3\kappa_2^2\kappa_3 e^{\kappa_3 s}}{\kappa_1^2+\kappa_3^2}, \tag{3.14}$$

Solving equation (3.14) we get,

$$\nu(s)=\frac{-c_3\kappa_2^2e^{\kappa_3 s}-2\kappa_2[(c_2\kappa_1-c_1\kappa_3)\sin(\kappa_1 s)+(c_1\kappa_1+c_2\kappa_3)\cos(\kappa_1 s)]}{2\left(k_1^2+k_3^2\right)}+c_4e^{-\kappa_3 s}.$$

(3.15)

where c_4 is an arbitrary constant.

Substituting from (3.7), (3.10), (3.12) and (3.15) in (3.1) we get the required position vector as.

$$\begin{aligned}\alpha(s)=\ &([c_1\cos(\kappa_1 s)+c_2\sin(\kappa_1 s)+\frac{c_3\kappa_1\kappa_2e^{\kappa_3 s}}{\kappa_1^2+\kappa_3^2}]T(s)+[-c_1\sin(\kappa_1 s)+c_2\cos(\kappa_1 s)+\frac{c_3\kappa_2\kappa_3e^{\kappa_3 s}}{\kappa_1^2+\kappa_3^2}]N(s)\\ &+[\frac{-c_3\kappa_2^2e^{\kappa_3 s}-2\kappa_2[(c_2\kappa_1-c_1\kappa_3)\sin(\kappa_1 s)+(c_1\kappa_1+c_2\kappa_3)\cos(\kappa_1 s)]}{2\left(k_1^2+k_3^2\right)}+c_4e^{-\kappa_3 s}]B_1(s)\\ &+[c_3e^{\kappa_3 s}]B_2(s)).\end{aligned}$$

(3.16)

We can take $\kappa_1>0$, $\kappa_2\neq 0$ are non zero-constants, $\kappa_3=0$ and $c_3=0$ we get the position vector in this case as

$$\begin{aligned}\alpha(s)=\ &([c_1\cos(\kappa_1 s)+c_2\sin(\kappa_1 s)]T(s)+[-c_1\sin(\kappa_1 s)+c_2\cos(\kappa_1 s)]N(s)\\ &+[\frac{-\kappa_2}{\kappa_1}(c_2\sin(\kappa_1 s)+c_1\cos(\kappa_1 s))+c_4]B_1(s)),\end{aligned}$$

(3.17)

which means that the partially null curve α lies fully in a three dimensional subspace. Considering the above obtained results, we can formulated the following remark:

Remark 3.1 *Let* $\alpha=\alpha(s)$ *be a spacelike curve with spacelike principal normal vector and null first binormal vector in* E_1^4 *with curvatures* $\kappa_1>0$ *and* $\kappa_2\neq 0$. *Then* α *has* $\kappa_3=0$ *if and only if* α *lies fully in a three dimensional subspace [5].*

4. Position vectors of a pseudo null W-curve in E_1^4

In this part, we get a position vector of a pseudo null W-curve in Minkowski 4-space.
If $\alpha(s)$ is a spacelike curve with null principal normal and spacelike first binormal in E_1^4.
Then we can write its position vector as follows:

$$\alpha(s)=\lambda(s)T(s)+\mu(s)N(s)+\nu(s)B_1(s)+\gamma(s)B_2(s), \tag{4.1}$$

for some differentiable functions λ, μ, ν and γ of $s\in I\subset \mathbb{R}$.

Differentiating (4.1) with respect to s and by using the corresponding Frenet equation (2.2), we get the system of linear differential equations as follows:

$$\begin{cases} \lambda'(s)-\gamma(s)\kappa_1(s)=1, \\ \gamma'(s)-\nu(s)\kappa_2(s)=0, \\ \mu(s)\kappa_2(s)+\nu'(s)-\gamma(s)\kappa_3(s)=0, \\ \lambda(s)\kappa_1(s)+\mu'(s)+\nu(s)\kappa_3(s)=0, \end{cases} \tag{4.2}$$

Solving the system of linear differential equations (4.2), we get

$$\lambda(s)=\frac{c_4e^{-As}+c_3e^{As}-e^{-Bs}(c_2+c_1e^{2Bs})\kappa_2}{C}-\frac{1}{\kappa_1}, \qquad C\neq 0, \quad \kappa_1\neq 0 \tag{4.3}$$

$$\mu(s)=\frac{e^{(A-B)s}[(c_2-c_1e^{2Bs})BC\kappa_1^2-c_3Ae^{As}(2\kappa_3^2(C+\kappa_3)+\kappa_1^2(C+2\kappa_3))]}{C^2\kappa_1(C+\kappa_3)} + \frac{c_4Ae^{-As}(2\kappa_3^2(C+\kappa_3)+\kappa_1^2(C+2\kappa_3))}{C^2\kappa_1(C+\kappa_3)}, \tag{4.4}$$

$$\nu(s) = c_4 e^{-As} + c_3 e^{As} + c_2 e^{-Bs} + c_1 e^{Bs} - \frac{\kappa_3}{\kappa_1 \kappa_2}, \qquad \kappa_2 \neq 0$$

(4.5)

$$\gamma(s) = \frac{1}{C}(-Bc_4 e^{-As} + c_3 A e^{As} + Be^{-Bs}(c_2 - c_1 e^{2Bs}))$$

(4.6)

where c_1, c_2, c_3 and c_4 are an arbitrary constants.

Also,

$$A = \sqrt{\kappa_2(\kappa_3 - \sqrt{\kappa_1^2 + \kappa_3^2})} \ , \quad \kappa_3 \neq 0$$

$$B = \sqrt{\kappa_2(\kappa_3 + \sqrt{\kappa_1^2 + \kappa_3^2})} \text{ and } C = \sqrt{\kappa_1^2 + \kappa_3^2} \ , \quad \kappa_1^2 \neq -\kappa_3^2$$

Substituting from (4.3), (4.4), (4.5) and (4.6) in (4.1) we get the required position vector as.

$$\begin{aligned} \alpha(s) = \quad & ([\frac{c_4 e^{-As} + c_3 e^{As} - e^{-Bs}(c_2 + c_1 e^{2Bs})\kappa_2}{C} - \frac{1}{\kappa_1}]T(s) \\ & +[\frac{e^{(A-B)s}[(c_2 - c_1 e^{2Bs})BC\kappa_1^2 - c_3 A e^{As}(2\kappa_3^2(C+\kappa_3) + \kappa_1^2(C+2\kappa_3))]}{C^2\kappa_1(C+\kappa_3)} \\ & + \frac{c_4 A e^{-As}(2\kappa_3^2(C+\kappa_3) + \kappa_1^2(C+2\kappa_3))}{C^2\kappa_1(C+\kappa_3)}]N(s) \\ & +[c_4 e^{-As} + c_3 e^{As} + c_2 e^{-Bs} + c_1 e^{Bs} - \frac{\kappa_3}{\kappa_1\kappa_2}]B_1(s) \\ & +[\frac{1}{C}(-Bc_4 e^{-As} + c_3 A e^{As} + Be^{-Bs}(c_2 - c_1 e^{2Bs}))]B_2(s)). \end{aligned}$$

(4.7)

Taking Case 2 into account, we can take the curvature $\kappa_1 = 1$, then (4.7) takes the following form

$$\begin{aligned}\alpha(s) = \quad & ([\frac{c_4 e^{-\hat{A}s} + c_3 e^{\hat{A}s} - e^{-\hat{B}s}(c_2 + c_1 e^{2\hat{B}s})\kappa_2}{\hat{C}}]T(s) \\ & +[\frac{e^{(\hat{A}-\hat{B})s}[(c_2 - c_1 e^{2\hat{B}s})\hat{B}\hat{C} - c_3 \hat{A} e^{\hat{A}s}(2\kappa_3^2(\hat{C}+\kappa_3) + (\hat{C}+2\kappa_3))]}{\hat{C}^2(\hat{C}+\kappa_3)} \\ & + \frac{c_4 \hat{A} e^{-\hat{A}s}(2\kappa_3^2(\hat{C}+\kappa_3) + (\hat{C}+2\kappa_3))}{\hat{C}^2(\hat{C}+\kappa_3)}]N(s) \\ & +[c_4 e^{-\hat{A}s} + c_3 e^{\hat{A}s} + c_2 e^{-\hat{B}s} + c_1 e^{\hat{B}s} - \frac{\kappa_3}{\kappa_2}]B_1(s) \\ & +[\frac{1}{\hat{C}}(-\hat{B}c_4 e^{-\hat{A}s} + c_3 \hat{A} e^{\hat{A}s} + \hat{B}e^{-\hat{B}s}(c_2 - c_1 e^{2\hat{B}s}))]B_2(s)).\end{aligned}$$

(4.8)

where $\hat{A} = \sqrt{\kappa_2(\kappa_3 - \sqrt{1+\kappa_3^2})}$, $\hat{B} = \sqrt{\kappa_2(\kappa_3 + \sqrt{1+\kappa_3^2})}$ and $\hat{C} = \sqrt{1+\kappa_3^2}$.

From the foregoing results, we give the following remark:

Remark 4.1 *Let* $\alpha = \alpha(s)$ *be a spacelike curve with spacelike first binormal vector and null principal normal vector in the Minkowski 4-space* E_1^4 *with curvatures* $\kappa_1 > 0$ *and* $\kappa_2 \neq 0$*, then the curve* α *lies fully in the space* E_1^4 *[3].*

5. Conclusion

In this paper, we introduced the position vectors of a partially null W-curve in E_1^4 (given by 3.16). The geometric meaning explains that the position vector (3.16) lies fully in a three dimensional subspace. Also, we introduced the Position vectors of a pseudo null W-curve in E_1^4 (given by 4.7). The geometric meaning explains that the position vector (4.8) lies fully in the space E_1^4.

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في هذا البحث تم إيجاد متجهات الموضع ل partially null and pseudo null W-curves في فضاء منكوفيسكي الرباعي، كما تم إثبات أن منحنى partially null في فضاء منكوفيسكي الرباعي واقع بأكملة في الفضاء الثلاثي، أيضا تم إثبات أن منحنى pseudo null في فضاء منكوفيسكي الرباعي واقع بأكملة في الفضاء الرباعي.