Assiut Univ. J. of Mathematics and Computer Science *Printed ISSN 1687-4897*
47(2), pp10- 26(2018)

ON COUPLED FIXED POINTS FOR TWO MULTI-VALUED MAPPINGS IN ORDERED S-METRIC SPACES

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Received: 8/3/2018 **Accepted:** 24/6/2018 **Available Online:** 20/12/2018

In the present paper, we propose a multi-valued version of weakly mixed monotone property for two single-valued mappings in partially ordered S-metric spaces. Also, we state and prove some coupled fixed point theorems using this property. These theorems extend the corresponding results in [10].

AMS Mathematics Subject Classification (2010): 47H10, 54H25

Keywords and phrases: Ordered S-metric space, coupled fixed point, multi-valued mappings, weakly mixed monotone property.

1. INTRODUCTION

The metric fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instance, matrix, differential and functional equations (see, e.g. [21, 22, 23]). There are different generalizations of metric spaces. One of them, Gahler [8] introduced the concept of 2-metric space. On the other hand, Dhage [6] gave the concept of D-metric space. On the third hand, Mustafa and Sims [20] presented some remarks on topological structure of D-metric spaces. Consequently, they defined more generalized metric spaces so-called G-metric spaces as follows.

Definition 1.1 *[19] Let X be a nonempty set and $G: X^3 \to [0,\infty)$ be a function satisfying the following conditions, for all $x, y, z, a \in X$,*

(G_1) $G(x, y, z) = 0$ if $x = y = z$,

(G_2) $0 < G(x, x, y)$ whenever $x \neq y$,

(G_3) $G(x, x, y) \leq G(x, y, z)$ whenever $z \neq y$,

(G_4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$,

(G_5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$.

Then the pair (X, G) is called a G**-metric space**.

Also, in 2012, Sedghi et al. [26] established the concept of an S-metric space in the following way.

Definition 1.2 *Let X be a non-empty set. An S-metric on X is a function $S: X^3 \to [0,\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(S_1) $\quad S(x,y,z) = 0 \Leftrightarrow x = y = z$,

(S_2) $\quad S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$.

Then the pair (X,S) is called an S **-metric space**.

Lemma 1.1 *[26] If (X,S) is an S-metric space, then* $S(x,x,y) = S(y,y,x)$.

Lemma 1.2 *[7] Let (X,S) be an S- metric space. Then*

$$S(x,x,z) \le 2S(x,x,y) + S(y,y,z),$$

for all $x, y, z \in X$.

Definition 1.3 *[26] Let (X,S) be an S-metric space. For $x \in X$ and $r > 0$, we recall the* ***open ball*** *$B_S(x,r)$ and the* ***closed ball*** *$\overline{B}_S(x,r)$ with center x and radius r as follows*

$$B_S(x,r) = \{y \in X : S(x,x,y) < r\}, \overline{B}_S(x,r) = \{y \in X : S(x,x,y) \le r\}.$$

Definition 1.4 *[26] Let (X,S) be an S-metric space.*

(1) A sequence $\{x_n\}$ in X **converges** to x iff $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

(2) A sequence $\{x_n\}$ in X is called a **Cauchy** iff $S(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$.

(3) An S-metric space X is said to be **complete** iff every Cauchy sequence is convergent.

Lemma 1.3 *[26] Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x,x,y)$.*

In recent years, there has been a growing interest in studying the existence of fixed points for contractive mappings satisfying monotone properties in ordered metric spaces. This trend was initiated by Ran and Reurings in [22] where they extended Banach Contraction Principle (BCP) in partially ordered metric spaces.

Definition 1.5 *[17] A* ***partially ordered space*** *is a nonempty set* X *with a binary relation* $\leq$, *which satisfies the three conditions, for all* $x, y, z \in X$,

(1) $x \leq x$ (reflexivity),

(2) if $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry),

(3) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

Definition 1.6 *[3] Let* $(X,^\circ\,)$ *be an ordered space.* X *is said to have the* ***sequential monotone property*** *if it verifies the following properties:*

$I.$ if $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \leq x$, for all $n \in N$,

$II.$ if $\{y_n\}$ is a decreasing sequence with $y_n \to y$, then $y_n \geq y$, for all $n \in N$.

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [18] who extended the BCP to multi-valued setting. Later many authors developed the existence of fixed points for various multi-valued contractions. For example, see [1, 4, 5, 11, 12, 13, 16, 24, 25]. On the other hand, in 2006, Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point and proved some fixed point results under certain conditions in a complete metric space endowed with a partial order. They applied their results to study the existence of a unique solution for a periodic boundary value problem associated with a first order ordinary differential equation. Later, Lakshmikantham and $\mathrm{C}'\mathrm{iri\,c}'$ [15] generalized the results in [3].

Definition 1.7 *[3] Let* $(X, \leq)$ *be a partially ordered space and* $F : X \times X \to X$. *We say that* F *has the* **mixed monotone property** *iff* $F(x, y)$ *is monotone non-decreasing in* x *and monotone non-increasing in* y, *that is, for any* $x, y \in X$,

$$x_1, x_2 \in X,\ x_1 \leq x_2 \text{ implies } F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X,\ y_1 \leq y_2 \text{ implies } F(x, y_1) \geq F(x, y_2).$$

Definition 1.8 *[3] An element* $(x, y) \in X \times X$ *is called a coupled fixed point of the mapping* F *if*

$$F(x, y) = x, \ F(y, x) = y.$$

Following Bhaskar and Lakshmikantham [3], Beg and Butt [2] proved some coupled fixed point results for multi-valued mappings in partially ordered metric spaces. For this purpose, they gave a generalized mixed monotone property for a multi-valued mapping.

Definition 1.9 *[2] Let* $(X, \leq)$ *be a partially ordered space and* $F : X \times X \to CB(X)$ *be a multi-valued mapping.* F *is said to be a mixed monotone mapping if* F *is order-preserving in* x *and order-reversing in* y*, i.e.,* $x_1 \leq x_2$, $y_2 \leq y_1$, $x_i, y_i \in X \,(i = 1,2)$ *imply for all* $u_1 \in F(x_1, y_1)$ *there exists* $u_2 \in F(x_2, y_2)$ *such that* $u_1 \leq u_2$ *and for all* $v_1 \in F(y_1, x_1)$ *there exists* $v_2 \in F(y_2, x_2)$ *such that* $v_2 \leq v_1$.

Definition 1.10 *[2] A point* $(x, y) \in X \times X$ *is said to be a coupled fixed point of the multi-valued mapping* F *if* $x \in F(x, y)$ *and* $y \in F(y, x)$.

On the third hand, in 2012, Gordii et al. [9] generalized the concept of mixed monotone property to two single-valued mappings. They proved coupled common fixed point results using this property. Therefore, Gupta and Deep [10] used altering distance function generalizing these results to S-metric spaces.

Definition 1.11 *[9] Let* $(X, \leq)$ *be a partially ordered space and* $F, G : X \times X \to X$ *be mappings. We say that a pair* F, G *has the mixed weakly monotone property on* X *if, for any* $x, y \in X$

$$x \leq F(x, y), \ y \geq F(y, x),$$

$$\Rightarrow F(x, y) \leq G(F(x, y), F(y, x)) \quad , F(y, x) \geq G(F(y, x), F(x, y))$$

and

$$x \leq G(x, y), \ y \geq G(y, x),$$

$$\Rightarrow G(x, y) \leq F(G(x, y), G(y, x)) \quad , G(y, x) \geq F(G(y, x), G(x, y)).$$

Theorem 1.1 *[10] Let* $(X, \leq, S)$ *be a partially ordered complete* S*-metric space and* $F, G : X \times X \to X$ *satisfies the mixed weakly monotone property on* X, $x_0 \leq F(x_0, y_0)$, $y_0 \geq F(y_0, x_0)$ *or* $x_0 \leq G(x_0, y_0)$, $y_0 \geq G(y_0, x_0)$ *for some* $x_0, y_0 \in X$. *Consider a function* $\Phi : [0, \infty) \to [0, \infty)$ *with* $\Phi(t) < t$ *and* $\lim_{r \to t^+} \Phi(r) < t$, $\forall t > 0$, *such that*

$$S(F(x,y)\quad,F(x,y),G(u,v)) \le \Phi\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right),$$

for all $x,y,u,v \in X$ with $x \le u$ and $y \ge v$.

Also, assume that either F or G is continuous or X has the sequential monotone property, then F and G have a coupled common fixed point in X.

In this paper, we state and prove extension of Theorem 1.1 to multi-valued arena. Our theorem extends some known results in S-metric spaces to multi-valued setting (see, [14, 27]).

2. MAIN RESULT

Firstly, we define the Hausdorff S-metric as follows.

Definition 2.1 *Let (X,S) be an S-metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A,B \in CB(X)$, define the Hausdorff S-metric $H_S : CB(X)\times CB(X)\times CB(X) \to [0,\infty)$ by*

$$H_S(A,B,C) = \max\{\sup_{a\in A} S(a,B,C), \sup_{b\in B} S(b,C,A), \sup_{c\in C} S(c,A,B)\},$$

where

$$S(a,B,C) = d_S(a,B)+d_S(a,C)+d_S(B,C),\ d_S(A,B) = \inf_{a\in A, b\in B} d_S(a,b).$$

Secondly, we give the following definition.

Definition 2.2 *Let A,B be two subsets of X, we define the binary relation between A and B as:*

- $A \le^1 B$ if for any $a \in A$ we can find $b \in B$ such that $a \le b$,
- $A \le^2 B$ if for any $b \in B$ we can find $a \in A$ such that $a \le b$,
- $A \le B$ if $A \le^1 B$ and $A \le^2 B$.

Therefore, we extend Definition 1.11 to multi-valued setting by the following way.

Definition 2.3 *Let $(X,\le)$ be a partially ordered space and $F,G : X\times X \to CB(X)$ be multi-valued mappings. We say that a pair (F,G) has the mixed weakly monotone property on X if for any $x,y \in X$*

$$\{x\} \le F(x,y) \text{ and } \{y\} \ge F(y,x)$$
$$\Rightarrow F(x,y) \le G(F(x,y),F(y,x)) \text{ and } F(y,x) \ge G(F(y,x),F(x,y))$$

and

$$\{x\} \le G(x,y) \text{ and } \{y\} \ge G(y,x)$$
$$\Rightarrow G(x,y) \le F(G(x,y),G(y,x)) \text{ and } G(y,x) \ge F(G(y,x),G(x,y)).$$

Example 2.1 *Let* $X=[0,\infty)$ *be endowed with its usual order* "$\le$" *and* $F,G : X\times X \to CB(X)$ *defined by*

$$F(x,y)=G(x,y)=[0,\max\{x,y\}].$$

We find that,

$$\begin{aligned}
& \{x\} \le F(x,y) \text{ and } \{y\} \ge F(y,x) \\
\Rightarrow\quad & \{x\} \le [0,\max\{x,y\}] \text{ and } \{y\} \ge [0,\max\{y,x\}] \\
\Rightarrow\quad & x=0 \text{ and } y=\max\{x,y\} \\
\Rightarrow\quad & F(x,y)=[0,y] \le G(F(x,y),F(y,x)) \text{ and } F(y,x) \ge G(F(y,x),F(x,y)).
\end{aligned}$$

Similarly, one can show that

$$\{x\} \le G(x,y) \text{ and } \{y\} \ge G(y,x)$$
$$\Rightarrow G(x,y) \le F(G(x,y),G(y,x)) \text{ and } G(y,x) \ge F(G(y,x),G(x,y)).$$

Now, we are ready to state and prove our main theorem as follows.

Theorem 2.1 *Let* $(X,\le,S)$ *be a partially ordered complete* S*-metric space and* $F,G : X\times X \to CB(X)$ *be multi-valued mappings such that* F *and* G *have the mixed weakly monotone property on* X. *Assume that there exists a function* $\Phi:[0,\infty)\to[0,\infty)$ *with* $\Phi(t)<t$ *and* $\lim_{r\to t^+}\Phi(r)<t$, $\forall t>0$, *such that*

$$H_S(F(x,y),F(x,y),G(u,v)) \le \Phi\left(\frac{S(x,x,u)+S(y,y,v)}{2}\right), \tag{2.1}$$

for all $x,y,u,v\in X$ with $x\le u$ and $y\ge v$.

Suppose that one of the following conditions is satisfied:

(i) F is continuous,

(ii) G is continuous,

(iii) X has the sequential monotone property.

If there exist $x_0, y_0 \in X$ such that

$\{x_0\} \leq F(x_0, y_0),\ \{y_0\} \geq F(y_0, x_0)\ \ or\ \ \{x_0\} \leq G(x_0, y_0),\ \{y_0\} \geq G(y_0, x_0),$
then F and G have a coupled common fixed point in X. Furthermore, if we assume that the set of coupled common fixed points is totally ordered and

$$S(x, x, x^*) \leq H_S(F(x, y), F(x, y), G(x^*, y^*)),$$

for two coupled common fixed points (x, y) and (x^*, y^*), then F and G have a unique coupled common fixed point.

Proof. Assume that $\{x_0\} \leq F(x_0, y_0)$ and $\{y_0\} \geq F(y_0, x_0)$. Since F and G satisfy the mixed weakly monotone property, then

$$F(x_0, y_0) \leq G(F(x_0, y_0), F(y_0, x_0))\ and\ F(y_0, x_0) \geq G(F(y_0, x_0), F(x_0, y_0)).$$

Let $x_1 \in F(x_0, y_0)$ and $y_1 \in F(y_0, x_0)$, then we have
$F(x_0, y_0) \leq G(x_1, y_1)\ and\ F(y_0, x_0) \geq G(y_1, x_1)$
$\Rightarrow \{x_1\} \leq G(x_1, y_1)\ and\ \{y_1\} \geq G(y_1, x_1).$

$$\tag{2.2}$$

Again by monotonicity

$$G(x_1, y_1) \leq F(G(x_1, y_1), G(y_1, x_1))\ and\ G(y_1, x_1) \geq F(G(y_1, x_1), G(x_1, y_1)).$$

Let $x_2 \in G(x_1, y_1)$ and $y_2 \in G(y_1, x_1)$, then we have

$$\{x_2\} \leq F(x_2, y_2)\ and\ \{y_2\} \geq G(y_2, x_2). \tag{2.3}$$

By (2.2), for $x_2 \in G(x_1, y_1)$ and $y_2 \in G(y_1, x_1)$ we have

$$x_1 \leq x_2\ and\ y_1 \geq y_2. \tag{2.4}$$

Also, by (2.3), for $x_3 \in F(x_2, y_2)$ and $y_3 \in F(y_2, x_2)$ we have

$$x_2 \leq x_3\ and\ y_2 \geq y_3. \tag{2.5}$$

Continuing in this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X for which

$$\begin{aligned} x_{2n+1} &\in F(x_{2n}, y_{2n}) \quad , \quad x_{2n+2} \in G(x_{2n+1}, y_{2n+1}), \\ y_{2n+1} &\in F(y_{2n}, x_{2n}) \quad , \quad y_{2n+2} \in G(y_{2n+1}, x_{2n+1}) \end{aligned} \tag{2.6}$$

and

$$x_n \leq x_{n+1},\ y_n \geq y_{n+1}. \tag{2.7}$$

By definition of Hausdorff S-distance, we obtain that for

$x_{2n+1} \in F(x_{2n}, y_{2n})$ there exists $x_{2n+2} \in G(x_{2n+1}, y_{2n+1})$ such that $S(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq H_S(F(x_{2n}, y_{2n}), F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}))$.
Therefore, by (2.1), we have

$$\begin{aligned} S(x_{2n+1}, x_{2n+1}, x_{2n+2}) &\leq H_S(F(x_{2n}, y_{2n}), F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1})) \\ &\leq \Phi\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}\right). \end{aligned} \tag{2.8}$$

Also, for $y_{2n+1} \in F(y_{2n}, x_{2n})$ there exists $y_{2n+2} \in G(y_{2n+1}, x_{2n+1})$ such that $S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \leq H_S(F(y_{2n}, x_{2n}), F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1}))$. Then, by (2.1), we get

$$S(y_{2n+1}, y_{2n+2}, y_{2n+2}) \leq \Phi\left(\frac{S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1})}{2}\right). \tag{2.9}$$

Adding (2.8) and (2.9) to obtain

$$\begin{aligned} \frac{\omega_{2n+1}}{2} &\leq \Phi\left(\frac{S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})}{2}\right), \\ \frac{\omega_{2n+1}}{2} &\leq \Phi\left(\frac{\omega_{2n}}{2}\right). \end{aligned} \tag{2.10}$$

Interchanging the role of mappings F and G and using (2.1), yield that

$$\begin{aligned} S(x_{2n+2}, x_{2n+2}, x_{2n+3}) &\leq H_S(G(x_{2n+1}, y_{2n+1}), G(x_{2n+1}, y_{2n+1}), F(x_{2n+2}, y_{2n+2})) \\ &\leq \Phi\left(\frac{S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})}{2}\right) \end{aligned} \tag{2.11}$$

and

$$S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \leq \Phi\left(\frac{S(y_{2n+1}, y_{2n+1}, y_{2n+2}) + S(x_{2n+1}, x_{2n+1}, x_{2n+2})}{2}\right). \tag{2.12}$$

Adding (2.11) and (2.12) to obtain

$$\frac{S(x_{2n+2},x_{2n+2},x_{2n+3})+S(y_{2n+2},y_{2n+2},y_{2n+3})}{2}$$
$$\le \Phi\left(\frac{S(x_{2n+1},x_{2n+1},x_{2n+2})+S(y_{2n+1},y_{2n+1},y_{2n+2})}{2}\right),$$
$$\frac{\omega_{2n+2}}{2} \le \Phi\left(\frac{\omega_{2n+1}}{2}\right).$$

From (2.10) and (2.13) and using the fact that $\Phi(t)\le t$ give

$$\begin{aligned}\frac{\omega_{n+1}}{2} &\le \Phi\left(\frac{\omega_n}{2}\right),\\ \omega_{n+1} &\le \omega_n.\end{aligned} \tag{2.14}$$

That is, $\{\omega_n\}$ is decreasing sequence of nonnegative real numbers. Therefore there exists some $\omega \ge 0$ such that

$$\lim_{n\to\infty}\omega_n = \omega.$$

Now we want to show that $\omega = 0$. Assume the contrary that $\omega > 0$. By taking limit as n tends to infinity in equation (2.14) and having in mind $\lim_{r\to t^+}\Phi(r) < t$, we have

$$\omega = \lim_{n\to\infty}\omega_{n+1} \le 2\lim_{n\to\infty}\Phi\left(\frac{\omega_n}{2}\right) = 2\lim_{\frac{\omega_n}{2}\to\frac{\omega^+}{2}}\Phi\left(\frac{\omega_n}{2}\right) < \omega. \tag{2.15}$$

By repeatedly use of property of S-metris space, for every $n,m\in N$ with $m>n$, we get

$$\begin{aligned}S(x_n,x_n,x_m)+S(y_n,y_n,y_m) &\le 2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_m)\\ &\quad +2S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_m)\\ &\le 2S(x_n,x_n,x_{n+1})+2S(x_{n+1},x_{n+1},x_{n+2})+S(x_{n+2},x_{n+2},x_m)\\ &\quad 2S(y_n,y_n,y_{n+1})+2S(y_{n+1},y_{n+1},y_{n+2})+S(y_{n+2},y_{n+2},y_m)\\ &\quad \vdots\\ &\le 2[S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_{n+2})+\ldots+S(x_{m-2},x_{m-2},x_{m-1})]\\ &\quad +2[S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_{n+2})+\ldots+S(y_{m-2},y_{m-2},y_{m-1})]\\ &\quad +S(x_{m-1},x_{m-1},x_m)+S(y_{m-1},y_{m-1},y_m)\\ &\to 0 \ as\ n\to\infty.\end{aligned}$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, then there exist $x,y\in X$ such that

$$x_n \to x \ \ and \ \ y_n \to y \ \ as \ \ n \to \infty. \tag{2.16}$$

Using the continuity of F to obtain

$$\begin{aligned} S(x,x,F(x,y)) &\le 2S(x_{2n+1},x_{2n+1},x)+S(x_{2n+1},x_{2n+1},F(x,y)) \\ &\le 2S(x_{2n+1},x_{2n+1},x)+H_S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(x,y)) \to 0 \ as \ n \to \infty \end{aligned}$$

and

$$\begin{aligned} S(y,y,F(y,x)) &\le 2S(y_{2n+1},y_{2n+1},y)+S(y_{2n+1},y_{2n+1},F(y,x)) \\ &\le 2S(y_{2n+1},y_{2n+1},y)+H_S(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),F(y,x)) \to 0 \ as \ n \to \infty. \end{aligned}$$

Hence, $x \in F(x,y)$ and $y \in F(y,x)$. From (2.1) we get

$$\begin{aligned} &H_S(F(x,y),F(x,y),G(x,y)+H_S(F(y,x),F(y,x),G(y,x)) \\ &\le \Phi\left(\frac{S(x,x,x)+S(y,y,y)}{2}\right)+\Phi\left(\frac{S(y,y,y)+S(x,x,x)}{2}\right) \end{aligned}$$

$$S(x,x,G(x,y))+H_S(y,y,G(y,x))=0, \Rightarrow x \in G(x,y) \ and \ y \in G(y,x).$$

Hence (x,y) is coupled common fixed point of F and G. Similarly, the result follows when G is assumed to be continuous.

Now, consider that X has the sequential monotone property. If $x_{2n}=x$ and $y_{2n}=y$ for some $n \ge 0$, then $x=x_{2n} \le x_{2n+1} \le x = x_{2n}$ and $y=y_{2n} \le y_{2n+1} \le y_{2n}$ imply that $x_{2n}=x_{2n+1} \in F(x_{2n},y_{2n})$ and $y_{2n}=y_{2n+1} \in F(y_{2n},x_{2n})$. Also, from (2.1) we get

$$\begin{aligned} S(x,x,G(x,y)) &\le 2S(x_{2n+1},x_{2n+1},x)+H_S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),G(x,y)) \\ &\le 2S(x_{2n+1},x_{2n+1},x)+\Phi\left(\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}\right) \\ &\le 2S(x_{2n+1},x_{2n+1},x)+0 \to 0 \end{aligned}$$

and

$$S(y,y,G(y,x)) \ \le 2S(y_{2n+1},y_{2n+1},y)+H_S(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),G(y,x)) \to 0.$$

So, (x_{2n},y_{2n}) is a coupled common fixed point of F and G.

Suppose that $(x_{2n},y_{2n}) \ne (x,y)$ for all n.

Thus, $$\Phi\left(\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}\right) < \frac{S(x_{2n},x_{2n},x)+S(y_{2n},y,y)}{2} \quad .$$

From (2.1) we have

$$
\begin{aligned}
S(x,x,G(x,y)) &\le 2S(x_{2n+1},x_{2n+1},x)+H_S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),G(x,y))\\
&\le 2S(x_{2n+1},x_{2n+1},x)+\Phi\left(\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}\right)\\
&< 2S(x_{2n+1},x_{2n+1},x)+\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}\to 0.
\end{aligned}
$$

Therefore, $x\in G(x,y)$. Similarly, $y\in G(y,x)$. By interchanging the role of functions F and G, we get the same result for F. Thus (x,y) is the common coupled fixed point of F and G.

Let (x,y) and (x^*,y^*) be two coupled common fixed points for F and G. Without loss of generality we may assume that $(x,y)\le(x^*,y^*)$. Then from (2.1), we have

$$
S(x,x,x^*)=H_S(F(x,y),F(x,y),G(x^*,y^*))\le\Phi\left(\frac{S(x,x,x^*)+S(y,y,y^*)}{2}\right)
$$

and

$$
S(y,y,y^*)=H_S(F(y,x),F(y,x),G(y^*,x^*))\le\Phi\left(\frac{S(y,y,y^*)+S(x,x,x^*)}{2}\right).
$$

Assume that $x\ne x^*$ and $y\ne y^*$ and adding the above inequalities imply

$$
\begin{aligned}
\frac{S(x,x,x^*)+S(y,y,y^*)}{2} &\le \Phi\left(\frac{S(x,x,x^*)+S(y,y,y^*)}{2}\right)\\
&< \frac{S(x,x,x^*)+S(y,y,y^*)}{2},
\end{aligned}
$$

which is a contradiction. Hence $x=x^*$ and $y=y^*$. This proves that the coupled common fixed point of F and G is unique. Again from (2.1), we have

$$
\begin{aligned}
S(x,x,y) = H_S(F(x,y),F(x,y),G(y,x)) &\leq \Phi\left(\frac{S(x,x,y)+S(y,y,x)}{2}\right) \\
&\leq \frac{S(x,x,y)+S(y,y,x)}{2} \ (if \ \ x \neq y) \\
&\leq \frac{S(x,x,y)+S(x,x,y)}{2} \\
&\leq S(x,x,y).
\end{aligned}
$$

This implies to $x = y$.

Finally, we establish a fixed point result in ordered complete S-metric space involving contractive conditions of integeral type.

Theorem 2.2 *Let* $(X, \leq, S)$ *be an ordered complete* S*-metric space and* $F, G : X \times X \to CB(X)$ *be multi-valued mappings such that* F *and* G *have the mixed weakly monotone property on* X. *Assume that there exists a function* $\Phi : [0,\infty) \to [0,\infty)$ *with* $\Phi(t) < t$ *and* $\lim_{r \to t^+} \Phi(r) < t$, $\forall t > 0$, *such that*

$$
\int_0^{H_S(F(x,y),F(x,y),G(u,v))} \phi(t)dt \leq \Phi \int_0^{\frac{S(x,x,u)+S(y,y,v)}{2}} \phi(t)dt, \tag{2.17}
$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$. Here $\phi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable function as a summable for each compact R^+, non-negative and such that for each $\varepsilon > 0$, $\int \phi(t)dt > 0$.

Suppose that one of the following conditions is satisfied:

(i) F is continuous,

(ii) G is continuous,

(iii) X has the sequential monotone property.

If there exist $x_0, y_0 \in X$ with $\{x_0\} \leq F(x_0, y_0)$, $\{y_0\} \geq F(y_0, x_0)$ or $\{x_0\} \leq G(x_0, y_0)$, $\{y_0\} \geq G(y_0, x_0)$. Then F and G have coupled common fixed point in X.

Proof. As in Theorem 2.1, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$
\begin{aligned}
x_{2n+1} \in F(x_{2n}, y_{2n}) \quad &, \quad x_{2n+2} \in G(x_{2n+1}, y_{2n+1}), \\
y_{2n+1} \in F(y_{2n}, x_{2n}) \quad &, \quad y_{2n+2} \in G(y_{2n+1}, x_{2n+1})
\end{aligned} \tag{2.18}
$$

and

$$x_n \le x_{n+1},\ y_n \ge y_{n+1}. \tag{2.19}$$

Using (2.17), we have

$$\begin{aligned}\int_0^{S(x_{2n+1},x_{2n+1},x_{2n+2})}\phi(t)dt &\le \int_0^{H_S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),G(x_{2n+1},y_{2n+1}))}\phi(t)dt \\ &\le \Phi\int_0^{\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}}\phi(t)dt \\ &< \int_0^{\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}}\phi(t)dt.\end{aligned} \tag{2.20}$$

This implies,

$$S(x_{2n+1},x_{2n+1},x_{2n+2}) < \frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}. \tag{2.21}$$

By a similar way, we get

$$S(y_{2n+1},y_{2n+1},y_{2n+2}) < \frac{S(y_{2n},y_{2n},y_{2n+1})+S(x_{2n},x_{2n},x_{2n+1})}{2}. \tag{2.22}$$

Adding (2.21) and (2.22) to obtain

$$\frac{\omega_{2n+1}}{2} < \frac{\omega_{2n}}{2}, \tag{2.23}$$

where $\omega_n = S(x_n,x_{n+1},x_{n+1})+S(y_n,y_{n+1},y_{n+1})$ as in Theorem 2.1. Interchanging the role of mappings F and G and using (2.17), yield that

$$\frac{\omega_{2n+2}}{2} < \frac{\omega_{2n+1}}{2}. \tag{2.24}$$

So we get $\{\omega_n\}$ be decreasing sequence and $\lim_{n\to\infty}\omega_n = \omega \ge 0$. Assume that $\omega > 0$ and then take limits as $n\to\infty$ in (2.20) to get

$$\begin{aligned}\int_0^{\omega}\phi(t)dt &\le \lim_{n\to\infty}\Phi\int_0^{\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}}\phi(t)dt \\ &< \int_0^{\frac{\omega}{2}}\phi(t)dt.\end{aligned}$$

Note that $\int_0^{\frac{S(x_{2n},x_{2n},x_{2n+1})+S(y_{2n},y_{2n},y_{2n+1})}{2}}\phi(t)dt \to \int_0^{\frac{\omega^+}{2}}\phi(t)dt = (\int_0^{\frac{\omega}{2}}\phi(t)dt)^+$.

Which is contradiction, then $\omega = 0$. By repeatedly use of property of S -metris space we observe that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and

$$x_n \to x \ \ and \ \ y_n \to y \ \ as \ \ n\to\infty, \tag{2.25}$$

for some $x, y \in X$. By continuity of F, we have $x \in F(x,y)$ and

$y \in F(y,x)$.

Now from (2.17), we get

$$\int_0^{H_S(F(x,y),F(x,y),G(x,y))} \varphi(t)dt + \int_0^{H_S(F(y,x),F(y,x),G(y,x))} \varphi(t)dt$$
$$\le \Phi\int_0^{\frac{S(x,x,x)+S(y,y,y)}{2}} \varphi(t)dt + \Phi\int_0^{\frac{S(y,y,y)+S(x,x,x)}{2}} \varphi(t)dt \Rightarrow$$
$$S(x,G(x,y),G(x,y)+S(y,G(y,x),G(y,x)) = 0 \Rightarrow x \in G(x,y),\ y \in G(y,x).$$

Hence (x,y) is coupled common fixed point of F and G. Similarly, the result follows when G is assumed to be continuous.

Now, consider that X has the sequential monotone property. If $x_{2n} = x$ and $y_{2n} = y$ for some $n \ge 0$, then $x_{2n} = x_{2n+1} \in F(x_{2n}, y_{2n})$ and $y_{2n} = y_{2n+1} \in F(y_{2n}, x_{2n})$. Also, from (2.17) we get

$$\begin{aligned}\int_0^{S(x,x,G(x,y))}\phi(t)dt &\le \int_0^{2S(x_{2n+1},x_{2n+1},x)}\phi(t)dt + \int_0^{H_S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),G(x,y))}\phi(t)dt\\ &\le \int_0^{2S(x_{2n+1},x_{2n+1},x)}\phi(t)dt + \Phi\int_0^{\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}}\phi(t)dt\\ &= 0\end{aligned}$$

and

$$\int_0^{S(y,y,G(y,x))}\phi(t)dt \le \int_0^{2S(y_{2n+1},y_{2n+1},y)}\phi(t)dt + \int_0^{H_S(F(y_{2n},x_{2n}),F(y_{2n},x_{2n}),G(y,x))}\phi(t)dt = 0.$$

So, (x_{2n}, y_{2n}) is a coupled common fixed point of F and G.

Suppose that $(x_{2n}, y_{2n}) \ne (x,y)$ for all n. Thus,

$$\Phi\int_0^{\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}}\phi(t)dt < \int_0^{\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}}\phi(t)dt.$$

From (2.17) we have

$$\begin{aligned}\int_0^{S(x,x,G(x,y))}\phi(t)dt &\le \int_0^{2S(x_{2n+1},x_{2n+1},x)}\phi(t)dt + \int_0^{H_S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),G(x,y))}\phi(t)dt\\ &\le \int_0^{2S(x_{2n+1},x_{2n+1},x)}\phi(t)dt + \Phi\int_0^{\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}}\phi(t)dt\\ &< \int_0^{2S(x_{2n+1},x_{2n+1},x)}\phi(t)dt + \int_0^{\frac{S(x_{2n},x_{2n},x)+S(y_{2n},y_{2n},y)}{2}}\phi(t)dt \to 0.\end{aligned}$$

Therefore, $x \in G(x,y)$. Similarly, $y \in G(y,x)$. By interchanging the role of functions F and G, we get the same result for F. Thus (x, y) is the common coupled fixed point of F and G.

Remark 2.1 *If we put* $\phi(t) = 1$ *for all* $t \in [0, \infty)$, *then Theorem 2.2 reduces to Theorem 2.1 as a special case.*

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في هذا البحث نقدم تعريف لخاصية الاطراد المختلطة (weakly mixed monotone property) لدوال متعددة القيم في فراغات متريه ذات ترتيب جزئي (partially ordered S- metric spaces) و كذلك نقوم بإثبات بعض نظريات علي وجود ووحدانية النقطة الثابته المزدوجة (coupled fixed point) والمشتركة لهذه الدوال في وجود شرط تضاغطي. نتائج هذا البحث تعمم بشكل أساسي النتائج المناظره في المرجع [١٠] .