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ESTIMATION FOR THE WEIBULL-GEOMETRIC DISTRIBUTION BASED ON CONSTANT PARTIALLY ACCELERATED LIFE TESTS VIA MCMC TECHNIQUE

Z.F. Jaheen and Sara M.A.M. Ali
Department of Mathematics, Faculty of Science, Assiut University, Egypt.

E-mail: zjaheen@aun.edu.eg

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In this article, constant partially accelerated life tests are considered. Based on a progressive first-failure censoring scheme, the maximum likelihood and the Bayes estimates for the parameters of the Weibull-Geometric distribution as well as the acceleration parameter are obtained. The Bayes estimates are derived using the Markov Chain Monte Carlo (MCMC) technique. A Monte Carlo simulation study has been conducted to compare the different estimates.

Keywords: Weibull-Geometric distribution; progressive first-failure censoring scheme; partially accelerated life test; Markov Chain Monte Carlo technique; Metropolis-Hastings.

1. INTRODUCTION

Most modern products are designed to have long lifetimes. So, it is too difficult to obtain reliable information about the lifetimes of these products at the time of testing under normal conditions due to high coasts. For this reason, accelerated life tests (ALTs) are used to estimate the lifetime of these products within a reasonable testing time. The test products are run at higher than usual levels of stress which include pressure, temperature, load, etc. The stress can be applied in different ways: commonly used methods are constant stress, progressive stress and step stress, see Nelson (1990).

In ALTs, the test items are tested only at accelerated conditions and the data collected are then extrapolated through a physically appropriate statistical model to estimate the life distribution at normal use conditions. On the other hand, in partially accelerated life tests (PALTs) items are tested at both normal and accelerated conditions. There are two major types of PALTs, constant PALT (C-PALT) and step-stress PALT (SS-PALT). Under SS-PALT, a test involves two levels of stress with the first one being at the normal level and at a specific time point, the stress changes. In a constant PALT, which is the main topic of this article, each item runs at either use condition, or accelerated condition only. PALTs have been extensively studied in recent years, see for example, Cheng and

Wang (2012), Zarrin et. al. (2012), Ismail (2014), Jaheen et. al. (2014), Abd El-Monem and Jaheen (2015), Abushal and Soliman (2015), Hyun and Lee (2015), Abdel-Hamid (2016), Abd-Elmougod and Mahmoud (2016) and Ismail (2016).

The Weibull-Geometric (WG) distribution was first introduced by Barreto-Souzaa et. al. (2011). With different parametrization, the same distribution has been studied by Tojeiro et al. (2014), under the name, the complementary Weibull-geometric distribution. The WG distribution generalizes the exponential-geometric (EG) distribution, proposed by Adamidis and Loukas (1998), and also the Weibull distribution. The hazard rate function of the EG distribution is monotone decreasing while that of the WG distribution can take more general forms. Unlike the Weibull distribution, the WG distribution is useful for modeling unimodal failure rates.

The Weibull-Geometric distribution with the parameters $\alpha > 0$, $\beta > 0$ and $p \in (0,1)$ *(denoted by $WG(\alpha,\beta,p)$)* has the following probability density function (pdf) and cumulative distribution function (cdf)

$$f_1(x;\alpha,\beta,p) = \alpha\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}\{1-pe^{-(\beta x)^{\alpha}}\}^{-2}, \quad x > 0, \tag{1.1}$$

$$F_1(x) = (1-e^{-(\beta x)^{\alpha}})(1-pe^{-(\beta x)^{\alpha}})^{-1}, \quad x > 0, \tag{1.2}$$

respectively.

As it can be seen from (1.1), when $p = 0$ we obtain the two-parameter Weibull distribution. Another special case is obtained for $\alpha = 1$, which corresponds to the exponential-geometric (EG) distribution with parameters (β, p).

The corresponding reliability and failure rate functions are given, respectively, by

$$S_1(x) = 1 - F_1(x) = ((1-p)e^{-(\beta x)^{\alpha}})(1-pe^{-(\beta x)^{\alpha}})^{-1}, \quad x > 0 \tag{1.3}$$

and

$$H_1(x) = \frac{f_1(x)}{S_1(x)} = \alpha\beta^{\alpha}x^{\alpha-1}\{1-pe^{-(\beta x)^{\alpha}}\}^{-1}, \quad x > 0. \tag{1.4}$$

The hazard rate function (1.4) is decreasing for $0 < \alpha \leq 1$. However, for $\alpha > 1$ it can take different forms.

Hamedani and Ahsanullah (2011) presented various characterizations of the Weibull-geometric distribution. Jodra and Jimenez-Gamero (2014) obtain explicit expressions for the moments of order statistics from the half-logistic distribution, the Weibull-geometric distribution and the long-term Weibull-geometric distribution. In (2015), Elhag et al. discussed the Bayesian inferences of unknown parameters of the progressively Type-II censored Weibull-geometric (WG) distribution. Jaheen and Ali (2016) estimated the parameters of the Weibull-Geometric distribution based on progressive first-failure censoring scheme. Also, in (2017) Jaheen and Ali predicted future observables from the Weibull-Geometric model based on progressively Type-II censored data.

The rest of the article is as follows: The model description and basic assumptions are described. The maximum likelihood and Bayes methods of estimation are used for estimating the unknown parameters of the WG model and the acceleration factor based on progressive first-failure censored data. Metropolis-Hastings (MH) algorithm is used to draw Markov Chain Monte Carlo (MCMC) samples from the posterior distributions, and they are in turn used to compute the Bayes estimates under two different loss functions. Monte Carlo simulation study is used to compare the different estimates.

2. THE CONSTANT PARTIALLY ACCELERATED LIFE TEST MODEL

According to constant PALTs, there are two groups of test items. The first group is under normal conditions while the second one is under accelerated condition. Progressive first-failure censoring is applied as follows. The first group has n_1 sets, each set has k_1 items. The second group has n_2 sets, each of which has k_2 items. In group j, $j = 1,2$, as soon as the first failure (say $X_{j1:m_j:n_j:k_j}$) has occurred R_{j1} groups with the group in which the first failure is observed are randomly removed from the test, when the second failure (say $X_{j2:m_j:n_j:k_j}$) has occurred R_{j2} groups and the group in which the second first failure is observed are removed from the test, and finally as soon as the $m_j - th$ failure (say $X_{jm_j:m_j:n_j:k_j}$) has occurred R_{jm_j} $(m_j \leq n_j)$ groups and the group in which the $m_j - th$ first failure is observed are randomly removed from the test. The life times $X_{j1:m_j:n_j:k_j} < X_{j2:m_j:n_j:k_j} <\ldots< X_{jm_j:m_j:n_j:k_j}$, $j = 1,2$ are called progressive first-failure censored order statistics with the progressive censoring scheme $R_j = (R_{j1}, R_{j2}, \ldots, R_{jm_j})$. It is clear that m_j, $j = 1,2$, is number of the observed first-failures $(1 < m_j \leq n_j)$ and $\sum_{i=1}^{m_j} R_{ji} +$

$m_j = n_j$. If the failure times of the $n_j \times k_j$ items originally in the test are from a continuous population with cdf s $F_j(x)$ and pdf s $f_j(x)$, the joint probability density function for $X_{j1:m_j:n_j:k_j} < X_{j2:m_j:n_j:k_j} <\ldots< X_{jm_j:m_j:n_j:k_j}$ is given by

$$L(\alpha,\beta,p,\gamma;\underline{x}) = \prod_{j=1}^{2} C_j k_j^{m_j}[\prod_{i=1}^{m_j} f_j\left(x_{ji:m_j:n_j:k_j}\right)\left(1-F_j\left(x_{ji:m_j:n_j:k_j}\right)\right)^{k_j(R_{ji}+1)-1}\Big], \quad (2.1)$$

where

$$0 < x_{j1:m_j:n_j:k_j} < x_{j2:m_j:n_j:k_j} <\ldots< x_{jm_j:m_j:n_j:k_j} < \infty$$

and

$$C_j = n_j(n_j - R_{j1} - 1)(n_j - R_{j1} - R_{j2} - 2)\ldots(n_j - R_{j1} - R_{j2} -\ldots- R_{jm_j-1} - m_j + 1).$$

It is clear from (2.1) that the progressive first-failure censored scheme includes the first-failure censored scheme, the progressive Type-II censored order statistics, usual Type-II censored ordered statistics and the complete ordered sample as special cases.

2.1 BASIC ASSUMPTIONS

In this study, the lifetimes of items under normal conditions are assumed to follow the WG distribution having pdf, cdf, reliability and failure rate functions given in (1.1)-(1.4). The failure rate function of an item tested at accelerated condition is given by $H_2(x) = \gamma H_1(x)$ where γ is an acceleration parameter satisfying $\gamma > 1$. Therefore the failure rate and the reliability functions are given, respectively, by

$$H_2(x;\alpha,\beta,p,\gamma) = \gamma\alpha\beta^\alpha x^{\alpha-1}(1-pe^{-(\beta x)^\alpha})^{-1}, \quad (2.2)$$

and

$$\begin{aligned} S_2(x;\alpha,\beta,p,\gamma) &= \exp[-\int_0^x H_2(u)du] \\ &= ((1-p)e^{-(\beta x)^\alpha}(1-pe^{-(\beta x)^\alpha})^{-1})^\gamma. \end{aligned} \quad (2.3)$$

Then the cdf and pdf under accelerated condition can be written, respectively, by

$$\begin{aligned} F_2(x;\alpha,\beta,p,\gamma) &= 1 - S_2(x;\gamma,\alpha,\beta,p) \\ &= 1 - (((1-p)e^{-(\beta x)^\alpha})(1-pe^{-(\beta x)^\alpha})^{-1})^\gamma, \end{aligned} \quad (2.4)$$

and

$$f_2(x;\alpha,\beta,p,\gamma) = \gamma\alpha\beta^\alpha(1-p)^\gamma x^{\alpha-1}e^{-\gamma(\beta x)^\alpha}(1-pe^{-(\beta x)^\alpha})^{-(\gamma+1)}. \quad (2.5)$$

3. MAXIMUM LIKELIHOOD ESTIMATION

In this section we derive the maximum likelihood estimates (MLEs) of the acceleration parameter and the unknown parameters α and p of the WG distribution when β is known. For $j = 1,2$, the life times $X_{j1:m_j:n_j:k_j} < X_{j2:m_j:n_j:k_j} <\ldots< X_{jm_j:m_j:n_j:k_j}$ denote two progressive first-failure censored samples from two populations with (pdf s) and (cdf s) given by (1.1), (1.2), (2.4) and (2.5), with the progressive censoring scheme $R_j = (R_{j1}, R_{j2}, \ldots, R_{jm_j})$. Thus, from (1.1), (1.2), (2.4), (2.5) and (2.1) the likelihood function takes the following form

$$\begin{aligned} L(\alpha,\beta,p,\gamma;\underline{x}) &= C_1C_2k_1^{m_1}k_2^{m_2}(\alpha\beta^\alpha)^{m_1+m_2}\gamma^{m_2}\prod_{j=1}^2\prod_{i=1}^{m_j} x_{ji}^{\alpha-1}(1-p)^{A_{ji}\gamma^{j-1}} \\ &\times e^{-(\beta x_{ji})^\alpha(A_{ji}\gamma^{j-1})}(1-pe^{-(\beta x_{ji})^\alpha})^{-(A_{ji}\gamma^{j-1}+1)}, \end{aligned} \quad (3.1)$$

where $A_{ji} = k_j(R_{ji}+1)$ and

$$C_j = n_j(n_j - R_{j1} - 1)(n_j - R_{j1} - R_{j2} - 2)\ldots(n_j - R_{j1} - R_{j2} - \ldots - R_{jm_j-1} - m_j + 1).$$

The logarithm of (3.1) can be written as

$$\begin{aligned} l(\alpha,\beta,p,\gamma;\underline{x}) &= \sum_{j=1}^2 \ln(C_jk_j^{m_j}) + (m_1+m_2)(\ln\alpha + \alpha\ln\beta) + m_2\ln\gamma \\ &+ \sum_{j=1}^2\sum_{i=1}^{m_j} [(\alpha-1)\ln x_{ji} - (\beta x_{ji})^\alpha(A_{ji}\gamma^{j-1}) + (A_{ji}\gamma^{j-1})\ln(1-p) \\ &-(A_{ji}\gamma^{j-1}+1)\ln(1-pe^{-(\beta x_{ji})^\alpha})]. \end{aligned} \quad (3.2)$$

Taking the derivatives with respect to α, p and γ of (3.2), assuming β is known, and putting them equal to zero we get

$$\begin{aligned} \frac{\partial l}{\partial\alpha} &= (m_1+m_2)(\frac{1}{\alpha} + \ln\beta) + \sum_{j=1}^2\sum_{i=1}^{m_j} [\ln(x_{ji}) \\ &-(\beta x_{ji})^\alpha((pe^{-(\beta x_{ji})^\alpha} + A_{ji}\gamma^{j-1})(1-pe^{-(\beta x_{ji})^\alpha})^{-1})\ln(\beta x_{ji})] = 0, \end{aligned} \quad (3.3)$$

$$\frac{\partial l}{\partial p} = \sum_{j=1}^2\sum_{i=1}^{m_j} [\frac{A_{ji}\gamma^{j-1}}{1-p} - \frac{(A_{ji}\gamma^{j-1}+1)e^{-(\beta x_{ji})^\alpha}}{1-pe^{-(\beta x_{ji})^\alpha}}] = 0. \quad (3.4)$$

$$\frac{\partial l}{\partial\gamma} = \frac{m_2}{\gamma} - \sum_{j=1}^2\sum_{i=1}^{m_j} (j-1)\gamma^{j-2}A_{ji}[(\beta x_{ji})^\alpha - \ln(1-p) + \ln(1-pe^{-(\beta x_{ji})^\alpha})] = 0. \quad (3.5)$$

It follows from (3.5) that

$$\gamma = \frac{m_2}{\sum_{i=1}^{m_2} A_{2i}[(\beta x_{2i})^\alpha - \ln(1-p) + \ln(1-pe^{-(\beta x_{2i})^\alpha})]}. \quad (3.6)$$

By solving the non-linear equations (3.3), (3.4) and (3.6) together, numerically, we get the maximum likelihood estimates of the parameters α, p and γ.

4. BAYESIAN ESTIMATION

Assume that the parameter α is a random variable with Gamma prior distribution with pdf of the form

$$g_1(\alpha) = \frac{b_1^{a_1}}{\Gamma(a_1)} \alpha^{a_1-1} e^{-b_1\alpha}, \quad a_1, b_1 > 0, \quad \alpha > 0. \tag{4.1}$$

Assuming also that the parameter p is independent of α and has a Beta prior distribution with pdf given by

$$g_2(p) = \frac{1}{B(a_2,b_2)} p^{a_2-1}(1-p)^{b_2-1}, \quad 0 \leq p \leq 1. \tag{4.2}$$

The prior density for the acceleration factor γ can be taken as

$$g_3(\gamma) = a e^{-a(\gamma-1)}, \quad \gamma > 1, \quad a > 0. \tag{4.3}$$

Hence, the pdf for the joint prior distribution of α, p and γ is

$$g(\alpha, p, \gamma) = \frac{a b_1^{a_1}}{\Gamma(a_1) B(a_2,b_2)} \alpha^{a_1-1} p^{a_2-1} (1-p)^{b_2-1} e^{-(b_1\alpha + a(\gamma-1))}. \tag{4.4}$$

From (3.1) and (4.4), the joint posterior distribution takes the form

$$\begin{aligned} q(\alpha, p, \gamma | \underline{x}) &= K C_1 C_2 k_1^{m_1} k_2^{m_2} \alpha^{m_1+m_2+a_1-1} \beta^{\alpha(m_1+m_2)} \gamma^{m_2} p^{a_2-1} (1-p)^{b_2-1} \\ &\times e^{-(b_1\alpha + a(\gamma-1))} \prod_{j=1}^{2} \prod_{i=1}^{m_j} x_{ji}^{\alpha-1} (1-p)^{A_{ji}\gamma^{j-1}} e^{-(\beta x_{ji})^{\alpha}(A_{ji}\gamma^{j-1})} \\ &\times (1 - p e^{-(\beta x_{ji})^{\alpha}})^{-(A_{ji}\gamma^{j-1}+1)}, \end{aligned} \tag{4.5}$$

where K is the normalizing constant given by

$$K^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty g(\alpha, \beta, p) L(\alpha, \beta, p; \underline{x}) dp d\alpha d\beta,$$

and $L(\alpha, p, \gamma; \underline{x})$ is the likelihood function given by (3.1).

The marginal posterior distributions and hence the Bayes estimates are computed from the posterior distribution (4.5) which includes complicated integrals that cannot be obtained in closed forms. Therefore the MCMC sampling procedure will be used to compute these Bayes estimates.

The most two often utilized techniques of the MCMC methods are the Gibbs sampler and the MH techniques. The Gibbs sampler technique needs the conditional posterior distributions to be in closed forms that can be simply generated from them. On the other hand, the MH technique needs only to use a jumping or a proposal distribution to generate from it instead of some complex distribution. For the algorithm to be efficient, the jumping distribution should be easy to sample from it. These techniques have been established in a number of references, see for example, Upadhyay and Gupta (2010) and Jaheen and Al Harbi (2011).

From (4.5) the conditional posterior density functions are given, respectively, by

$$q_1(\alpha|\beta,p,\gamma;\underline{x}) \propto \alpha^{m_1+m_2+a_1-1}\beta^{\alpha(m_1+m_2)}e^{-b_1\alpha} \times \prod_{j=1}^{2}\prod_{i=1}^{m_j} x_{ji}^{\alpha-1}e^{-(\beta x_{ji})^{\alpha}(A_{ji}\gamma^{j-1})}(1-pe^{-(\beta x_{ji})^{\alpha}})^{-(A_{ji}\gamma^{j-1}+1)}, \quad (4.6)$$

$$q_2(p|\alpha,\beta,\gamma;\underline{x}) \propto p^{a_2-1}(1-p)^{b_2-1} \times \prod_{j=1}^{2}\prod_{i=1}^{m_j} (1-p)^{A_{ji}\gamma^{j-1}}(1-pe^{-(\beta x_{ji})^{\alpha}})^{-(A_{ji}\gamma^{j-1}+1)}, \quad (4.7)$$

$$q_3(\gamma|\alpha,\beta,p;\underline{x}) \propto \gamma^{m_2}e^{-a\gamma}\prod_{i=1}^{m_2}[(1-p)e^{-(\beta x_{2i})^{\alpha}}(1-pe^{-(\beta x_{2i})^{\alpha}})^{-1}]^{A_{2i}\gamma}. \quad (4.8)$$

As can be seen from (4.6), (4.7) and (4.8) these conditional posteriors cannot be reduced to closed forms and therefore we cannot sample directly from them applying the Gibbs sampler technique. So, we will consider the MH algorithm to generate samples from these conditional posterior distributions and then compute the Bayes estimates under the squared error and Linex loss functions. For this purpose, we use the MH algorithm, described in Metropolis and Ulam (1949) and Metropolis et al. (1953), as follows:

1. Start with initial values $\alpha^{(0)}$, $p^{(0)}$ and $\gamma^{(0)}$ and set $i=1$.
2. Generate a candidate point p^* from a proposal Uniform (0,1) distribution, and calculate the ratio

$$r_1 = \frac{q_1(p^*|\alpha^{(0)},\gamma^{(0)};\underline{x})}{q_1(p^{(i-1)}|\alpha^{(0)},\gamma^{(0)};\underline{x})}.$$

3. Generate u from a Uniform (0, 1) distribution, If $u \le r_1$, accept p^* and set $p^{(i)}=p^*$, else set $p^{(i)}=p^{(i-1)}$.
4. i=i+1.
5. Repeat steps from 2-4 N times.

6. Calculate the Bayes estimator of p under a squared error loss function from

$$\hat{p}_{BS}=\frac{1}{N-M}\sum_{i=M+1}^{N}p^{(i)},$$

where M is the burn-in period.

7. Repeat steps from 2-6 for the parameters α and γ. The proposal distribution for the parameter α is the normal distribution $N(\alpha^{(i-1)},1)$, and for the parameter γ is $Uniform(2,5)$. Calculate the ratio r_2 from

$$r_2=\frac{q_2(\alpha^*|\hat{p}_{BS},\gamma^{(0)};\underline{x})}{q_2(\alpha^{(i-1)}|\hat{p}_{BS},\gamma^{(0)};\underline{x})},$$

for the parameter α, and for the parameter γ calculate the ratio r_3 from

$$r_3=\frac{q_3(\gamma^*|\hat{\alpha}_{BS},\hat{p}_{BS};\underline{x})}{q_3(\gamma^{(i-1)}|\hat{\alpha}_{BS},\hat{p}_{BS};\underline{x})},$$

where $\hat{\alpha}_{BS}$ is the Bayes estimator of α under a squared error loss function which calculated from

$$\hat{\alpha}_{BS}=\frac{1}{N-M}\sum_{i=M+1}^{N}\alpha^{(i)}.$$

8. Calculate the Bayes estimator of γ under a squared error loss function as follows

$$\hat{\gamma}_{BS}=\frac{1}{N-M}\sum_{i=M+1}^{N}\gamma^{(i)},$$

and under a Linex loss function, with the asymmetric parameter ξ, calculate the Bayes estimators of α, p and γ in the forms

$$\hat{\alpha}_{BL}=\frac{-1}{\xi}Ln[\frac{\sum_{i=M+1}^{N}e^{-\xi\alpha^{(i)}}}{N-M}],$$

$$\hat{p}_{BL}=\frac{-1}{\xi}Ln[\frac{\sum_{i=M+1}^{N}e^{-\xi p^{(i)}}}{N-M}],$$

$$\hat{\gamma}_{BL}=\frac{-1}{\xi}Ln[\frac{\sum_{i=M+1}^{N}e^{-\xi\gamma^{(i)}}}{N-M}].$$

It may be noted that we use the maximum likelihood estimates of the parameters as the initial values $\alpha^{(0)}$, $\beta^{(0)}$ and $p^{(0)}$ in step 1.

5. SIMULATION STUDY

The performance of the different methods cannot be compared theoretically. Therefore, in order to compare the estimators of the parameters, Monte Carlo simulations are performed. Based on progressive first-failure censoring scheme, the different estimators are computed and compared numerically. All computations are performed using Mathematica 7.0. We mainly compare the performance of the MLEs and Bayes estimators of the unknown parameters α, p and γ when β is known under two different losses.

The comparison between the estimates is taking place according to the following steps.

1. For given hyper parameters, a_1, b_1, a_2, b_2 and a, generate α, p and γ from the prior densities given by (4.1), (4.2) and (4.3).

2. For given values of n_j and the progressive schemes R_{ji}, $j = 1,2$, with the generated values α, p and γ in step (1), generate two progressive first-failure censored samples of size m_j using the algorithm described in Balakrishnan and Sandhu (1995) with the distribution functions $[1 - (1 - F_j(x))^{k_j}]$ where $F_j(x)$ is given by (1.2) and (2.4) for $j = 1,2$ respectively.

3. The maximum likelihood estimators are then obtained by solving the three nonlinear equations given by (3.3), (3.4) and (3.5) numerically.

4. The Bayes estimators under the two different loss functions are then obtained by applying the MCMC technique, as described above.

5. The above four steps are repeated 500 times and the mean squared errors (MSE) are then computed for the different estimators.

The Bayes estimates are computed based on 10,000 MCMC samples, where the first 1000 values discarded as burn-in. Two different values of the asymmetric parameter $\xi(-5,5)$ are considered to get the corresponding Bayes estimates.

Different combinations of n_j, m_j and the progressive schemes R_{ji} are considered. A special case from the progressive first-failure censored scheme, which is progressive censored scheme, are considered as well. The results of this simulation are presented in tables (1) and (2).

Table 1: Mean squared error of the ML and Bayes estimators with hyperparameters $a_1 = 2, b_1 = 3, a_2 = 2, b_2 = 4, a = 2$, and $k_1 = k_2 = 3$

n_1 n_2	m_1 m_2	R_1 R_2		MLE	MCMC		
					SEL	Linex	
						$\xi = 5$	$\xi = -5$
20 20	10 15	(1^{10}) $(1^5, 0^{10})^*$	α p γ	0.1674 0.0468 1.4595	0.1017 0.0343 0.1261	0.1039 0.0384 0.0342	0.1079 0.0314 0.8712
		(1^{10}) $(0^{10}, 1^5)$	α p γ	0.2568 0.0470 3.0037	0.1580 0.0348 0.1793	0.1500 0.0376 0.0512	0.1742 0.0330 0.9601
25 35	15 30	$(1^{10}, 0^5)$ $(1^5, 0^{25})$	α p γ	0.0861 0.0397 0.6636	0.0629 0.0328 0.1438	0.0639 0.0360 0.0440	0.0651 0.0303 0.5766
		$(0^5, 1^{10})$ $(0^{25}, 1^5)$	α p γ	0.0925 0.0344 0.7469	0.0666 0.0286 0.1596	0.0661 0.0310 0.0558	0.0700 0.0268 0.5617
30 40	20 35	$(1^{10}, 0^{10})$ $(1^5, 0^{30})$	α p γ	0.0689 0.0304 0.5458	0.0524 0.0261 0.1596	0.0537 0.0284 0.0783	0.0536 0.0242 0.4754
		$(0^{10}, 1^{10})$ $(0^{30}, 1^5)$	α p γ	0.0868 0.0341 0.5965	0.0640 0.0287 0.1579	0.0629 0.0306 0.0659	0.0673 0.0272 0.4853

*The scheme: $(1^5, 0^{10})$ means $R = (1,1,1,1,1,0,0,0,0,0,0,0,0,0,0)$

Table 2: Mean squared error of the ML and Bayes estimators with hyper parameters $a_1 = 2, b_1 = 3, a_2 = 2, b_2 = 4, a = 2$, and $k_1 = k_2 = 1$

n_1 n_2	m_1 m_2	R_1 R_2		MLE	MCMC		
					SEL	Linex	
						$\xi = 5$	$\xi = -5$
20 20	10 15	(1^{10}) $(1^5, 0^{10})^*$	α p γ	0.2566 0.0499 1.2699	0.1028 0.0317 0.1112	0.1236 0.0362 0.0499	0.1228 0.0288 0.7758
		(1^{10}) $(0^{10}, 1^5)$	α p γ	0.2752 0.0481 2.1335	0.1258 0.0324 0.1571	0.1420 0.0375 0.0678	0.1475 0.0290 0.8396
25 35	15 30	$(1^{10}, 0^5)$ $(1^5, 0^{25})$	α p γ	0.1332 0.0419 0.7388	0.0802 0.0317 0.1653	0.0904 0.0359 0.0694	0.0854 0.0285 0.5678
		$(0^5, 1^{10})$ $(0^{25}, 1^5)$	α p γ	0.1262 0.0368 0.7464	0.0755 0.0288 0.1612	0.0848 0.0326 0.0644	0.0815 0.0260 0.5463
30 40	20 35	$(1^{10}, 0^{10})$ $(1^5, 0^{30})$	α p γ	0.0764 0.0400 0.6292	0.0542 0.0308 0.1583	0.0661 0.0352 0.0654	0.0528 0.0273 0.5233
		$(0^{10}, 1^{10})$ $(0^{30}, 1^5)$	α p γ	0.1025 0.0353 0.5759	0.0657 0.0284 0.1549	0.0726 0.0315 0.0761	0.0701 0.0259 0.4699

6. CONCLUSIONS

In this article we discussed the estimation of the two unknown parameters (α, p) of the Weibull-Geometric distribution and the acceleration parameter γ, when the parameter β is known, based on constant partially accelerated life tests. Based on a progressive first-failure censored sample the maximum likelihood and the Bayes estimates are obtained. It is observed that the Bayes estimators cannot be obtained in explicit forms and they need complicated integrals to be performed numerically. Because of that, the MCMC method, namely the MH sampling technique, is applied to obtain the Bayes estimates under squared error and Linex loss functions.

From the results, in tables (1) and (2), it can be observed that the Bayes estimates under the symmetric (SEL) and the asymmetric (Linex) loss functions are generally better than their corresponding MLEs. It can also be seen that the mean squared errors decrease as the sample sizes increase. Also there is no large effect of exchanging the censoring scheme on results.

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تقدير معالم توزيع وايبل - الهندسي على أساس اختبارات الحياة المسرعة تحت إجهاد ثابت جزئيا باستخدام طريقة MCMC

أ.د. زينهم فكري جاهين [*] *و د.سارة محمد عادل محمد علي* [*]
[*]قسم الرياضيات - كلية العلوم- جامعة أسيوط

تم في هذا البحث إيجاد مقدرات بييز والترجيح الأعظم لمعالم توزيع وايبل - الهندسي بالاضافة إلى مَعلمة التسارع (γ) على أساس اختبارات الحياة المسرعة تحت اجهاد ثابت جزئيا .تم حساب مقدرات بييز باستخدام طريقة Markov Chain Monte Carlo. وقمنا بالمقارنة بين التقديرات المختلفة باستخدام طرق محاكاة مونت كارلو (Monte Carlo simulation study).