Assiut Univ. J. of Mathematics and Computer Science
46(1), p-p1- 12(2017)

EXACT SOLUTIONS OF THE ONE-DIMENSIONAL CONVECTION-DIFFUSION EQUATION USING LIE GROUP METHOD

M. A. Assabaai* and Salim F. Bamsaoud

[*]Department of Mathematics, Faculty of Science, Hadhramout University, Yemen

Tel: 00967775437743, (assabaai@hotmail.com)

[**]Department of Physics, Faculty of Science, Hadhramout University, Yemen

Tel: 00967770771896, (saalem88@yahoo.com)

Received: 22/1/2017 **Accepted:** 14/5/2017

Lie symmetry group analysis is applied to determine the exact solution of the one-dimensional convection-diffusion equation. The similarity transformation is found using symmetries, and the invariant solution of the original partial differential equation (PDE) is produced from the solution of transformed ordinary differential equation (ODE). The analytical solutions are obtained using symmetries and summarized in tabulated form.

Keywords: convection equation; diffusion equation; Lie group; similarity transformation; invariant solution.

1. Introduction

Convection-diffusion equation is one of the most important partial differential equations. The equation appears in a wide range of engineering and various fields of science, for instance: radial physics, hydrology, building physics, chemistry. Daga and Pradhan [1] presented an analytical solution to describe the uniform dispersion of a solute in uniform flow. Fallahzadeh and Shakibi [2] solved the convection-diffusion equation using homotopy analysis method. Veling [3] presented an analytical solution of the convection-diffusion equation in radial physics. Tracy [6] applied an analytical solution to study the relation between moisture content and relative conductivity against pressure head to unsaturated flow in groundwater. Hu et al. [8] used Fourier series to obtain new analytical solution for convection-diffusion equation to calculate soil thermal diffusivity, water flux density and soil temperature. Svoboda [19] showed that the modern construction containing permeable thermal insulation are very sensitive to the convective component of the heat transfer.

Many researchers have used various numerical methods to solve convection-diffusion equation. Baza'n [5] studied numerical solution of convection-diffusion equation by Chebyshev pseudospectral method. In Boztosun and Charafi [10], mesh-free and mesh-dependent methods were used to solve convection-diffusion equation. Ghasemi and Kajani [12] solved the convection-diffusion equation using He's homotopy perturbation method. Olayiwola [13] presented Variational iteration method to solve convection-diffusion equation. Feng [17], Explicit finite difference method was used to solve convection-diffusion equation. EL-Wakil and Elhanbaly [18] solved the convection-diffusion equation using Adomian decomposition method.

The purpose of this paper is to use Lie group analysis method, also called classical symmetries method (CSM) to obtain the exact solution of convection-diffusion equation. Some applications of this method in differential equations can be found in [4; 7; 9; 11; 14; 15 and 16].

2. A Model Problem

We consider the one-dimensional convection-diffusion equation

$$\frac{\partial u}{\partial t} + \alpha \frac{\partial u}{\partial x} - \varepsilon \frac{\partial^2 u}{\partial x^2} = 0, \ 0 \le x \le 1, 0 \le t \le T, \tag{1.1}$$

With initial condition

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{1.2}$$

and boundary conditions

$$u(0,t) = g_0(t), \quad 0 \le t \le T,$$

$$u(1,t) = g_1(t), \quad 0 \le t \le T \tag{1.3}$$

where f, g_0, and g_1 are known functions, the parameters α and ε are convection and diffusion coefficients respectively and both are assumed to be positive.

In fluid mechanics, Eq. (1.1) describes the transport occurring through the combination of convection and diffusion. The analytical solution of convection-diffusion equation (1.1) along with the initial and boundary conditions (1.2), (1.3) describe practically the behavior of the pollutant concentration distribution through an open medium like rivers [1].

3. Lie Group Transformation Method

The Lie group transformation method [4, 9, 14, 16] is one of the group theoretic methods, which is used to transform the partial differential equation (PDE) to ordinary differential equation (ODE) by so-called similarity transformation. In this study, we present the method of Lie group of transformations, which makes Eq. (1.1) invariant.

We start with the system of m differential equations

$$\Delta^{i}(x,u,u_{1},...,u_{(k)})=0, \quad i=1,2,\ldots,m,$$

(1.4)

of order k , with p independent variables $x=(x_1,x_2,...,x_p)\in R^p$, and q dependent variables $u=(u^1,u^2,...,u^q)\in R^q$. where m,k, p *and* q are positive integers, and u_k is the set corresponding to all $k\underline{th}$ order partial derivatives of u with respect to x.

Suppose that the one-parameter ε-Lie group point of transformations are given by

$$x_i^{*}=X_i(x,u;\varepsilon)=x_i+\varepsilon\,\xi_i(x,u)+\mathrm{O}(\varepsilon^2),$$

(1.5)

$$u^{*\alpha}=U^{\alpha}(x,u;\varepsilon)=u^{\alpha}+\varepsilon\,\phi^{\alpha}(x,u)+\mathrm{O}(\varepsilon^2),$$

(1.6)

$$u_i^{*\alpha}=U_i^{\alpha}(x,u,u_1;\varepsilon)=u_i^{\alpha}+\varepsilon\,\phi_i^{(1)\alpha}(x,u,u_1)+\mathrm{O}(\varepsilon^2),$$

(1.7)

$$\vdots$$

$$\begin{aligned} u^{*\alpha}_{i_1 i_2\cdots i_k} &= U^{\alpha}_{i_1 i_2\cdots i_k}(x,u,u_1,...,u_k;\varepsilon) \\ &= u^{\alpha}_{i_1 i_2\cdots i_k}+\varepsilon\,\phi^{(k)\alpha}_{i_1 i_2\cdots i_k}(x,u,u_1,...,u_k)+\mathrm{O}(\varepsilon^2) \end{aligned}$$

(1.8)

where ε is the group parameter, and $\xi_i(x,u)$, $\phi^{\alpha}(x,u)$ are the infinitesimals and $\phi^{(k)\alpha}_{i_1 i_2\cdots i_k}$ are given by

$$\phi_i^{(1)\alpha}=D_i\,\phi^{\alpha}-(D_i\,\xi_i)\,u_j^{\alpha}, \qquad i,j=1,2,\ldots,p;\ \alpha=1,2,\ldots,q$$

(1.9)

and

$$\phi^{(k)\alpha}_{i_1 i_2 \cdots i_k} = D_i\,\phi^{(k-1)\alpha}_{i_1 i_2 \cdots i_{k-1}} - (D_i\,\xi_i)\,u^{\alpha}_{i_1 i_2 \cdots i_{k-1} j},$$

$$i_\ell = 1,2,\ldots,p \ \ for \ \ \ell = 1,2,\ldots,k \ \ with \ k = 2,3,\ldots$$

(1.10)

where D is the total derivative operator defined as

$$D_i = \frac{\partial}{\partial x_i} + u^{\alpha}_i \frac{\partial}{\partial u^{\alpha}} + u^{\alpha}_{ij} \frac{\partial}{\partial u^{\alpha}_j} + \ldots + u^{\alpha}_{i i_1 i_2 \cdots i_p} \frac{\partial}{\partial u^{\alpha}_{i_1 i_2 \ldots i_p}} + \ldots$$

(1.11)

with summation over a repeated index.

The infinitesimal generator of the one-parameter Lie group of transformations for the system (1.4) is

$$\mathrm{X} = \sum_{i=1}^{P} \xi_i(x,u)\frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{q} \phi^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}$$

(1.12)

and the k^{th} prolongation of the infinitesimal generator (1.12) is

$$\mathrm{Pr}^{(k)}\,\mathrm{X} = \mathrm{X} + \phi^{(1)\alpha}_i(x,u,u_1)\frac{\partial}{\partial u^{\alpha}_i} + \ldots + \phi^{(k)\alpha}_{i_1 i_2 \cdots i_k}(x,u,u_1,\ldots,u_k)\frac{\partial}{\partial u^{\alpha}_{i_1 i_2 \cdots i_k}} \qquad (1.13)$$

The invariance condition of the system (1.4) is given by the following equations:

$$\mathrm{Pr}^{(k)}\,\mathrm{X}(\Delta^i)\big|_{\Delta^j=0} = 0, \qquad i,j = 1,2,\ldots,m \qquad (1.14)$$

Thus, the system of differential equations (1.4) is an invariant under the transformations of a one-parameter group with the infinitesimal generator (1.12) if the ξ_i's and ϕ^{α}'s are determined from (1.14). Hence, condition (1.14) expresses that $pr^{(k)}\mathrm{X}$ vanishes on the solution set of system (1.4).

Now, we consider

$$\Delta = \frac{\partial u}{\partial t} + \alpha\frac{\partial u}{\partial x} - \varepsilon\frac{\partial^2 u}{\partial x^2} \qquad (1.15)$$

and the following Lie group of transformations with independent variables x,t and dependent variable u

$$\bar{x} = \bar{x}(x,t,u;\varepsilon),\ \ \bar{t} = \bar{t}(x,t,u;\varepsilon),\ \ \bar{u} = \bar{u}(x,t,u;\varepsilon) \qquad (1.16)$$

The infinitesimal generator (1.12) for (1.16) can be expressed in the following form

$$\mathrm{X} = \xi_1(x,t,u)\frac{\partial}{\partial x} + \xi_2(x,t,u)\frac{\partial}{\partial t} + \phi\,(x,t,u)\frac{\partial}{\partial u} \qquad (1.17)$$

Here $u^1 = u,\ x_1 = x,\ x_2 = t$ and $\phi^1 = \phi$.

To calculate the infinitesimals ξ_1, ξ_2 and ϕ from the condition (1.14) we need to write the second prolongation of the infinitesimal generator given by (1.12) since the governing equation include the second order partial derivative. Then from the formula (1.13), we get

$$\mathrm{Pr}^{(2)}\,\mathrm{X} = \mathrm{X} + \phi_x\frac{\partial}{\partial u_x} + \phi_t\frac{\partial}{\partial u_t} + \phi_{xt}\frac{\partial}{\partial u_{xt}} + \phi_{xx}\frac{\partial}{\partial u_{xx}} + \phi_{tt}\frac{\partial}{\partial u_{tt}} \qquad (1.18)$$

where $\phi_x ,, \phi_t , \phi_{xt} , \phi_{xx}$ *and* ϕ_{tt} are given from the expressions (1.9) and (1.10).

Applying the second prolongation (1.18) to (1.15), that is,

$$\mathrm{Pr}^{(2)}\,\mathrm{X}(\Delta) = 0, \quad when \quad \Delta = 0$$

(1.19) Eq. (1.19) then leads to

$$\phi_t + \alpha\phi_x - \varepsilon\phi_{xx} = 0 \qquad (1.20)$$

Conditions on the infinitesimals ξ_1, ξ_2 and ϕ are determined by equating coefficients of similar derivatives of monomials in u_x, u_t and higher derivatives by zero. This leads to a system of partial differential equations from which we can determine ξ_1, ξ_2 and ϕ. Solving these equations to get the infinitesimals solutions ξ_1, ξ_2 and ϕ in the following forms:

$$\xi_1(x,t,u) = \frac{1}{2}xt\,c_1 + \frac{1}{2}(\alpha t + x)c_2 - 2\varepsilon t c_5 + c_8,$$

$$\xi_2(x,t,u) = \frac{1}{2}t^2 c_1 + t c_2 + c_3,$$

$$\phi(x,t,u) = c_6 \exp(at + \frac{1}{2\varepsilon}(\alpha - \sqrt{\alpha^2 + 4a\varepsilon})x) + c_7 \exp(at + \frac{1}{2\varepsilon}(\alpha + \sqrt{\alpha^2 + 4a\varepsilon})x)$$

$$+ u c_4 - (\alpha t - x)u c_5 - \frac{1}{4}(t - \frac{1}{2\varepsilon}(\alpha t - x)^2)u c_1$$

(1.21)

where c_i, $i = 1,2,\dots,8$ are arbitrary constants.

Eqs. (1.21) show that the convection-diffusion equation has the following generators symmetry group:

$$X_1 = \frac{1}{2} x t \partial_x + \frac{1}{2} t^2 \partial_t - \frac{1}{4}\left(t - \frac{1}{2\varepsilon}(\alpha t - x)^2\right) u \partial_u,$$

$$X_2 = \frac{1}{2}(\alpha t + x)\partial_x + t\partial_t,$$

$$X_3 = \partial_t,\ X_4 = u\partial_u,\ X_5 = -2\varepsilon t \partial_x - (\alpha t - x) u \partial_u,$$

$$X_6 = \exp\left(a t + \frac{1}{2\varepsilon}\left(\alpha - \sqrt{\alpha^2 + 4 a \varepsilon}\right) x\right)\partial_u,$$

$$X_7 = \exp\left(a t + \frac{1}{2\varepsilon}\left(\alpha + \sqrt{\alpha^2 + 4 a \varepsilon}\right) x\right)\partial_u,\ X_8 = \partial_x$$

(1.22)

4. Similarity Reduction And Similarity Solutions

We consider the generators $(X_3 + cX_8), (X_4 + X_8), (X_3 + X_4),$ and (X_2).

Now, we classify and organize the similarity transformations and the exact solutions of Eq. (1.1) by using above generators in the following tables.

Table 1. (For $X_3 + cX_8$)

$X_3 + cX_8$	$\partial_t + c\partial_x$
Similarity transformation	$\eta = x - ct,\ u(x,t) = f(\eta)$
Similarity reduced ODE	$\varepsilon f''(\eta) + (c - \alpha) f'(\eta) = 0$
Similarity solution	$f(\eta) = c_1 + c_2 \exp\left(\frac{\alpha - c}{\varepsilon}\eta\right)$
Exact solution	$u(x,t) = c_1 + c_2 \exp\left(\frac{\alpha - c}{\varepsilon}(x - ct)\right)$

Table 2. (For $X_4 + X_8$)

$X_4 + X_8$	$\partial_x + u\partial_u$
Similarity transformation	$\eta = t,\ u(x,t) = \exp(x + f(\eta))$
Similarity reduced ODE	$f'(\eta) + (\alpha - \varepsilon) = 0$
Similarity solution	$f(\eta) = (\varepsilon - \alpha)\eta + c_1$
Exact solution	$u(x,t) = c_1 \exp(x + (\varepsilon - \alpha)t)$

Table 3. (For $X_3 + X_4$)

$X_3 + X_4$	$\partial_t + u\partial_u$
Similarity transformation	$\eta = x,\ u(x,t) = \exp(t + f(\eta))$
Similarity reduced ODE	$\varepsilon(f''(\eta) + f'^2(\eta)) - \alpha f'(\eta) - 1 = 0$
Similarity solution	$f(\eta) = (\frac{\alpha - \sqrt{A}}{2\varepsilon})\eta - Ln\left\{\frac{\sqrt{A}}{\varepsilon\left(c_1 \exp(\frac{\sqrt{A}}{\varepsilon})\eta - c_2\right)}\right\}$
Exact solution	$u(x,t) = \exp\left\{t + (\frac{\alpha - \sqrt{A}}{2\varepsilon})x - Ln\left[\frac{\sqrt{A}}{\varepsilon\left(c_1 \exp(\frac{\sqrt{A}}{\varepsilon})x - c_2\right)}\right]\right\}$

Where $A = \alpha^2 + 4\varepsilon$

Table 4. (For X_2)

X_2	$\frac{1}{2}(\alpha t + x)\partial_x + t\partial_t$
Similarity transformation	$\eta = \frac{x}{\sqrt{t}} - \alpha\sqrt{t}$, $u(x,t) = f(\eta)$
Similarity reduced ODE	$2\varepsilon f''(\eta) + \eta f'(\eta) = 0$
Similarity solution	$f(\eta) = c_1 + c_2\, erf(\frac{\eta}{2\sqrt{\varepsilon}})$
Exact solution	$u(x,t) = c_1 + c_2\, erf(\frac{x-\alpha t}{2\sqrt{\varepsilon t}})$

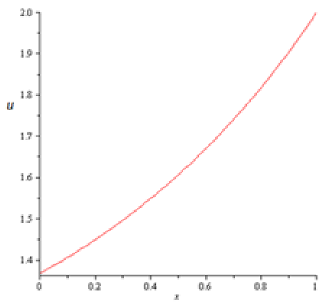

Figure (1a): $c_1 = c_2 = 1$, $\alpha = 2$, $\varepsilon = 1$, $t = 1$

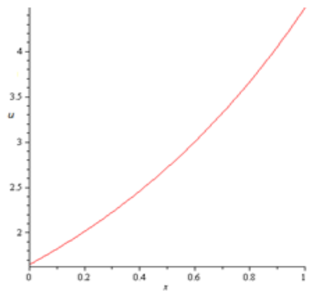

Figure (2a): $c_1 = 1$, $\alpha = 0.5$, $\varepsilon = 1$, $t = 1$

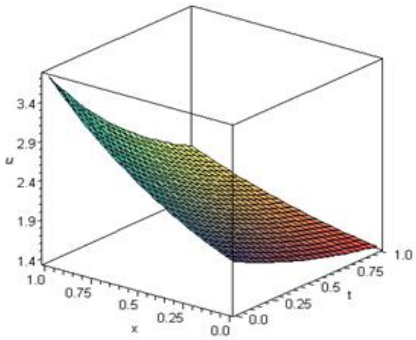

Figure (1b): $c_1 = c_2 = 1$, $\alpha = 2$, $\varepsilon = 1$

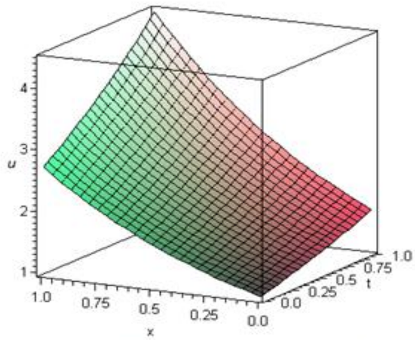

Figure (2b): $c_1 = 1$, $\alpha = 0.5$, $\varepsilon = 1$

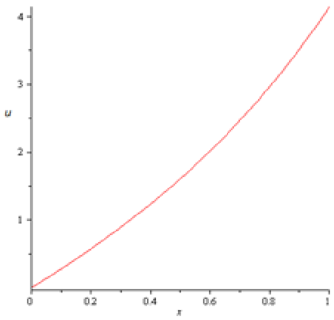

Figure (3a): $c_1 = c_2 = 1,\ \alpha = 0.5,\ \varepsilon = 1,\ t = 1$

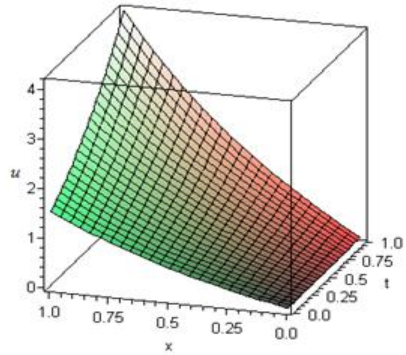

Figure (3b): $c_1 = c_2 = 1,\ \alpha = 0.5,\ \varepsilon = 1$

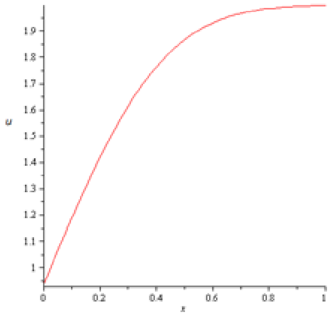

Figure (4a): $c_1 = c_2 = 1,\ \alpha = 0.5,\ \varepsilon = 1,\ t = 0.05$

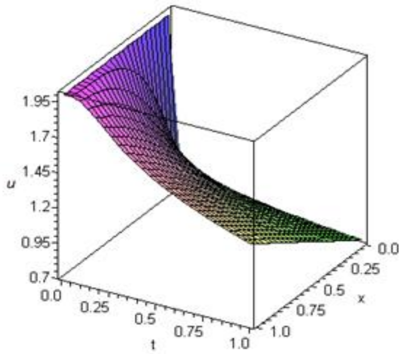

Figure (4b): $c_1 = c_2 = 1,\ \alpha = 0.5,\ \varepsilon = 1$

5. Symmetry Reduction And Exact Solution Of The Initial Boundary Value Problem (Ibvp) (1.1)-(1.3)

In this section, we present the similarity transformations and exact solution for one of the above generators for the convection-diffusion equation (1.1) with initial and boundary conditions (1.2) and (1.3). We will consider a linear combination of generators X_3 and X_8 ($X_3 + cX_8$) as in the table 1, we get

$$\eta = x - ct,\ u(x,t) = f(\eta) \qquad (1.23)$$

It is interesting to note that, if $c = 0$ equation (1.1) becomes steady, in such case $\eta = x$ and $f(\eta) = u(x)$.

The similarity transforms in (1.23) change the original governing equations to the reduced system of the BVP in the following form of ordinary differential equation:

$$\varepsilon f''(\eta) + (c - \alpha) f'(\eta) = 0, \quad \eta \in [-ct, 1 - ct] \qquad (1.24)$$

with the moving boundary conditions

$$f = g_0(t) \quad at\ \eta \to -ct, \qquad f = g_1(t) \quad at\ \eta \to 1 - ct \qquad (1.25)$$

The ordinary differential equation (1.24) with boundary conditions (1.25) will be solved by any suitable method, and then we can obtain the solution of the original problem.

6. Discusion And Concludig Remarks

In this paper, we apply Lie group method for Eq. (1.1), then we have found the infinitesimals (1.21) and its similarity generators (1.22) for Eq. (1.1). The similarity transformation and exact solutions for some generators are presented in tables as follow:

Tables (1), (2) and (3), contain the similarity transformation, similarity reduction, similarity solution and exact solutions of Eq. (1.1) for the linear combinations of generators (X_3, X_8), (X_4, X_8), and (X_3, X_4) respectively.

Table (4) contains the similarity transformation, similarity reduction, similarity solution and exact solution of Eq. (1.1) for the generator X_2.

Figures (1a,b)- (4a,b) show the behavior of the exact solutions for the original partial differential equation as in tables (1)-(4) respectively for various arbitrary values of quantities.

Finally, its demonstrated by using the similarity transforms (1.23), the original partial differential equation (1.1) with initial and boundary conditions (1.2) and (1.3) respectively that can be transformed to boundary value problem (1.24) and (1.25) of ordinary differential equation.

ACKNOWLEDGEMENT

The authors are indebted to Professor H. M. El-Hawary for various valuable suggestions and constructive criticism.

References

[1] A. Daga, V. H. Pradhan, Analytical Solution of Advection Diffusion Equation in Homogeneous Medium, PRATIBHA: International Journal of Science, Spirituality, Business and Technology (IJSSBT), 2 (1) (2013), 2277—7261.

[2] A. Fallahzadeh, K. Shakibi, A method to solve Convection Diffusion equation based on Homotopy analysis method, Journal of Interpolation and Approximate in Scientific Computing pp: 1-8 (2015).

[3] E.D. Veling, Analytical solution and numerical evaluation of the radial symmetric convection–diffusion equation with arbitrary initial and boundary data, Impact of Human Activityon Groundwater Dynamics (Proceedings of a symposium held during the Sixth IAHS Scientific Assembly at Maastricht, The Netherlands, IAHS Publl. no. 269 (2001).

[4] E. Hizel, N.C. Turgay, Symmetry group analysis and similarity solutions for nonlinear Reaction-Diffusion system of Gray-Scott type, Int. Math. Forum. 2, 2847-2858 (2007).

[5] F.S.V. Baza'n, Chebyshev pseudospectral method for computing numerical solution of convection–diffusion equation, Appl. Math. and compt. 200, 537–546 (2008).

[6] F. T. Tracy, 1D, 2D, 3D analytical solutions of unsaturated flow in groundwater, J. Hydrol. 170: 199-214(1995).

[7] G. Bluman, S. Kumei, Symmetries and Differential Equations, Springer-Verlag, (1989).

[8] G. Hu, L. Zhao, X. Wu, R. Li, T. Wu, C. Xie, Y. Qiao, J. Shi, W. Li, G. Cheng, New Fourier-series-based analytical solution to the conduction-convection equations to calculate soil temperature, determine soil thermal properties, or estimate water flux, Int. J. Heat Mass Transf. 95:815–823 (2016).

[9] G.W. Bluman, S.C. Anco, Symmetry and Integration Methods for Differential Equations, Springer, New York (2002).

[10] I. Boztosun, A. Charafi, An analysis of the linear advection-diffusion equation using mesh-free and mesh-dependent methods, Engineering Analysis with Boundary Elements 26, 889–895 (2002).

[11] M.A.A. Hamad, I.A. Hassanien, E.Kh.H. El-Nahary, Discrete Symmetries Analysis of Burgers Equation with Time Dependent Flux at the Origin, World Applied Sciences Journal, 12, 2291-2300 (2011).

[12] M. Ghasemi, M. Tavassoli Kajani, Application of He's homotopy perturbation method to solve a diffusion-convection problem, Mathematical Sciences 4, 171-186 (2010).

[13] M. O. Olayiwola, Application of Variational Iteration Method to the Solution of Convection-Diffusion Equation, J Appl. Computat. Math 5: 299. doi:10.4172/2168-9679.1000299 (2016).

[14] N.H. Ibragimov, (Ed.), CRC Handbook of Lie groups analysis of Differential Equations, II. Application in engineering and physical sciences, CRC Press, Boca Raton (1994).

[15] P.E. Hydon, symmetry methods for differential equations. A beginners' guide, Cambridge Texts in Applied Mathematics (2000).

[16] P.J. Olver, Applications of Lie Groups to Differential Equations, Grad. Texts in Math., vol. 107, Springer-Verlag, New York (1993).

[17] Q. Feng, Explicit Finite Difference Method for Convection-Diffusion Equation, Proceedings of the World Congress on Engineering, Vol II, London, U. K (2009).

[18] S.A. EL- Wakil, A. Elhanbaly, Adomian decomposition method for solving the diffusion convection reaction equations, Appl. Math. and compt. 177, 729-736 (2006).

[19] Z. Svoboda, The convection–diffusion equation and its use in building physics, International Journal on Architectural Science, 1(2): 68-79 (2000).

مبارك عوض السباعي* وسالم فرج بامسعود**

قسم الرياضيات- كلية العلوم- جامعة حضرموت (اليمن)

هاتف: ٠٠٩٦٧٧٧٥٤٣٧٧٤٣ - بريد إليكتروني **assabaai@hotmail.com**

*****قسم الفيزياء- كلية العلوم- جامعة حضرموت (اليمن)***

هاتف: ٠٠٩٦٧٧٧٠٧٧١٨٩٦ - بريد إليكتروني **Saalem88@yahoo.com**

في هذا البحث طبقنا طريقة تناظر لي لتحديد الحل المضبوط لمعادلة الانتقال- الانتشار في بعد واحد. باستخدام التحويلات المتماثلة تم تحويل المعادلة التفاضلية الجزئية الى معادلة تفاضلية عادية ومن ثم حصلنا على الحل اللاتغيري للمعادلة التفاضلية الجزئية الأصلية من حل المعادلة التفاضلية العادية. الحلول التحليلية التي حصلنا عليها بالتناظر لخصت في جداول.

كلمات مفتاحيه: معادلة الانتقال، معادلة الانتشار، مجموعة لي، التحويلات المتماثلة، حل لاتغيري.