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TOPOLOGIES INDUCED BY COVERING SOFT APPROXIMATION SPACE

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The concept of covering is considered as a one of fundamental concepts in topological spaces for its contribution in the study of many of topological problems. Our goal in this work is to introduce and study the covering soft approximation space as a new type of approximation space and investigating some of its properties. Also, we define the topology that induced by the covering of soft approximation space and some concepts related to this topology such as neighborhood, closure, connectedness and separation axioms. Finally we define several pairs of soft approximation operations and investigate the relationship among them.

Keywords :Soft approximation space, Soft rough sets, Soft open sets, Soft closed sets.

1 INTRODUCTION

In 1999, Molodtsov [7] proposed soft set theory as a new mathematical tool for dealing with uncertainties problems. A wide range of applications of soft sets have been developed in many different fields, including the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory and measurement theory. There has been a rapid growth of interest in soft set theory and its applications in recent years.

Rough set theory was initiated by Pawlak [9] for dealing with vagueness and granularity in information systems. This theory deals with the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations.

Covering rough set theory as one of the extension models of rough set theory, which was first formulated by W. Zakowski [12], has been studied extensively[12, 6, 16, 15, 13]. Many useful concepts have been proposed and generalized in the study of covering rough set theory. For example, the concept of complementary neighborhoods was first proposed by L.Ma [6, 16, 15, 13]who has studied the properties and topological importance of this concept.

In[5], soft set theory is utilized, for the first time, to generalize Pawlak's rough set model. Based on the novel granulation structures called soft approximation spaces, soft rough approximations and soft rough sets are introduced.

Topological methods have been applied to the study of covering rough sets [10, 14]. This paper focuses on some topological structures of a covering soft approximation space. Given a covering soft approximation space, we can construct a new type of topology called the topology induced by the covering. We use this topology to define some concepts such as neighborhoods, closures and connectedness. Drawing on these concepts, we define several pairs of approximation operators. We not only investigate the relationships among them, but also give clear explanations there concepts. We think that topological methods are useful tools for the study of covering soft rough set theory. From topological points of view, we can get a good insight into the essence of covering soft sets and make our discussions concise and profound. The rest of this paper is organized as follows. In Section2, we outline some fundamental concepts in rough set theory and soft set theory. In Section3, we introduce a new type of approximation space in soft set theory called covering soft approximation space and investigate some properties of this new approximation. Also, in Section4, we introduce a topology that induced by the covering on a covering soft approximation space. We also define some important concepts such as soft subspace topologies, connectedness, etc. Finally, we also show how to judge whether a covering soft approximation space is connected or not.

2 Preliminaries

This section presents a review of some fundamental notions of rough sets and soft sets.

Definition 2.1[9]. Suppose U is a finite nonempty set, called the universe, and R is an equivalence relation on U. The pair (U, R) is called the approximation space. Let X be a subset of U and $R(x)$ denotes the equivalence class determined by $x \in U$. Then:

(i) The lower approximation of a subset X is defined as
$\underline{R}(X) = \cup_{x \in U}\{R(x) : R(x) \subseteq X\}$.

(ii) The upper approximation of a subset X is defined as
$\overline{R}(X) = \cup_{x \in U}\{R(x) : R(x) \cap X \neq \emptyset\}$.

(iii) The boundary region of X with respect to R is defined as $B_R(X) = \overline{R}(X) - \underline{R}(X)$.

Moreover, the sets $Pos_R X = \underline{R}(X)$ and $Neg_R X = U - \overline{R}(X)$ are referred to as the R-positive, and the R-negative region of X,respectively. The set X is said to be rough with respect to R if $\underline{R}(X) \neq \overline{R}(X)$. That is, if $B_R(X) \neq \emptyset$. Note that sometimes the pair $(\overline{R}(X), \underline{R}(X))$ is also referred to as the rough set of X with respect to R.

Definition 2.2[11, 3] Let U be a universe and C be a collection of nonempty subsets of U. C is said to be a covering of U if $U = \cup_{c \in C} C$. The ordered pair (U, C) is called a covering approximation space.

Throughout this work, U refers to an initial universe, E is a set of parameters,

$P(U)$ is the power set of U, and $A \subseteq E$.

Definition 2.3[7]. A soft set F_A on the universe U is defined by the set of ordered pairs $F_A = \{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\}$, where $f_A : E \to P(U)$ such that $f_A(x) = \phi$ if $x \notin A$. Here, f_A is called an approximate function of the soft set F_A. The value of $f_A(x)$ may be arbitrary. Some of them may be empty, some may have nonempty intersection.

Definition 2.4[5]. Let F_A be a soft set over U.
(1) F_A is called full, if $\cup_{a\in A} F(a) = U$.
(2) F_A is called partition, if $\{F(a) : a \in A\}$forms a partition of U.

Definition 2.5[7]. A soft set F_A over U is said to be a null soft set, denoted by $\emptyset$, if $\forall e A, F(e) = \phi$.

Definition 2.6[2]. A soft set (F, E) over U is called a soft element, denoted by e/x or e_F if $F(e) = \{x\}, F(e') = \phi$, for all $e' \in E - \{e\}$.

3 Covering soft approximation space

Definition 3.1. Let F_A be a full soft set over U. Then the pair $P = (U, F_A)$ is called a covering soft approximation space, and the following two operations are defined as follows

$\underline{R_P}(X) = \{u \in U : \exists a \in A, [u \in F(a) \subseteq X]\}$,

$\overline{R_P}(X) = \{u \in U : \exists a \in A, \; [u \in F(a) \cap X \neq \emptyset]\}$

assigning to every subset $X \subset U$ two sets $\underline{R_P}(X)$ and $\overline{R_P}(X)$, which are called the soft P-lower approximation and the soft P-upper approximation of X, respectively. In general, we refer to $\underline{R_P}(X)$ and $\overline{R_P}(X)$ as soft rough approximations of X with respect to P. If $\underline{R_P}(X) = \overline{R_P}(X)$, X is said to be soft P-definable; otherwise X is called a soft P-rough set.

Theorem 3.1. Let F_A be a full soft set over U, and $P = (U, F_A)$ be a covering soft approximation space. Then, the following properties hold.
(1) $\underline{R_P}(\phi) = \overline{R_P}(\phi) = \phi$
(2) $\underline{R_P}(U) = \overline{R_P}(U) = U$
(3) $\underline{R_P}(X) \subseteq X \subseteq \overline{R_P}(X)$
(4) $\underline{R_P}(X) = \underline{R_P}(\underline{R_P}(X))$
(5) $\overline{R_P}(X) = \underline{R_P}(\overline{R_P}(X))$
(6) $\underline{R_P}(X \cap Y) = \underline{R_P}(X) \cap \underline{R_P}(Y))$
Proof:
(1), (2) are obvious.
(3) $\underline{R_P}(X) = \{u \in U : \exists a \in A, [u \in F(a) \subseteq X]\} \subseteq X$. Suppose that $X - \overline{R_P}(X) \neq \phi$, $x \in X - \overline{R_P}(X) \neq \phi$. Since F_A is a full soft set over U, so $x \in F(a)$ for some $a \in A, x \in X$ implies $F(a) \cap X \neq \phi$. Therfore $x \in \overline{R_P}(X) \neq \phi$, which is a contradiction, hence $X \subseteq \overline{R_P}(X)$
(4)Let $Y = \underline{R_P}(X)$ and $u \in Y$. Then $u \in F(a) \subseteq X$ for some $a \in A$, but

$Y = \underline{R_P}(X) = \cup_{a\in A}\{F(a) : F(a) \subseteq X\}$, we deduced that $u \in F(a) \subseteq Y$ for $a \in A$. Thus $u \in \underline{R_P}(Y)$, so $Y \subseteq \underline{R_P}(Y)$. Since $\underline{R_P}(Y) \subseteq Y$ for any $Y \subseteq U$, we conclude that $Y = \underline{R_P}(Y)$.
(5)Let $Y = \overline{R_P}(X)$ and $u \in Y$. Then $u \in F(a)$ and $F(a) \cap X \neq \phi$ for some $a \in A$, but $Y = \overline{R_P}(X) = \cup_{a\in A}\{F(a) : F(a) \cap X \neq \phi\}$. There exists $a \in A$such that $u \in F(a) \subseteq Y$. Hence $u \in \underline{R_P}(Y)$, so $Y \subseteq \underline{R_P}(Y)$. On other hand we know that $\underline{R_P}(Y) \subseteq Y$ holds for any $Y \subseteq U$. It follows that $Y = \underline{R_P}(Y)$.
(6) Suppose that $\underline{R_P}(X) \cap \underline{R_P}(Y) - \underline{R_P}(X \cap Y) \neq \phi, x \in \underline{R_P}(X) \cap \underline{R_P}(Y) - \underline{R_P}(X \cap Y)$. Then, there exist $a, b \in A$ such that $x \in F(a) \subseteq X$ and $x \in F(b) \subseteq Y$. Since F_A is a full soft set over U, then $F(a) = F(b)$. This implies $x \in F(a) \subseteq X \cap Y$. So, $x \in \underline{R_P}(X \cap Y)$, which is a contradiction. Thus, $\underline{R_P}(X \cap Y) \supseteq \underline{R_P}(X) \cap \underline{R_P}(Y)$. Therefore $\underline{R_P}(X \cap Y) = \underline{R_P}(X) \cap \underline{R_P}(Y)$.

Corollary 3.1. Let F_A be a full soft set over U, and $P = (U, F_A)$ be a covering soft approximation space. Then for any $X \subseteq U$, X is soft P-definable if and only if $\overline{R_P}(X) = X$.

Proposition 3.1.Let F_A be a full soft set over U, and $P = (U, F_A)$ be a covering soft approximation space. Then for any $X \subseteq U$, we have the following:

1. $-\underline{R_P}(X) \subset \overline{R_P}(-X)$

2. $Neg_P(X) = -\overline{R_P}(X) \subset \underline{R_P}(-X)$, where $-$ means the complement.

Proof:

1. Clearly if the set $-\underline{R_P}(X)$ is empty, then we have that the inclusion $-\underline{R_P}(X) \subset \overline{R_P}(-X)$ holds. So suppose that $-\underline{R_P}(X) \neq \phi$, and u be any element of $-\underline{R_P}(X)$. Since F_A is a full soft set, there exists some $a_0 \in A$ such that $u \in F(a_0)$. Note also that $-\underline{R_P}(X) = \{u \in U : \forall a \in A, u \in F(a) \Rightarrow F(a) \cap (-X) \neq \phi\}$. Thus it follows that $F(a_0) \cap (-X) \neq \phi$ since $u \in F(a_0)$. Hence we have that $u \in \overline{R_P}(-X)$ as required.

2. It is clear that the inclusion $Neg_P(X) = -\overline{R_P}(X) \subset \underline{R_P}(-X)$ holds when the set $-\overline{R_P}(X)$ is empty. So suppose that $-\overline{R_P}(X) \neq \phi$, and $u \in -\overline{R_P}(X)$. Since F_A is a full soft set, there exists some $a_0 \in A$ such that $u \in F(a_0)$. But we have that $Neg_P(X) = -\overline{R_P}(X) = \{u \in U : \forall a \in A, u \in F(a) \Rightarrow F(a) \subset (-X)\}$. Thus we deduce that $F(a_0) \subset (-X)$ since $u \in F(a_0)$. Hence we have that $u \in \underline{R_P}(-X)$ as required.

Example 3.1. Let $U = \{p_1, p_2, ..., p_8\}, E = \{e_1, e_2, ..., e_6\}, A = \{e_1, e_2, e_3, e_4\} \subset E$ and $F_A = \{(e_1, \{p_7, p_8\}), (e_2, \{p_1, p_5, p_6\}), (e_3, \{p_2, p_3, p_4, p_5\}), (e_4, \{p_4, p_7, p_8\})\}$ be full soft set. For $X = \{p_4, p_7, p_8\} \subset U$, we have $\overline{R_P}(X) = Y = \{p_2, p_3, p_4, p_5, p_7, p_8\}$, $\underline{R_P}(X) = X$. Hence $-\underline{R_P}(X) = -X = \{p_1, p_2, p_3, p_5, p_6\}$, and $Neg_P(X) = -\overline{R_P}(X) = -Y = \{p_1, p_6\}$. In addition, we have that $\underline{R_P}(-X) = \{p_1, p_5, p_6\}$ and $\overline{R_P}(-X) = \{p_1, p_2, p_3, p_4, p_5, p_6\}$.

Remark 3.1.Generally the inclusions in Proposition(3.1)need not be hold when the soft set is not full as shown by the following example.

Example 3.2. Let $U = \{p_1, p_2, ..., p_6\}, E = \{e_1, e_2, ..., e_6\}, A = \{e_1, e_2, e_3, e_4\} \subset E$ and $F_A = \{(e_1, \{p_1, p_6\}), (e_2, \{p_3\}), (e_3, \phi), (e_1, \{p_2, p_5\})\}$ is not full soft set. For $X = \{p_3, p_4\} \subset U$, we have $\overline{R_P}(X) = \underline{R_P}(X) = \{p_3\}$. Hence $-\overline{R_P}(X) = -\underline{R_P}(X) = \{p_1, p_2, p_4, p_5, p_6\}$. Moreover, we have that $\underline{R_P}(-X) = \overline{R_P}(-X) = \{p_1, p_2, p_5, p_6\}$. Therefore $-\underline{R_P}(X) \nsubseteq \overline{R_P}(-X)$ and $Neg_P(X) = -\overline{R_P}(X) \nsubseteq \underline{R_P}(-X)$.

Definition 3.2 Let (U, F_A) be a covering soft approximation space and $e/x \in U$. The neighborhood $N(e/x)$ of e/x is the intersection of all the elements of F_A containing e/x; that is

$$N(e/x) = \cap\{F_a \in F_A : e/x \in F_a\},$$

and the adhesion $P_{e/x}^{F_A}$ of e/x is defined by

$$P_{e/x}^{F_A} = \{y \in U : \forall F_a \in F_A(e/x \in F_a \leftrightarrow y \in F_a)\}.$$

Obviously, $\{P_{e/x}^{F_A} : e/x \in U\}$ forms a partition of U, and we have $P_{e/x}^{F_A} = \{y \in U : N(e/x) = N(y)\}$. Let $P(U)$ be the class of all subsets of U. Based on neighborhoods and adhesions of elements of U, three types of soft lower and soft upper approximations will define in this paper.

Definition 3.3 Let (U, F_A) be a covering soft approximation space and X be a subset of U. The soft lower approximation $l^-(X)$ and the soft upper approximation $l^+(X)$ are defined by

$$l^-(X) = \{e/x \in U : N(e/x) \subseteq X\} \quad , \quad l^+(X) = \{e/x \in U : N(e/x) \cap X \neq \phi\}.$$

Definition 3.4 Let (U, F_A) be a covering soft approximation space and X be a subset of U. The soft lower approximation $r^-(X)$ and the soft upper approximation $r^+(X)$ are defined by

$$r^-(X) = \{e/x \in U : \forall u(e/x \in N(u) \to u \in X)\}, r^+(X) = \cup\{N(e/x) : e/x \in X\}.$$

Definition 3.5 Let (U, F_A) be a covering soft approximation space and X be a subset of U. The soft lower approximation $A^-(X)$ and the soft upper approximation $A^+(X)$ are defined by

$$A^-(X) = \cup\{P_{e/x}^{F_A} : P_{e/x}^{F_A} \subseteq X\}, A^+(X) = \cup\{P_{e/x}^{F_A} : P_{e/x}^{F_A} \cap X \neq \phi\}.$$

By the earliest definition of soft rough sets, a set $X \subseteq U$ is called a covering soft rough set if its covering-induced soft lower approximation and soft upper approximation are not equal. Thus the above definitions determine three different types of covering soft rough sets.

4 Topologies induced by covering soft sets

In this section, we first construct a type of topology called the topology induced by a covering soft approximation space. This notion is indeed in the core of this paper. We can use it to build relations between the concepts defined in the preceding sections and some notions in topological spaces and we can also use it to give clear explanations of the concepts defined in the preceding sections.

Theorem 4.1. Let $P = (U, F_A)$ be a covering soft approximation space. The topology τ induced by the covering F_A is defined as follows: A subset F_G of U is said to be soft open in U if for each $e/x \in F_G$, there are finite elements $F_{a1}, F_{a2}, ...F_{an}$ of F_A such that $e/x \in \cap_{i=1}^{n} F_{ai} \subset F_G$.

Proof .Let us check that the collection τ induced by the covering F_A is indeed a topology on U.

1. The empty set obviously satisfies the defining condition of openness vacuously. Likewise, U is in τ, because $U = \cup_{a \in A} F(a)$.

2. Suppose that there is an indexed family $\{F_{G\alpha}\}_{\alpha \in J}$ of elements of τ. We shall show that $F_G = \cup_{\alpha \in J} F_{G\alpha}$ belongs to τ. Given $e/x \in F_G$, there is an index α such that $e/x \in F_{G\alpha}$. Since $F_{G\alpha}$ is soft open, there exist finite elements $F_{a1}, F_{a2}, ...F_{an}$ of F_A such that $e/x \in \cap_{i=1}^{n} F_{ai} \subset F_{G\alpha}$. Then $e/x \in \cap_{i=1}^{n} F_{ai} \subset F_G$, so that F_G is soft open by definition.

3. Let F_{G1} and F_{G2} be two elements of τ. We shall prove that $F_{G1} \cap F_{G2}$ belongs to τ. Given $e/x \in F_{G1} \cap F_{G2}$, we can choose $F_{ai1}, F_{ai2}, ...F_{ain}$ of F_A such that $e/x \in \cap_{k=1}^{n} F_{aik} \subset F_{G1}$ and $F_{aj1}, F_{aj2}, ...F_{ajm}$ of F_A such that $e/x \in \cap_{l=1}^{m} F_{ajl} \subset F_{G2}$. It is obvious that $e/x \in (\cap_{k=1}^{n} F_{aik}) \cap (\cap_{l=1}^{m} F_{ajl}) \subset F_{G1} \cap F_{G2}$. Any finite intersections of elements of τ are obviously in τ by induction. Thus the desired result is proved.

From topological points of view, we can clarify the concepts defined in the previous sections and explore the relations between these concepts and some notions in topological spaces.

Proposition 4.1. Let $P = (U, F_A)$ be a covering soft approximation space and τ be the topology induced by the full soft set F_A. Then the following statements hold:

1. For each $e/x \in U, N(e/x)$ is the smallest soft open subset of U containing e/x.

2. If F_G is a soft open subset of U, then $F_G = \cup_{e/x \in F_G} N(e/x)$.

3. The closure of $\{e/x\}$ denoted by $\overline{\{e/x\}}$ is equal to $U - \cup_{e/x \notin F(a), F(a) \in F(A)} F_A$.

Proof.

1. $N(e/x)$ is a soft open set containing e/x by Definition 3.2. Let F_G be an soft open subset of U containing e/x. By Theorem4.1., there exist finite elements $F_{a1}, F_{a2}, ...F_{an}$ of F_A such that$e/x \in \cap_{i=1}^{n} F_{ai} \subset F_G$. Since $N(e/x) \subset \cap_{i=1}^{n} F_{ai}$ by definition, we must have $N(e/x) \subset F_G$; so that $N(e/x)$ is the smallest soft open subset of U containing e/x.

2. Let F_G be an soft open subset of U. Given $e/x \in F_G$, we must have $N(e/x) \subset F_G$ because $N(e/x)$ is the smallest soft open subset of U containing e/x. It is obvious that $F_G \subset \cup_{e/x \in F_G} N(e/x) \subset F_G$, so that $F_G = \cup_{e/x \in F_G} N(e/x) \subset F_G$.

3. Let $D(e/x) = \cup_{e/x \notin F(a), F(a) \in F(A)} F(A)$. Since each element of $F_A$is an soft open subset of U, the set $U - D(e/x)$ is a soft closed subset of U containing e/x. This implies that $\overline{\{e/x\}} \subset U - D(e/x)$ because $\overline{\{e/x\}}$ is the smallest soft closed subset of U containing e/x. To prove the converse inclusion relation, we suppose that $e/y \in U - \overline{\{e/x\}}$. Since $U - \overline{\{e/x\}}$ is an soft open set containing e/y, we must have $N(e/y) \subset U - \overline{\{e/x\}}$ by statement (1); therefore, $N(e/y) \cap \overline{\{e/x\}} = \phi$ and $e/x \notin N(e/y)$. There must exist an element $F(a)$ of $F(A)$ such that $e/y \in F(a)/x \notin F(a)$ because $N(e/y) = \cap\{F(a) \in F(A) : e/y \in F(a)\}$. Hence $e/y \in D(e/x)$ and $U - \overline{\{e/x\}} \subset D(e/x)$. Thus $U - D(e/x) \subset \overline{\{e/x\}}$, as desired.

Proposition 4.2. Let $P = (U, F_A)$ be a covering soft approximation space and τ be the topology induced by the covering F_A. Then the following statements hold:

1. Let A be a subset of U and e/x be a point of U. Then $e/x \in \overline{A}$ if and only if $N(e/x) \cap A \neq \phi$.

2. Let $e/x, e/y$ be two distinct points of U. Then $N(e/x) = N(e/y)$ if and only if $\overline{\{e/x\}} = \overline{\{e/y\}}$.

3. Let e/x be a point of U. Then $P^{F_A}_{e/x} = N(e/x) \cap \overline{\{e/x\}}$.

4. If V is a soft closed subset of U, then $V = \cup_{e/x \in V} \overline{\{e/x\}} = \cup_{e/x \in V} P^{F_A}_{e/x}$.

Proof.

1. Given $e/x \in A$, it is obvious that $N(e/x) \cap A \neq \phi$. Suppose that $N(e/x) \cap A \neq \phi$. Any soft open subset F_G of U containing e/x must intersect A in at least one point, because $N(e/x)$ is the smallest soft open subset of U containing e/x by Proposition 4.1; therefore, $e/x \in A$.

2. If $N(e/x) = N(e/y)$, then $e/y \in \overline{\{e/x\}}$ and $e/x \in \overline{\{e/y\}}$ by Statement(1); Since $\overline{\{e/x\}}$ is the smallest soft closed subset of U containing e/x, one must have $\overline{\{e/x\}} \subset \overline{\{e/y\}}$; similarly, one has $\overline{\{e/y\}} \subset \overline{\{e/x\}}$, so that $\overline{\{e/x\}} = \overline{\{e/y\}}$. Next, we suppose that $\overline{\{e/x\}} = \overline{\{e/y\}}$. Since $e/x \in \overline{\{e/y\}}$, one must have $e/y \in N(e/x)$ by Statement (1); therefore, $N(e/y) \subset N(e/x)$, for $N(e/y)$ is the smallest soft open set containing e/y; similarly, one has $N(e/x) \subset N(e/y)$. Hence $N(e/x) = N(e/y)$.

3. By Definition 3.2, $P^{F_A}_{e/x} = \{e/y \in U | N(e/x) = N(e/y)\}$. If e/y is an element of $P^{F_A}_{e/x}$, then $e/x \in N(e/y)$, so that $e/y \in \overline{\{e/x\}}$ by Statement (1) and hence $e/y \in N(e/x) \cap \overline{\{e/x\}}$. Therefore, $P^{F_A}_{e/x} \subset N(e/x) \cap \overline{\{e/x\}}$. To prove the converse inclusion relation, we suppose that $e/y \in N(e/x) \cap \overline{\{e/x\}}$. It follows from Statement (1) that $e/y \in N(e/x) \wedge e/x \in N(e/y)$; therefore, $N(e/x) = N(e/y)$ by the definition of neighborhoods, so that $e/y \in P^{F_A}_{e/x}$.

4. Given $e/x \in V$, one always has $\overline{\{e/x\}} \subset V$, for $\overline{\{e/x\}}$ is the smallest soft closed set containing e/x. We can easily get $V \subset \cup_{e/x \in V} \overline{\{e/x\}}$; therefore, $V = \cup_{e/x \in V} \overline{\{e/x\}}$. It follows from Statement (3) that $V \subset \cup_{e/x \in V} P^{F_A}_{e/x} \subset \cup_{e/x \in V} \overline{\{e/x\}} = V$. Hence the desired result is proved.

Proposition 4.3. Let $P = (U, F_A)$ be a covering soft approximation space with the topology τ induced by the covering F_A , and X be a subset of U. Then the following statements hold:

1. $l^-(X) = IntX, l^+(X) = \overline{X}$.
2. $r^-(X) = \{e/x \in U|\overline{\{e/x\}} \subset X\}, r^+(X) = \{e/x \in U|\overline{\{e/x\}} \cap X \neq \phi\}$.

Proof.

1. If e/x is an element of $l^-(X)$, then $N(e/x) \subset X$ by Definition 3.3, so that $e/x \in N(e/x) \subset IntX$. Therefore, $l^-(X) \subset IntX$. On the other hand, if $e/x \in IntX$, then there is an soft open subset F_G of U such that $e/x \in F_G \subset IntX$. Since $N(e/x)$ is the smallest soft open subset containing e/x, we must have $e/x \in N(e/x) \subset F_G \subset IntX \subset X$, so that $e/x \in l^-(X)$ by Definition 3.3. Therefore, $IntX \subset l^-(X)$ and thus $l^-(X) = IntX$. If e/x is an element of $l^+(X)$, then $N(e/x) \cap X \neq \phi$ by Definition 3.3, so that $e/x \in \overline{e/x}$ by (1) of Proposition 4.2. Therefore, $l^+(X) \subset \overline{e/x}$. On the other hand, if $e/x\overline{e/x}$, then $N(e/x) \cap X \neq \phi$ by (1) of Proposition 4.2, so that $e/x \in l^+(X)$. Therefore, $\overline{e/x} \subset l^+(X)$ and thus $l^+(X) = \overline{e/x}$.

2. If e/x is an element of $r^-(X)$, then for every $e/u, e/x \in N(e/u) \to e/u \in X$ by Definition 3.4. But $e/x \in N(e/u)$ is equivalent to $e/u \in \overline{\{e/x\}}$ by (1) of Proposition 4.2. It is obvious that $e/x \in N(e/u) \to e/u \in X \Leftrightarrow \overline{\{e/x\}} \subset X$, so that $r^-(X) = \{e/x \in U|\overline{\{e/x\}} \subset X\}$. If e/x is an element of $r^+(X)$, then there is an element e/y of X such that $e/x \in N(e/y)$ by Definition 3.4. By (1) of Proposition 4.2, $e/y \in \overline{\{e/x\}}$ holds, so that $e/y \in \overline{\{e/x\}} \cap X$; that is, $\overline{\{e/x\}} \cap X \neq \phi$. Therefore, $r^+(X) \subset \{e/x \in U|\overline{\{e/x\}} \cap X \neq \phi\}$. On the other hand, if $\overline{\{e/x\}} \cap X \neq \phi$ then there is an element e/y of X such that $e/y \in \overline{\{e/x\}} \cap X$, so that $e/x \in N(e/y)$ by (1) of Proposition 4.2. Therefore, $e/x \in r^+(X)$ by Definition 3.4 and $\{e/x \in U|\overline{\{e/x\}} \cap X \neq \phi\} \subset r^+(X)$. Thus the desired result is proved.

Definition 4.1 Let χ be a soft topological space with topology τ. If Y is a subset of χ, then the collection $\tau_Y = \{Y \cap F_G | F_G \in \tau\}$ is a topology on Y called the soft subspace topology. With this topology, Y is called a soft subspace of χ; its soft open sets consist of all intersections of soft open sets of χ with Y.

A soft set F_a is said to be closed in Y if F_a is a soft subset of Y and if F_a is soft closed in the soft subspace topology of Y.

The following proposition explore the relation between soft closed sets in soft subspace Y of χ and soft closed sets in χ.

Proposition 4.4 Let Y be a soft subspace of χ. Then a soft set $F_a \subset Y$ is soft closed in Y if and only if it equals the intersection of a soft closed subset of χ with Y.

Proof. Suppose that $F_a = F_b \cap Y$, where F_b is soft closed in χ. Then $\chi - F_b$ is soft open in χ, so that $\chi - F_b \cap Y$ is soft open in Y by the definition of soft subspace topology. Since $\chi - F_b \cap Y = Y - F_a, Y - F_a$ is soft open in Y; therefore, F_a is soft closed in Y.

Conversely, suppose that F_a is soft closed in Y. Then $Y - F_a$ is soft open in Y, so that by definition it equals the intersection of a soft open subset F_G of χ with Y. The soft set $\chi - F_G$ is soft closed in χ and $F_a = Y-(Y-F_a) = Y(F_G \cap Y = Y \cap (\chi - F_G)$, so that F_a equals the intersection of a soft closed subset of χ with Y, as desired.

separation and connectedness are important characteristics of a covering soft approximation space and are worth studying. The following definition tells us what is separation and what is connectedness.

Definition 4.2 Let χ be a soft topological space. A separation of χ is a pair of U, V of disjoint nonempty soft open subsets of χ whose union is χ. The space χ is said to be connected if there does not exist a separation of χ.

The following theorem illustrates which subsets of a covering soft approximation space are connected.

Theorem 4.2. Let $P = (U, F_A)$ be a covering soft approximation space and τ be the topology induced by the covering F_A. Then for each $e/x \in U, N(e/x)$ and $\overline{\{e/x\}}$ are connected soft subspaces of U.
Proposition 4.4 Let Y be a soft subspace of χ. Then a soft set $F_a \subset Y$ is soft closed in Y if and only if it equals the intersection of a soft closed subset of χ with Y.

Proof. Suppose that $F_a = F_b \cap Y$, where F_b is soft closed in χ. Then $\chi - F_b$ is soft open in χ, so that $\chi - F_b \cap Y$ is soft open in Y by the definition of soft subspace
Proof. Suppose that F_a and F_b form a separation of $N(e/x)$. Then F_a and F_b are disjoint nonempty soft open subsets in the soft subspace $N(e/x)$ whose union is $N(e/x)$. By the definition of soft subspaces, F_a and F_b are also disjoint nonempty soft open subsets of U because $N(e/x)$ is a soft open subset of U. Without loss of generality, we may assume that $e/x \in F_a$. Since $N(e/x)$ is the smallest soft open set containing e/x, one must have $N(e/x) \subset F_a$, so that $F_b = F_b \cap N(e/x) \subset F_b \cap F_a = \phi$, contradicting the fact that F_b is a nonempty soft set.

To prove that $\overline{\{e/x\}}$ is a connected soft subspace of U, we proceed by contradiction. Suppose that F_a and F_b form a separation of $\overline{\{e/x\}}$. Then F_a and F_b are disjoint nonempty soft open subsets in the soft subspace $\overline{\{e/x\}}$ whose union is $\overline{\{e/x\}}$. Since $F_a = \overline{\{e/x\}} - F_b$ and F_b is soft open in the soft subspace $\overline{\{e/x\}}$, one can easily get that F_a is soft closed in the soft subspace $\overline{\{e/x\}}$. By Proposition 4.4, there exists a soft closed subset F_V of U such that $F_a = \overline{\{e/x\}} \cap F_V$, so that F_a is a soft closed subset of U. Similarly, we can prove that F_b is a soft closed subset of U. Without loss of generality, we may assume that $e/x \in F_a$. Since $\overline{\{e/x\}}$ is the smallest soft closed set containing e/x, one must have $\overline{\{e/x\}} \subset F_a$, so that $F_b = F_b \cap \overline{\{e/x\}} \subset F_b \cap F_a = \phi$, contradicting the fact that F_b is a nonempty soft set.

The following two propositions present us sufficient conditions under which a covering soft approximation space is separated.

Proposition 4.5. Let $P = (U, F_A)$ be a covering soft approximation space and τ be the topology induced by the covering F_A. Suppose that the number of elements of F_A is greater than one. If there exists an element F_a that its intersections with all the other elements of F_A are empty, then U is not a connected soft topological space.

Proof. Since F_A is a soft covering of U, it is obvious that $U = F_a \cup [\cup_{F_b \in F_A, F_b \neq F_a} F_A]$. Since the intersections of F_a with all the other elements of F_A are empty sets and each element of F_A is a soft open subset of U, and the soft topological space U is a union of disjoint nonempty soft open subsets of U, so that U is not a connected soft topological space by Definition 4.2.

Proposition 4.6. Let $P = (U, F_A)$ be a covering soft approximation space and τ be the topology induced by the covering F_A. If there exists a non empty proper subset X of U such that $l^-(X) = l^+(X)$, then U is not a connect soft topological space.

Proof. If $l^-(X) = l^+(X)$, then $X = l^-(X) = l^+(X)$, for $l^-(X) \subset l^+(X)$. On the other hand, it follows from formula (1)(Proposition4.3) that $l^-(X) = IntX, l^+(X) = \overline{X}$, so $l^-(X)$ is a soft open subset of U and $l^+(X)$ is a soft closed subset of U. Therefore, X is a nonempty proper subset of U which is both soft open and soft closed. The disjoint nonempty soft open subsets of U X and $U - X$ constitute a separation of U. We have given the spaces that are not connected.

In the following we construct spaces that are connected.

Proposition 4.7 Let $P = (U, F_A)$ be a covering soft approximation space and τ be the topology induced by the covering F_A. If $\cap\{F_a : F_a \in F_A\} \neq \phi$, then U is a connected soft topological space.

Proof. Let F_G be a nonempty soft open subset of U and e/x be a point of F_G. Then there are finite elements $F_{a1}, F_{a2}, ...F_{an}$ of F_A such that $e/x \in \cap_{i=1}^{n} F_{ai} \subset F_G$. It is obvious that $\cap\{F_a : F_a \in F_A\} \subset \cap_{i=1}^{n} F_{ai} \subset F_G$. We have proved that $\cap\{F_a : F_a \in F_A\}$ is a subset of each nonempty soft open subset of U. Therefore, we can not separate U into two disjoint nonempty soft open subsets whose union is U, so that U is connected.

Theorem 4.3 The union of a collection of connected soft subspaces of a soft topological space χ that have a point in common is connected.

Proof. Let $\{F_{a_\alpha}\}$ be a collection of connected soft subspaces of the soft space χ; p be a point of $\cap F_{a_\alpha}$. We shall prove that the soft space $Y = \cup F_{a_\alpha}$ is connected. Suppose that F_C and F_D are disjoint nonempty soft open subsets of Y whose union is Y. Without loss of generality, we may assume that $p \in F_C$. Since $F_{a_\alpha} = (F_{a_\alpha} \cap F_C) \cup (F_{a_\alpha} \cap F_D)$ is the union of disjoint soft open subsets of F_{a_α} and since F_{a_α} is a connected soft subspace, one must have that either $(F_{a_\alpha} \cap F_C)$ or $(F_{a_\alpha} \cap F_D)$ is empty. Since $p \in (F_{a_\alpha} \cap F_C), (F_{a_\alpha} \cap F_D)$ must be an empty set. Therefore, we can easily get $F_D = F_D \cap Y = F_D \cap (\cup F_{a_\alpha}) = \cup(F_D \cap F_{a_\alpha}) = \phi$ contradicting the fact that F_D is a nonempty soft set.

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