Analysis of Earth's Geopotential and its Effect on GPS Orbits

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Abstract: In this paper, we study the GPS orbit as an orbit for navigational satellites; we are concerned with analyzing and simulating the EFFECT OF EARTH'S OBLATNESS, which gives an image for what a satellite will suffer due to this perturbing force. Finally, a numerical technique is used to calculate a real GPS satellite as an example.

Keywords: GPS orbit; Lagrange's planetary equations; earth's oblateness.

1. Introduction

The Global Positioning System (GPS) is a satellite-based navigation system that was developed by the U.S. Department of Defense (DOD) in the early 1970s. Initially, GPS was developed as a military system to fulfill U.S. military needs. However, it was later made available to civilians, and GPS now allows land, sea, and airborne users to determine their exact location, velocity, and time 24 hours a day, in all weather conditions, anywhere in the world.

Also GPS is used to support a broad range of military, commercial, and consumer applications. GPS provides continuous positioning and timing information, anywhere in the world under any weather conditions (El-Rabbany, 2002).
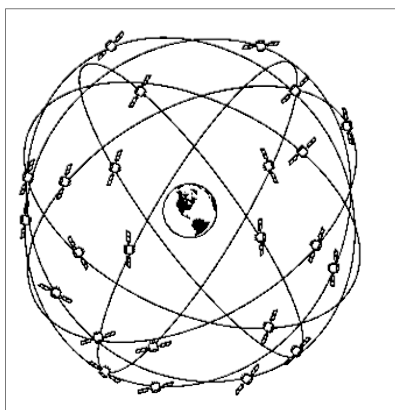
Fig.1 GPS constellation

At Present time, GPS is fully operational and meets the criteria established in the 1960s for an optimum positioning system (Kaplan and Hegarty, 2006). The system provides accurate, continuous, worldwide, three-dimensional position and velocity information to users with the

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appropriate receiving Equipment. GPS also disseminates a form of Coordinated Universal Time (UTC).The satellite constellation nominally consists of 24 satellites arranged in 6orbital planes with 4 satellites per plane as shown in Figure (1).

GPS satellite orbits are nearly circular (elliptical shape with a maximum eccentricity 0.01), with inclination of 55°. The semi-major axis of a GPS orbit is 26,560 km (i.e., the satellite altitude of about 20,200 km above the Earth's surface) (Langley, 1991). The corresponding GPS orbital period is about 12sidereal hours (~11 hours, 58 minutes).

GPS consists of three segments: **the space segment** consists of the 24-satellite constellation , **the control segment** consists of a worldwide network of tracking stations, with a master control station (MCS) located in the United States at Colorado Springs, Colorado , and **the user segment** includes all military and civilian users, the general architecture is depicted in figure (2) .
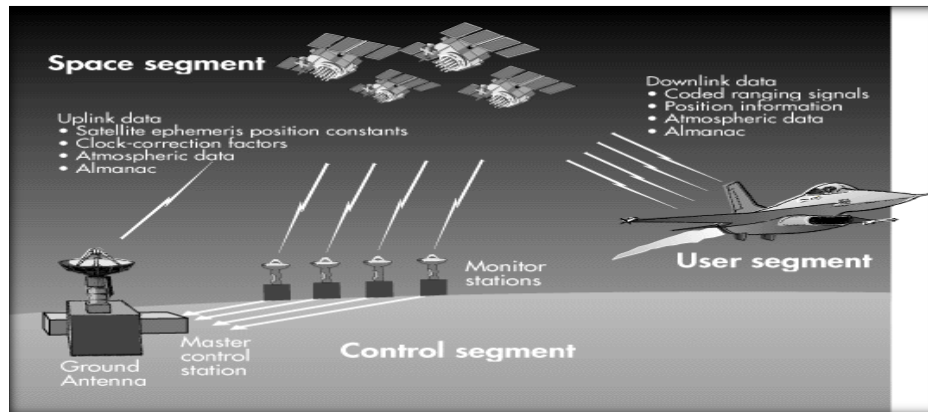
Fig.2. General architecture of GPS Segments

The idea behind GPS is simple. If the distances from a point on the Earth (a GPS receiver) to three GPS satellites are known along with the satellite locations, then the location of the point can be determined by simply applying the well-known concept of resection Figure (3) (Langley, 1991).
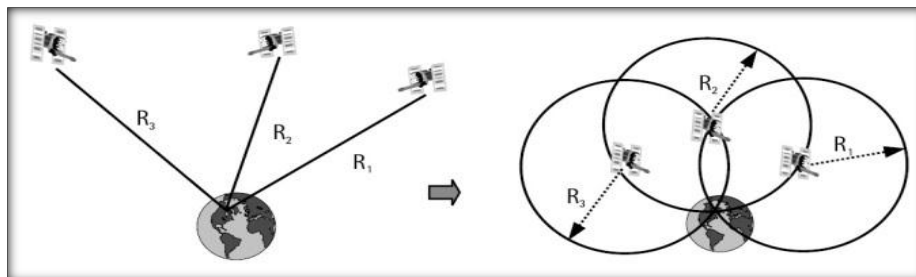
Fig.3. Concept of resection

2. Perturbations

From the characteristics of this orbit of GPS, the eccentricity zero or nearly zero (Geyling and Westerman , 1971). Then for this type of orbits (of low eccentricity), we will see in Lagrange's planetary Equations $\frac{\partial \check{R}}{\partial e}$, $\frac{\partial \check{R}}{\partial \omega}$ and $\frac{\partial \check{R}}{\partial M}$ should contain a multiplicative factor (e). And the change in e, M and ω are ill defined unless these conditions fulfilled. Noted that in such a case (GPS orbits) the time of perigee passage and the argument of perigee are meaningless. Consequently any change in the eccentricity causes difficulty in defining changes in ω and τ due to undefined angles with respect to reference axis measured from.

Such difficulty lies with the elements, and not with the Lagrange's Equations. One can form Equations to avoid this difficulty by introducing these three elements μ, ν and ζ. For the case of low eccentricity, a set of new elements is in Equation (1)

$$\left. \begin{array}{ll} a & , \quad I, \\ \mu = e \sin \omega & , \quad \Omega, \\ v = e \cos \omega & , \quad \zeta = \omega + M \end{array} \right\} \tag{1}$$

If we now employ μ , ζ and ν in place of e, ω and M in Lagrange's Equations we get the right hand sides which are easier to manipulate with ,

$$\check{R}(a, e, I, \omega, \Omega, M) = \check{R}[a, \mu(e,\omega), v(e,\omega), I, \Omega, \zeta(\omega,M)]$$

By using the following transformations:

$$\left. \begin{array}{ll} e = \sqrt{\mu^2 + v^2} & \sin \omega = \frac{\mu}{\sqrt{\mu^2+v^2}} \\[2mm] \cos \omega = \frac{v}{\sqrt{\mu^2+v^2}} & \tan \omega = \frac{\mu}{v} \\[2mm] \sin M = \frac{(v \sin \zeta - \mu \cos \zeta)}{\sqrt{\mu^2+v^2}} & \cos M = \frac{(\mu \sin \zeta + v \cos \zeta)}{\sqrt{\mu^2+v^2}} \end{array} \right\} \tag{2}$$

Expressing the distribution function in terms of the new elements

$$\frac{\partial \check{R}}{\partial a} = \frac{\partial \check{R}\prime}{\partial a} \tag{3.1}$$

$$\frac{\partial \check{R}}{\partial e} = \frac{1}{\sqrt{\mu^2+v^2}} (\mu \frac{\partial \check{R}\prime}{\partial \mu} + v \frac{\partial \check{R}\prime}{\partial v}) \tag{3.2}$$

$$\frac{\partial \check{R}}{\partial I} = \frac{\partial \check{R}\prime}{\partial I} \tag{3.3}$$

$$\frac{\partial \check{R}}{\partial \omega} = v \frac{\partial \check{R}\prime}{\partial \mu} - \mu \frac{\partial \check{R}\prime}{\partial v} + \frac{\partial \check{R}\prime}{\partial \zeta} \tag{3.4}$$

$$\frac{\partial \check{R}}{\partial \Omega} = \frac{\partial \check{R}\prime}{\partial \Omega} \tag{3.5}$$

$$\frac{\partial \check{R}}{\partial M} = \frac{\partial \check{R}\prime}{\partial \zeta} \tag{3.6}$$

The Lagrange's planetary Equations in terms of M are

$$\dot{a} = \frac{2}{na}\frac{\partial \check{R}}{\partial M} \tag{4.1}$$

$$\dot{e} = \frac{(1-e^2)}{nea^2}\frac{\partial \check{R}}{\partial M} - \frac{\sqrt{(1-e^2)}}{nea^2}\frac{\partial \check{R}}{\partial \omega} \tag{4.2}$$

$$\dot{I} = \frac{1}{na^2\sqrt{1-e^2}}(\cot I\,\frac{\partial \check{R}}{\partial \omega} - \frac{1}{\sin I}\frac{\partial \check{R}}{\partial \Omega}) \tag{4.3}$$

$$\dot{\Omega} = \frac{1}{(na^2\,\sin I\sqrt{1-e^2}}\frac{\partial \check{R}}{\partial I} \tag{4.4}$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{nea^2}\frac{\partial \check{R}}{\partial e} - \frac{\cot I}{(\sqrt{1-e^2})na^2}\frac{\partial \check{R}}{\partial I} \tag{4.5}$$

$$\dot{M} = n - \frac{2}{na}\frac{\partial \check{R}}{\partial a} - \frac{(1-e^2)}{nea^2}\frac{\partial \check{R}}{\partial e} \tag{4.6}$$

Using the transformation elements from Equation (2) and Equation(3) Lagrange's planetary Equations can be written as:

$$\dot{a} = \frac{2}{na}\frac{\partial \check{R}\prime}{\partial \zeta} \tag{5.1}$$

$$\dot{e} = \frac{-\sqrt{1-\mu^2-\nu^2}}{na^2\sqrt{\mu^2+\nu^2}}(\nu\,\frac{\partial \check{R}\prime}{\partial \mu} - \mu_1\frac{\partial \check{R}\prime}{\partial \nu} + (1-\sqrt{1-\mu^2-\nu^2})\frac{\partial \check{R}\prime}{\partial \zeta}) \tag{5.2}$$

$$\dot{I} = \frac{(1-\sqrt{1-\mu^2-\nu^2})^{-1}}{na^2\sin I}\left\{\cos I\left(\nu\,\frac{\partial \check{R}\prime}{\partial \mu} - \mu\frac{\partial \check{R}\prime}{\partial \nu} + \frac{\partial \check{R}\prime}{\partial \zeta}\right) - \frac{\partial \check{R}\prime}{\partial \Omega}\right\} \tag{5.3}$$

$$\dot{\Omega} = \frac{1}{(\sin I)na^2\sqrt{1-\mu^2-\nu^2}}\frac{\partial \check{R}\prime}{\partial I} \tag{5.4}$$

$$\dot{\omega} = \frac{\sqrt{1-\mu^2-\nu^2}}{na^2(\mu^2+\nu^2)}(\mu\,\frac{\partial \check{R}\prime}{\partial \mu_1} + \nu\,\frac{\partial \check{R}\prime}{\partial \nu_1}) - \frac{\cot I}{na^2\sqrt{1-\mu^2-\nu^2}}\frac{\partial \check{R}\prime}{\partial I} \tag{5.5}$$

$$\dot{M} = n - \frac{2}{na}\frac{\partial \check{R}\prime}{\partial a} - \frac{(1-\mu^2-\nu^2)}{na^2(\mu^2+\nu^2)}(\mu\,\frac{\partial \check{R}\prime}{\partial \mu} + \nu\,\frac{\partial \check{R}\prime}{\partial \nu}) \tag{5.6}$$

Differentiate the set of transformation elements Eqn.(1) to get the Lagrange's planetary Equations in terms of new elements:

$$\dot{a} = \frac{2}{na}\frac{\partial \check{R}\prime}{\partial \zeta} \tag{6.1}$$

$$\dot{\mu} = \frac{\sqrt{1-\mu^2-\nu^2}}{na^2\sqrt{\mu^2+\nu^2}}\left((\mu^2+\nu^2)\frac{\partial \check{R}\prime}{\partial \nu} - \mu(1-\mu^2-\nu^2)\frac{\partial \check{R}\prime}{\partial \zeta}\right) - \frac{\nu\cot I}{na^2\sqrt{1-\mu^2-\nu^2}}\frac{\partial \check{R}\prime}{\partial I} \tag{6.2}$$

$$\dot{\nu} = \frac{-\sqrt{1-\mu^2-\nu^2}}{na^2\sqrt{\mu^2+\nu^2}}\left((\mu^2+\nu^2)\frac{\partial \check{R}\prime}{\partial \nu} + \mu(1-\mu^2-\nu^2)\frac{\partial \check{R}\prime}{\partial \zeta}\right) + \frac{\nu\cot I}{na^2\sqrt{1-\mu^2-\nu^2}}\frac{\partial \check{R}\prime}{\partial I} \tag{6.3}$$

$$\dot{I} = \frac{1}{na^2\sqrt{1-\mu^2-\nu^2}}\left\{\cot I(\nu\,\frac{\partial \check{R}\prime}{\partial \mu} - \mu\frac{\partial \check{R}\prime}{\partial \nu} + \frac{\partial \check{R}\prime}{\partial \zeta}) - \frac{\partial \check{R}\prime}{\partial \Omega}\right\} \tag{6.4}$$

$$\dot{\Omega} = \frac{1}{na^2\sqrt{1-\mu^2-\nu^2}\sin I}\frac{\partial \check{R}\prime}{\partial I} \tag{6.5}$$

$$\dot{\zeta} = n - \frac{2}{na}\frac{\partial \check{R}\prime}{\partial a} + \frac{\sqrt{1-\mu^2-\nu^2}}{na^2\sqrt{1-\mu^2-\nu^2}}\left(1-\sqrt{1-\mu^2-\nu^2}\right)(\mu\,\frac{\partial \check{R}\prime}{\partial \mu} + \nu\,\frac{\partial \check{R}\prime}{\partial \nu}) - \frac{\cot I}{na^2\sqrt{1-\mu^2-\nu^2}}\frac{\partial \check{R}\prime}{\partial I} \tag{6.6}$$

2.1. The Effect of Earth's Oblateness
The new elements can now be evaluated without difficulty at low eccentricity.

Applying Equations (6.1) to (6.6) to solve the problem of the oblatness (Khatab, 2012).

The disturbing function in case of circular orbit (M=f) will be :

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$$\widetilde{R}g = \frac{-\mu}{r^3} J2 \, R2 \left(\frac{3}{2}\sin 2I \sin(\omega + M) - \frac{1}{2} \right) \tag{7}$$

where ;

$$r = \frac{a(1-\mu^2-\nu^2)}{1+\mu\sin\zeta+\nu\cos\zeta}$$

then the disturbing function (7) in terms of new elements can be written as:

$$\widetilde{R}'g = \frac{-\mu J_2 \, R^2 \, (1+\mu \sin\zeta+\nu \cos\zeta)^3}{2a^3 \, (1-\mu^2-\nu^2)^3} \, (3\sin^2 I \sin^2\zeta - 1) \tag{8}$$

The variations of the orbital elements are:

$$\dot{a} = \frac{-3\mu J_2 \, R^2 \, (1+\mu \sin\zeta+\nu \cos\zeta)^2}{na^4 \, (1-\mu^2-\nu^2)} \left\{ \begin{array}{c} (1 + \mu \sin\zeta + \nu \cos\zeta)\sin^2 I \sin 2\zeta \; + \; (\mu \cos\zeta - \nu \sin\zeta) \\ (3\sin^2 I \sin^2\zeta - 1) \end{array} \right\} \tag{9.1}$$

$$\dot{\mu} = \frac{-3\mu J_2 \, R^2 \, (1+\mu \sin\zeta+\nu\cos\zeta)^2}{2na^5 \, (1-\mu^2-\nu^2)^{7/2}} \left\{ \begin{array}{l} (3\sin^2 I\sin^2\zeta - 1) \left[\dfrac{2\nu(1 - \mu \sin\zeta) + (1 - \mu^2 + \nu^2)\cos\zeta - \frac{\mu}{(\mu^2+\nu^2)}\left(1 - \sqrt{1 - \mu^2 - \nu^2}\right)}{(1 - \mu^2 - \nu^2)(\mu \cos\zeta - \nu \sin\zeta)} \right] \\ -(1 + \mu\sin\zeta + \nu\cos\zeta)\left[\dfrac{\frac{\nu}{3}\cot I(3\sin^2 I\sin^2\zeta - 1) + \frac{\mu}{(\mu^2+\nu^2)}\left(1 - \sqrt{1 - \mu^2 - \nu^2}\right)}{(1 - \mu^2 - \nu^2)\sin^2 I\sin 2\zeta} \right] \end{array} \right\} \tag{9.2}$$

$$\dot{\nu} = \frac{-3\mu J_2 \, R^2 \, (1 + \mu \sin\zeta + \nu\cos\zeta)^2}{2na^5 \, (1 - \mu^2 - \nu^2)^{7/2}} \left\{ \begin{array}{l} (3\sin^2 I\sin^2\zeta - 1) \left[\dfrac{2\mu(1 + \nu\sin\zeta) + (1 + \mu^2 - \nu^2)\sin\zeta + \frac{\nu}{(\mu^2 + \nu^2)}\left(1 - \sqrt{1 - \mu^2 - \nu^2}\right)}{(1 - \mu^2 - \nu^2) \; (\mu \cos\zeta - \nu \sin\zeta)} \right] \\ -(1 + \mu \sin\zeta + \nu \cos\zeta)\left[\dfrac{\frac{\mu}{3}\cot I(3\sin^2 I\sin^2\zeta - 1) - \frac{\nu}{(\mu^2 + \nu^2)}\left(1 - \sqrt{1 - \mu^2 - \nu^2}\right)}{(1 - \mu^2 - \nu^2)\sin^2 I\sin 2\zeta} \right] \end{array} \right\} \tag{9.3}$$

$$\dot{I} = \frac{-3\mu J_2 \, R^2 \, \cot I(1+\mu\sin\zeta+\nu\cos\zeta)^3}{2na^5 \, (1-\mu^2-\nu^2)^{7/2}} \, \sin 2I \sin 2\zeta \tag{9.4}$$

$$\dot{\Omega} = \frac{-\mu J_2 \, R^2 \, (1+\mu \sin\zeta+\nu \cos\zeta)^3}{2na^5 \, (1-\mu^2-\nu^2)^{7/2}}(3\sin 2I \sin 2\zeta - 1) \tag{9.5}$$

$$\dot{\zeta} = n - \frac{3\mu J_2 \, R^2 \, \cot I(1+\mu \sin\zeta+\nu \cos\zeta)^2}{2na^5 \, (1-\mu^2-\nu^2)^{7/2}} \left\{ \begin{array}{l} (3\sin^2 I \sin^2\zeta - 1)\left[2a(1 + \mu\sin\zeta + \nu\cos\zeta)\sqrt{1 - \mu^2 - \nu^2}\right] \\ + \left(1 - \sqrt{1 - \mu^2 - \nu^2}\right)(2 + \mu\sin\zeta + \nu\cos\zeta) + \frac{\left(1-\sqrt{1-\mu^2-\nu^2}\right)}{(\mu^2+\nu^2)}(\mu \sin\zeta + \nu \cos\zeta) \\ -\frac{1}{2}\cot I(1 + \mu \sin\zeta + \nu \cos\zeta)(3\sin 2I\sin^2\zeta - 1) \end{array} \right\} \tag{9.6}$$

3. Numerical Application
Equations from (9.1) to (9.6) represent the rate of change of the Keplerian elements due to the effect of the Earth's Oblateness on a GPS orbit. Such Equations are solved by using the fourth order Runge–Kutta Method, and then applied to a satellite. A MATHEMATICA 5.2 program was used for calculating and plotting the following curves to describe the behaviors of the rate of change in two orbital elements with time (RAAN and argument of perigee).

Results
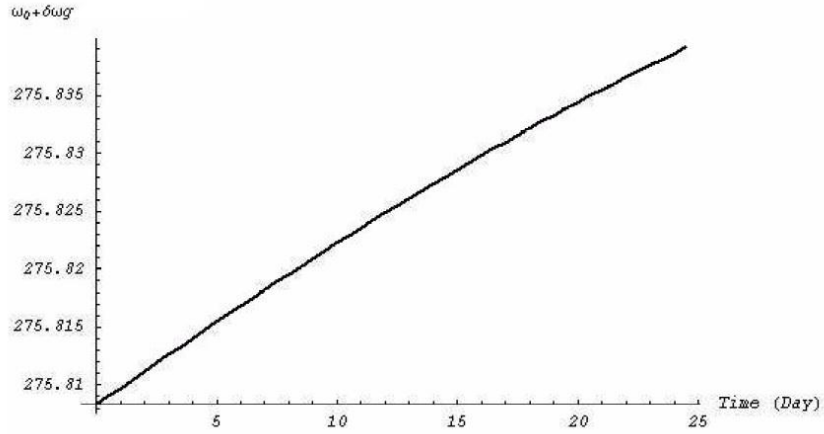
Fig.3. Variation of argument of perigee with time

Figure (3) illustrates the change of the argument of perigee with respect to time, it's noted that its value increases with time.
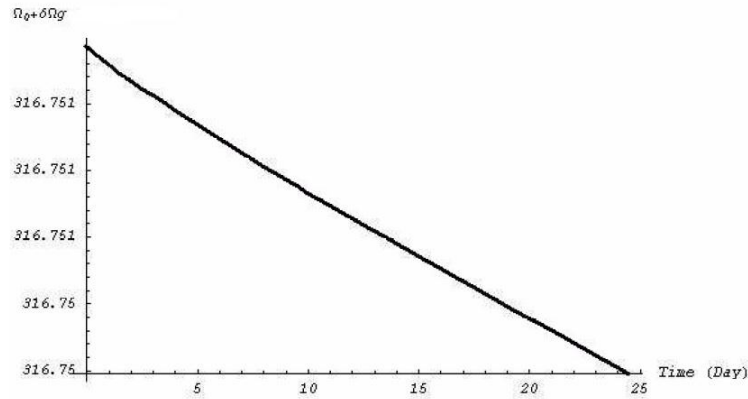
Fig.4. Variation of right ascension of ascending node with time

In the other hand figure (4) depicts the decrease of the right ascension of ascending node with respect to time.

Conclusion

For studying the six orbital elements for Keplerian orbits we noticed that there is no change in the elements (except M), but when including the perturbing forces there will be noticeable changes in these elements, in this paper we consider the perturbation caused by earth oblatness for Gps satellites orbits (medium Earth orbit) and get the results for varying the right ascension of ascending node and argument of perigee and in future work we'll increase the order of potential harmonics consider other perturbing forces such as Solar radiation pressure, air drag and third body effect.

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