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# Study effects of non-Darcy and heat source on MHD Rivlin-Ericksen Fluid of third-grade flow in a porous medium

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## ABSTRACT

The effects of non-Darcy and heat source on magneto hydrodynamics (MHD) Rivlin-Ericksen fluid of third-grade flow between two vertical flat plates through in a porous medium has been studied analytically and numerically in this paper. The system of highly non-linear partial differential equations is converted to a system of non-linear but ordinary differential equations via the similarity transformation. After that, the governing equations are solved analytically using Multi−step differential transform method (MDTM) and numerically using finite difference method (FDM) and shooting technique using Runge-Kutta fourth-order. The present study found that (MDTM), (FDM), and shooting are powerful approaches for answering highly non-linear differential equations such as this problem. The effects of different parameters on velocity and temperature are displayed through graphs and tables. Comparisons between current results and available previous results are listed and displayed in figures and tables. Current results prove the agreement with previously published results.

**Keywords:**

Magnetic field, Rivlin-Ericksen Fluid, Darcy and non-Darcy medium, Heat source, Analytical and Numerical methods.

## 1. INTRODUCTION

Many researchers have described the natural convection of non-Newtonian fluids [1-10]. A specific class of Rivlin–Ericksen fluid was investigated [6-7]. For the present nonlinear system of differential equations, there are various analytical and numerical methods that present different solutions [1-8]. Etbaeitabari et al. [7] introduced the solution analytically with heat transfer assessment and modeling, they created a novel technique that is independent of tiny parameters and is based on the (VIM). Rashidi [8] presented a new analytical solution by (DTM), Jyoti [5] investigated it by (HAM), and Kargar and Akbarzade [9] used (HPM). Toosi and Siavashi [6] studied the natural convection flow of a two-phase dusty non-Newtonian power-law fluid along a vertical surface.

When analytical solutions are unattainable or more difficult, and we need to compare analytical and experimental approaches, numerical methods are very useful tools for solving highly nonlinear differential equations. Because there is no accurate analytic solution for all nonlinear equations, numerical methods have been widely employed to solve them. To obtain the needed precision for linearized differential equations must be solved using iterative approaches methods. Because of its simplicity, (FDM) is commonly used to solve linear and non-linear differential equations. Rivlin–Ericksen fluid flow with the effect of non-Darcy medium and radiation investigated using (FDM) ([1] and [3]). Using the software MAPLE, the identical problem is solved numerically (Runge– Kutta ) as well.

The purpose of this work is to investigate the effects of non-Darcy and heat source on MHD Rivlin-Ericksen fluid analytically by (MDTM) and numerically (FDM) and Shooting method. In addition, the effect of non-Darcy parameter, porosity, Hartman number, Prandtl number and heat source parameters and the other different parameters on velocity and temperature of flow studied and presented graphically and tables. This approach provides highly accurate solution estimates in a series of steps. Finally, comparisons with previously published works are performed and showed that the present results have high accuracy and are found to be in a good agreement.

## 2. MATHEMATICAL FORMULATION

A schematic of the problem under study is shown in Fig (1). It contains of double smooth walls that can be located perpendicularly. A non-Newtonian fluid is contained on two smooth walls separated by 2b. At x = +b and x = b, the walls are kept at constant temperatures $T_1$ and $T_2$, respectively, with $T_1 > T_2$. The fluid near the wall is caused by the temperature difference, at x = −b to rise and the fluid near the wall at x = +b to fall [1-10].

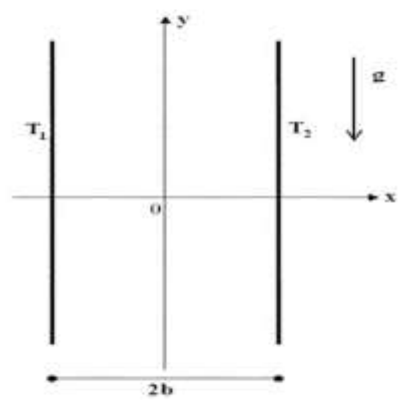



Figure 1: Schematic diagram of the problem

According to the above assumptions, the governing (momentum and energy) equations are written, respectively, as [1-10]:

$$\mu \frac{d^2u}{dx^2} + 6\beta_3 \left(\frac{du}{dx}\right)^2 \frac{d^2u}{dx^2} + \rho_0\gamma(\mathrm{T} - T_m)\,\mathrm{g} - \frac{\mu}{K}\,u - \frac{\sigma\beta_0^2}{\rho}u - \frac{\rho_{0\,B}}{K}u^2 = 0 \tag{1}$$

$$\kappa \frac{d^2T}{dx^2} + 2\beta_3 \left(\frac{du}{dx}\right)^4 + \mu\left(\frac{du}{dx}\right)^2 + Q_0(\mathrm{T} - T_m) = 0\ , \tag{2}$$

Rajagopal [2] has demonstrated that by using the similarity variables:

$$\upsilon=\frac{u}{u_0},\ \eta=\frac{x}{b}\ \text{ and }\ \theta=\frac{T-T_m}{T_1-T_2}, \tag{3}$$

The Navier − Stokes and energy equations can be reduced to the following pair of ordinary differential equations:

$$\frac{d^2v}{d\eta^2} + 6\,\delta\left(\frac{dv}{d\eta}\right)^2\frac{d^2v}{d\eta^2} + \theta - P\,v - H_a^2\,v - F_s v^2 = 0 \tag{4}$$

$$\frac{d^2\theta}{d\eta^2} + 2\delta\,E_c\,P_r\left(\frac{dv}{d\eta}\right)^4 + E_c\,P_r\left(\frac{dv}{d\eta}\right)^2 + \alpha\,\theta = 0, \tag{5}$$

Where $\delta = \frac{\beta_3\,u_0^2}{\mu\,b^2}$ Dimensionless non-Newtonian viscosity, $P = \frac{b^2}{K}$ porosity, $H_a^2 = \frac{\sigma\,\beta_0^2\,b^2}{\mu}$ Hartman number, $F_s = \frac{\rho_0\,B\,u_0\,b^2}{\mu\,K}$ non-Darcy, $E_c = \frac{u_0^2}{c\,(T_{1-}\,T_2)}$ Eckert number, $P_r = \frac{\mu\,c}{\kappa}$ Prandtl number and $\alpha = \frac{Q_0\,b^2}{\kappa}$ heat source are the parameters.

The following are the appropriate boundary conditions:

$$v(-1) = 0\,,\ \theta(-1) = \frac{1}{2}, \tag{6}$$

$$v(1) = 0\,,\ \theta(1) = -\frac{1}{2}. \tag{7}$$

## 3. METHODS OF SOLUTION

**3.1 Analytical method for solution:** (MDTM) has been created for the analytical solution of differential equations, and it is discussed in this section. By Appling differential transformation theorems on Eqs. (4) and (5), can be obtained the following recursive relations [1]:

$$(k+1)(k+2)V(k+2) + 6\delta\sum_{r_2=0}^{k}\sum_{r_1=0}^{r_2}(r_1+1)(r_2-r_1+1)(k-r_2+1)(k-r_2+2)V(r_1+1)V(r_2-r_1+1)V(k-r_2+2) + \Theta(k) - H_a^2V(k) - PV(k) - F_s\sum_{r=0}^{k}v(k)v(k-r) = 0\,, \tag{8}$$

$$(k+1)(k+2)\,\Theta(k+2) + 2\,\delta\,E_c\,P_r\sum_{r_3=0}^{k}\sum_{r_2=0}^{r_3}\sum_{r_1=0}^{r_2}(r_1+1)(r_2-r_1+1)(r_3-r_2+1)(k-r_3+1)V(r_1+1)V(r_2-r_1+1)V(r_3-r_2+1)\,V(k-r_3+1) + E_c\,P_r\sum_{r=0}^{k}(r+1)(k-r+1)V(r+1)\ V(k-r+1) + \alpha\,\Theta(k) = 0, \tag{9}$$

Where $V\,(k)$ and $\Theta(k)$ are the differential transforms of $v(\eta)$ and $\theta(\eta)$.

The differential transform of the boundary conditions (6−7) are as follows:

$$V(0)=0,\ \Theta\,(0)=\frac{1}{2}, \tag{10}$$

$$\sum_{k=0}^{i} v(k)2^k = 0,\ \sum_{k=0}^{i}\Theta(k)2^k = -\frac{1}{2}, \tag{11}$$

We can consider the following boundary conditions (6−7):

$$v(-1) = 0,\ \theta(-1) = \frac{1}{2}\ , \tag{12}$$

$$v`(-1) = \lambda\,,\ \theta`(-1) = \omega\,, \tag{13}$$

Then, differential transform of (12-13) are given by:

$$V(1)=\lambda,\ \Theta(1)=\omega, \tag{15}$$

Moreover, substituting equations (14) and (15) into equations (8) and (9) and by recursive method we can calculating other values of V(k) and Θ(K).

**3.2 Numerical method for solution:**

1. **Finite difference method**

The system of coupled non-linear ordinary differential equations (4-5) with boundary conditions (6-7) calculated for $v\ (\eta)$ and $\theta(\eta)$ by (FDM). The following linearized form should be applied because of nonlinearity in this system:

$$\frac{d^2v}{d\eta^2}\left(1+6\delta\left(\frac{d\bar{v}}{d\eta}\right)^2\right)+\theta-\left(\ +H_a^2+F_s\bar{v}\right)v=0 \tag{16}$$

$$\frac{d^2\theta}{d\eta^2}+\ E_c\,P_r\frac{dv}{d\eta}\left(2\delta\left(\frac{d\bar{v}}{d\eta}\right)^3+\ \frac{d\bar{v}}{d\eta}\right)^3+\alpha\,\theta=0, \tag{17}$$

Where the iterated terms are shown in bar notation that convert Eqs. (4-5) to a linearized one.

The central point are used in Eqs. (16 -17) to obtain system of algebraic equations [2].

$$\frac{dv_i}{d\eta}=\frac{v_{i+1}-v_{i-1}}{\Delta}+\mathrm{o}(\Delta^2) \tag{18}$$

$$\frac{d^2v_i}{d\eta^2}=\frac{v_{i+1}-2v_i+v_{i-1}}{\Delta^2}+\mathrm{o}(\Delta^2) \tag{19}$$

$$\frac{d^2\theta_i}{d\eta^2}=\frac{\theta_{i+1}-2\theta_i+\theta_{i-1}}{\Delta^2}+\mathrm{o}(\Delta^2) \tag{20}$$

Where i = 1, 2, 3, ……, m+ 1 and m the number of subintervals of the finite domain of solution

($-1<\eta<1$).

2. **Shooting method**

The standard Runge–Kutta technique is used to derive numerical solutions of the ordinary differential equations (4) – (5) according to Neumann boundary conditions (6) and (7) with Shooting techniques and MATLAB package (ode45). The set of coupled nonlinear ordinary differential equations along with boundary conditions have been reduced to a system of simultaneous equations of first order for the unknowns following the method of superposition in Baitharu [2] and Na [13].

Eqs. (4) – (5) can be written as follows:

$$z_1'=z_2, \tag{21}$$

$$z_2'=\frac{-z_3\ +\ H_a^2\ z_1+P\ z_1}{1+6\,\delta\,(z_2)^2} \tag{22}$$

$$z_3'=z_4, \tag{23}$$

$$z_4'=-\,2\delta\,E_c\,P_r(z_2)^4-\ (E_c\,P_r)\,(z_2)^2-\ \alpha\,z_3 \tag{24}$$

Where $z_1=v$ and $z_3=\theta$.

The initial conditions are:

$$z_1(-1)=0,\ z_2(-1)=i_1,\ z_3(-1)=\frac{1}{2},\ z_4(-1)=i_2, \tag{25}$$

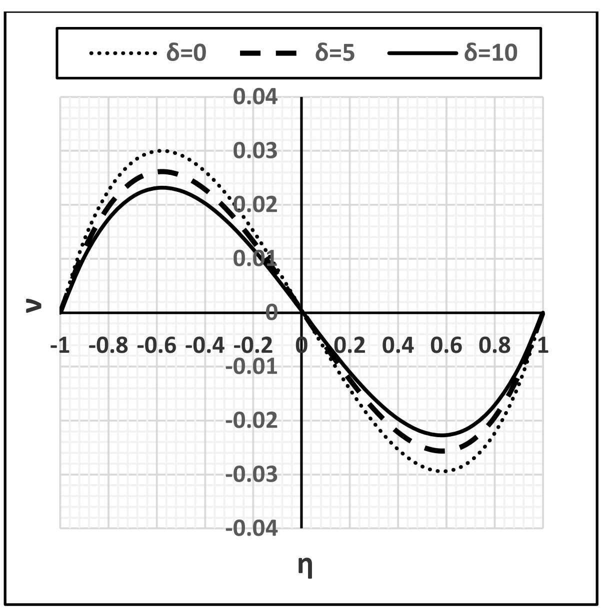
Where $i_1$ and $i_2$ are a priori unknowns that must be resolved as part of the solution.

(Ode45) integrates the system of differential Eqs. ((21)–(24)) with suitable guess values for initial conditions $i_1$ and $i_2$ . The calculated values of the velocity and temperature profiles are compared with the given boundary conditions.

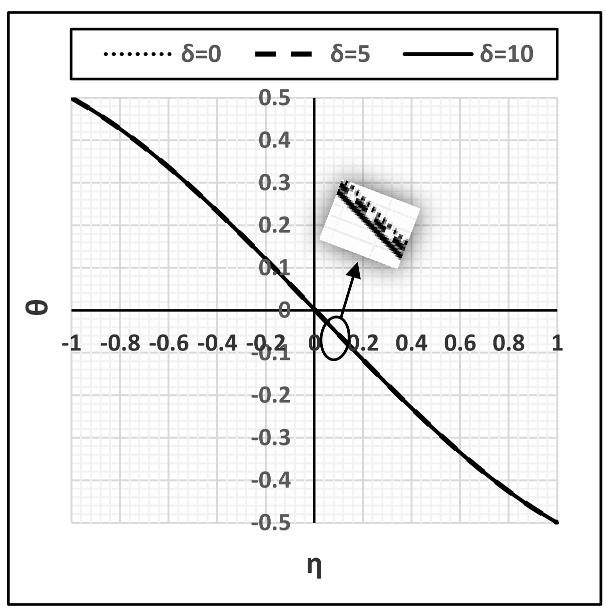
## ANALYSIS OF RESULTS

In this paper, the (MDTM), (FDM) and Shooting method are applied successfully to study effects of different parameters on Rivlin-Ericksen fluid. Tables and graphical representation of the results is very useful to demonstrate the efficiency and accuracy of (MDTM), (FDM) and Shooting method for the problem stated in this work. In order to ensure that the current results are accurate, we compared these results with the previously published work. Figures (2−8) (a) and (b) indicates the effects of δ, P, $H_a$, $F_s$, $E_C$, $P_r$ and α on v (η) and θ(η) profiles.

It can be noticed that there is an increase in δ, P, $H_a$ and $F_s$ lead to decrease in v (η) Figures (2−5) (a) but increasing in $P_r$, $E_C$ and α lead to increase in v (η) Figures (6−8) (a). It has also been noted that increases in δ cause reductions in θ(η) Figures (2) (b) but increasing in $P_r$, $E_C$ and α lead to increase in θ(η) Figures (6−8) (b). Also, it can be seen that the effect of P, $H_a$ and Fs on θ(η) is very little almost non-existent Figures (3−5) (b), because P, Ha and $F_s$ does not explicitly occur in the energy equation. Therefore, it can be concluded that P, $H_a$ and $F_s$ has negligible impact on the flow of θ(η).

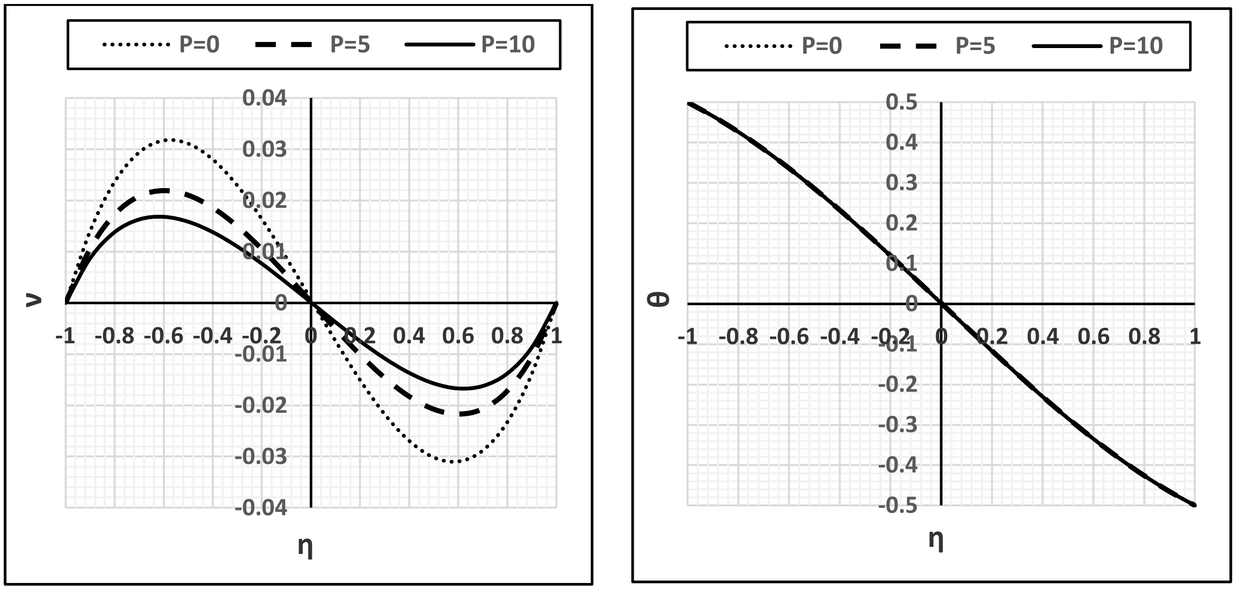



(a)



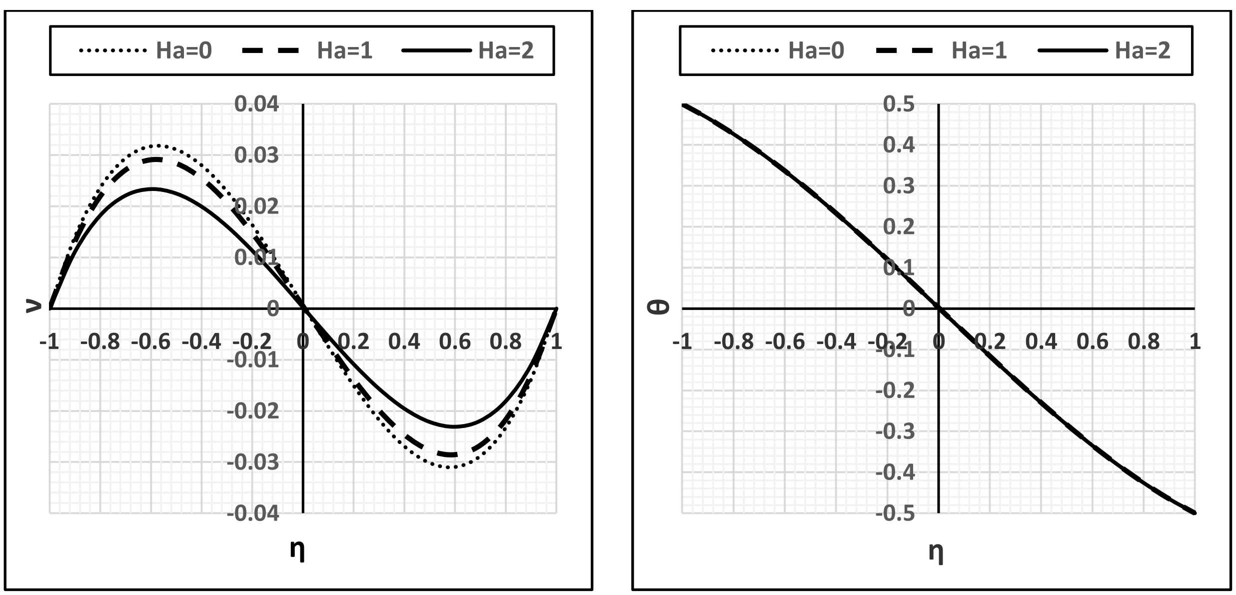

(b)

Figure 2: Variation of Velocity and Temperature at different values of δ.



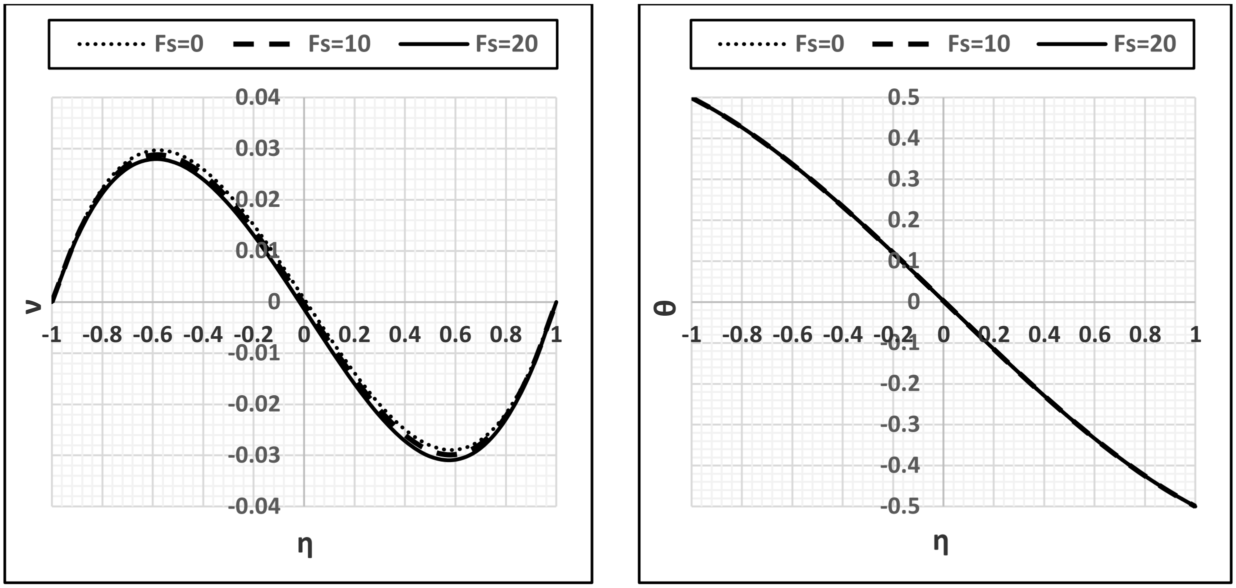

(a) (b)

Figure 3: Variation of Velocity and Temperature at different values of P.



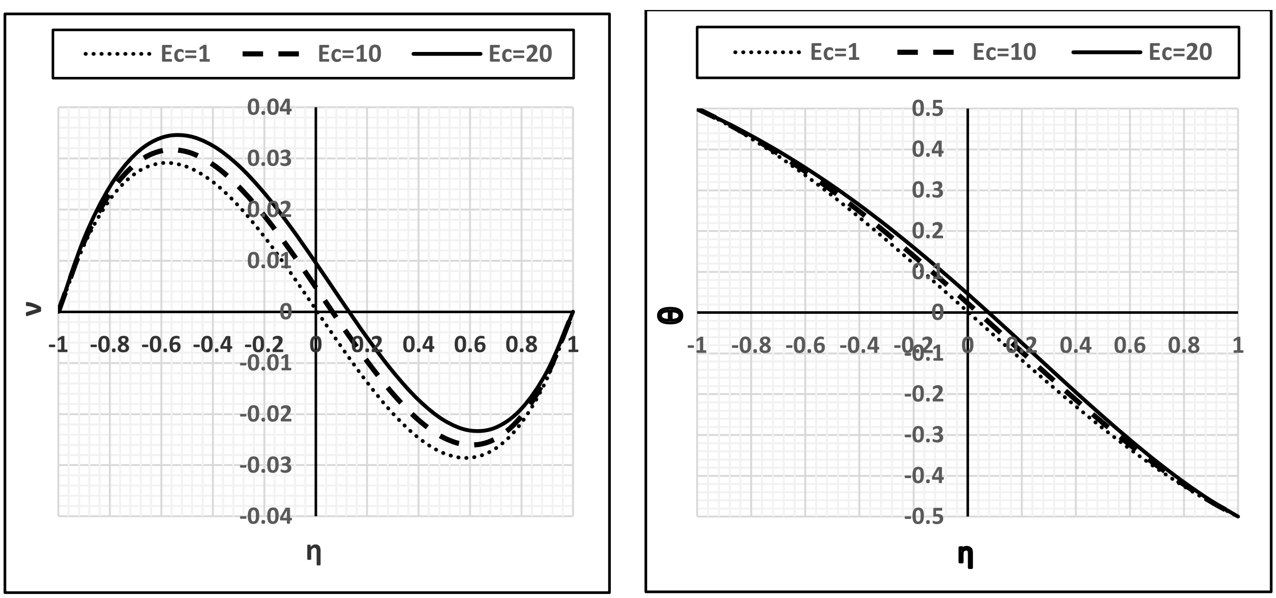

(a) (b)

Figure 4: Variation of Velocity and Temperature at different values of $H_a$.



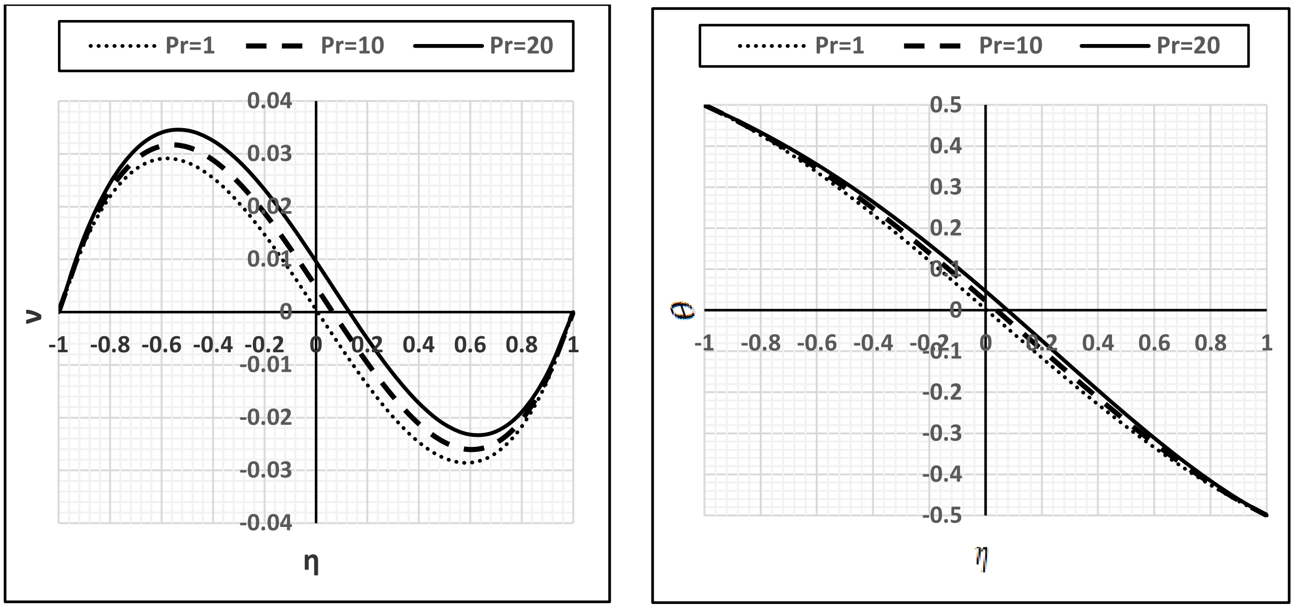

(a) (b)

Figure 5: Variation of Velocity and Temperature at different values of $F_s$.



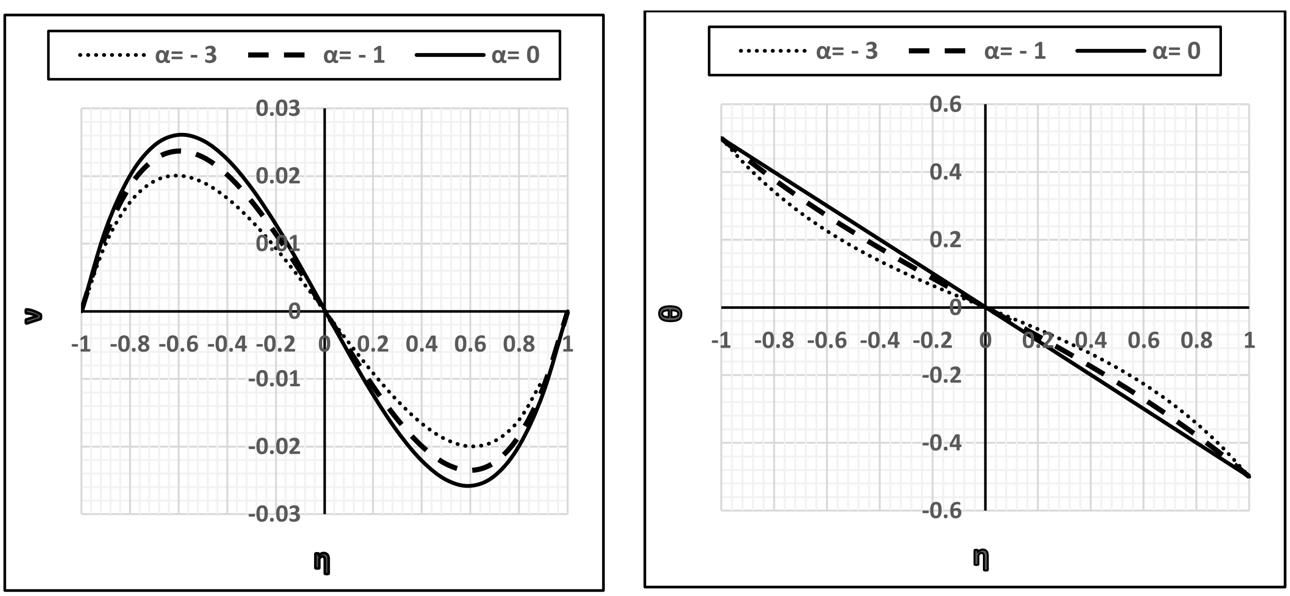

(a) (b)

Figure 6: Variation of Velocity and Temperature at different values of $E_c$.




(a) (b)

Figure 7: Variation of Velocity and Temperature at different values of $P_r$.




(a) (b)

Figure 8: Variation of Velocity and Temperature at different values of $\alpha$.

In addition, Tables (1−2) shows comparison between (MDTM), (FDM) and Shooting method with ((LSM) and (GM) [3])). As can be seen, this approximate analytical and numerical solution is in good agreement with the relevant answers. The following equations for v(η) and θ (η) distributions will be generated by algebraic computations:

$$v(\eta) = -1e-07\eta^{10} + 1.3e-05\eta^{9} + 6.9e-07\eta^{8} + 0.00021\eta^{7} - 1.5e-06\eta^{6} + 0.0024\,\eta^{5} + 2.5e-06\eta^{4} + 0.074\eta^{3} - 8.7e-06\eta^{2} - 0.076\eta + 7.1e-06 \quad (26)$$

$$\theta\,(\eta) = -6.7e-05\eta^{10} - 8.2e-05\eta^{9} + 0.00014\eta^{8} - 0.00033\eta^{7} - 0.00013\eta^{6} - 0.0042\eta^{5} + 7.3e-05\eta^{4} + 0.099\eta^{3} - 5e-05\eta^{2} - 0.59\eta + 3.2e-05 \quad (27)$$

Table 1: Comparison solution by (MDTM), (FDM) and Shooting method with ((LSM) and (GM) [3]) for v(η) when δ =1, P=1, $H_a$ =1, $F_s$=0, $E_c$ = 1, $P_r$ = 1 and α=1.

| $\eta$ | Present $v(\eta)$ | | | $v(\eta)$ [3] | |
|---|---|---|---|---|---|
| | MDTM | FDM | Shooting method | LSM | GM |
| −1 | 0 | 0 | 0 | 0 | 0 |
| −0.9 | 0.013146724 | 0.013094948 | 0.012932503 | 0.01296 | 0.013011 |
| −0.8 | 0.022204193 | 0.0219453880 | 0.022082051 | 0.021828 | 0.021913 |
| −0.7 | 0.027552442 | 0.0270982482 | 0.027400111 | 0.027059 | 0.027163 |
| −0.6 | 0.029639149 | 0.0290697213 | 0.029427599 | 0.029106 | 0.029218 |
| -0.5 | 0.028947913 | 0.0283513261 | 0.028678394 | 0.028426 | 0.028534 |
| -0.4 | 0.025971060 | 0.0254150876 | 0.025644843 | 0.025473 | 0.025568 |
| -0.3 | 0.021192239 | 0.0207179808 | 0.020802458 | 0.0207 | 0.020775 |
| -0.2 | 0.0150787593 | 0.0147057703 | 0.014613985 | 0.014563 | 0.014613 |
| -0.1 | 0.0080811050 | 0.0078163715 | 0.007532989 | 0.007517 | 0.007538 |
| 0 | 0.0006368290 | 0.0004828515 | 0.0000071 | 1.57E-05 | 6.53E-06 |
| 0.1 | - 0.006823275 | - 0.0068638171 | -0.007518963 | -0.00749 | -0.00753 |
| 0.2 | - 0.0138680651 | - 0.0137921405 | -0.014600473 | -0.01453 | -0.0146 |
| 0.3 | - 0.0200589440 | - 0.0198675177 | -0.020789786 | -0.02067 | -0.02076 |
| 0.4 | - 0.0249436890 | - 0.02464957683 | -0.02563331 | -0.02545 | -0.02556 |
| 0.5 | - 0.0280525328 | - 0.02768932697 | -0.028668274 | -0.0284 | -0.02852 |
| 0.6 | - 0.02889854812 | - 0.02852601206 | -0.029419133 | -0.02909 | -0.02921 |
| 0.7 | - 0.02698511095 | - 0.02668355321 | -0.027393515 | -0.02704 | -0.02716 |
| 0.8 | - 0.02182304241 | - 0.02166645779 | -0.022077515 | -0.02182 | -0.02191 |
| 0.9 | - 0.01295756242 | - 0.01295506647 | -0.012930187 | -0.01295 | -0.01301 |
| 1 | 0 | 0 | 0 | 0 | 0 |

Table 2: Comparison solution by (MDTM), (FDM) and Shooting method with ((LSM) and (GM) [3]) θ (η) when $\delta$ =1, P=1, Ha =1, Fs=0, $E_c$ = 1, $P_r$ = 1 and $\alpha$=1.

| $\eta$ | Present θ (η) | | | θ (η) [3] | |
|---|---|---|---|---|---|
| | MDTM | FDM | Shooting method | LSM | GM |
| −1 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 |
| −0.9 | 0.466012915 | 0.465645720 | 0.461505876 | 0.465147 | 0.465708161 |
| −0.8 | 0.427240839 | 0.426720201 | 0.422780584 | 0.425513 | 0.426457054 |
| −0.7 | 0.384145609 | 0.383581857 | 0.379795291 | 0.381631 | 0.382797641 |
| −0.6 | 0.337196762 | 0.336635359 | 0.332971998 | 0.334031 | 0.335280886 |
| -0.5 | 0.286877036 | 0.286327374 | 0.282781501 | 0.283244 | 0.28445775 |
| -0.4 | 0.233686538 | 0.233141923 | 0.229732991 | 0.229801 | 0.230879199 |
| -0.3 | 0.178145188 | 0.177595412 | 0.174365285 | 0.174232 | 0.175096193 |
| -0.2 | 0.120793326 | 0.120231361 | 0.117239457 | 0.117069 | 0.117659697 |
| -0.1 | 0.062190597 | 0.061614896 | 0.058932549 | 0.058842 | 0.059120673 |
| 0 | 0.002913268 | 0.002327053 | 0.000032 | 8.35E-05 | 3.01E-05 |
| 0.1 | - 0.056449821 | - 0.057041065 | -0.058869535 | -0.05868 | -0.059061105 |
| 0.2 | - 0.115302559 | - 0.115894206 | -0.117179239 | -0.11691 | -0.117601934 |
| 0.3 | - 0.173048004 | - 0.1736389001 | -0.174309274 | -0.17408 | -0.175041439 |
| 0.4 | - 0.229095968 | - 0.2296893664 | -0.229682149 | -0.22966 | -0.230828656 |
| 0.5 | - 0.282871753 | - 0.2834733124 | -0.282736476 | -0.28312 | -0.284412623 |
| 0.6 | - 0.3338261118 | - 0.3344375781 | -0.332933314 | -0.33392 | -0.335242377 |
| 0.7 | - 0.3814465042 | - 0.3820535624 | -0.379763469 | -0.38155 | -0.382766955 |
| 0.8 | - 0.4252696542 | - 0.4258223744 | -0.422756352 | -0.42545 | -0.426435393 |
| 0.9 | - 0.4648952048 | - 0.46527965267 | -0.461491453 | -0.46511 | -0.465696729 |
| 1 | -0.5 | - 0.5 | -0.5 | -0.5 | -0.5 |

## 4. CONCLUSION

In the present paper, we have applied (MDTM), (FDM) and Shooting method to compute effects of non-Darcy and heat source on MHD Rivlin-Ericksen Fluid of third-grade flow between two vertical flat plates through in a porous medium. Figures illustrate the effects of porosity parameter, Hartman number, non-Darcy parameter, Prandtl number and heat source parameter and the other parameters on the velocity and temperature. In particular, results for different parameters are summarized in the next points:

- It is observed that increasing in $\delta$, P, $H_a$ and $F_s$ lead to decrease in v(η) but increasing in $P_r$ , $E_c$ and $\alpha$ lead to increase in v (η).
- It is also noticed that an increase in $\delta$ lead to decreases in θ (η) but increasing in $P_r$, $E_c$ and $\alpha$ lead to increase in θ (η).

- Also, it can be seen that P, $H_a$ and $F_s$ has negligible impact on the flow $\theta$ ($\eta$).
- Furthermore, comparisons with available previously published works are performed and showed that the present methods for solutions and results have high accuracy and are found to be in good agreement as shown in tables.

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