مجلة الشروق للعلوم التجارية
الترقيم الدولي ISSN: 1687/8523
ترقيم دولي الكتروني Online :2682-356X
رقم الايداع بدار الكتب المصرية 2007/12870
البريد الالكتروني sjcs@sha.edu.eg
موقع المجلة : https://sjcs.sha.edu.eg/index.php

SH. A.
اكاديمية الشروق
عضو اتحاد الجامعات العربية

المعهد العالي للحاسبات وتكنولوجيا المعلومات

Hyers-Ulam-Rassias Stability of Abstract Dynamic Equations on Time Scales

د. نسرين عبد الحميد ياسين
مدرس الرياضيات والاحصاء – المعهد العالي للحاسبات وتكنولوجيا المعلومات
أكاديمية الشروق
dr.nisreen.yassin@sha.edu.eg

كلمات مفتاحية :

time scales; dynamic equations on time scales; Hyers–Ulam stability; Hyers–Ulam–Rassias stability

التوثيق المقترح وفقا لنظام APA :

Yassin, Nisreen Abd Elhamid(2022) Hyers-Ulam-Rassias Stability of Abstract Dynamic Equations on Time Scales, Al-Shorouk Journal of Commercial Sciences,V.14, The Higher institute for Computers and Information Technology, page 41 - 60

Hyers-Ulam-Rassias Stability of Abstract Dynamic equations On time scales

Nesreen A.Yaseen
Department of information systems
Higher Institute for Computers Science
El-Shorouk Academy
dr.nisreen . yassin @ sha.edu.eg

Abstract

This paper introduces the Hyers-Ulam-Rassias stability for abstract first order linear dynamic equation on time scales of the form,

$$x^{\Delta}(t) - A(t)x(t) - f(t) = 0\,, \qquad t \in [a,b]_{\mathbb{T}}, t > t_0$$
$$x(t_0) = x_0 \quad \in \mathbb{X}, \qquad .$$

and second order linear dynamic equation on time scales of the form,

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) + R(t)x(t) = f(t), \quad t \in \mathbb{T}$$

where $A, R: \mathbb{T} \to \mathrm{L}(\mathbb{X})$, the space of all bounded linear operators from a Banach space $\mathbb{X}$ into itself, and f is rd-continuous from a time scale $\mathbb{T}$ to $\mathbb{X}$. Illustrative example is given to show the applicability of the theoretical results.

Keywords: time scales; dynamic equations on time scales; Hyers–Ulam stability; Hyers–Ulam–Rassias stability

1. Introduction

S. M. Ulam gave a wide-ranging talk before a Mathematical Colloquium in1940 at the University of Wisconsin in which he discussed a number of important unsolved problems. He posed the following problem concerning the stability of functional equations, and established conditions which guarantee a linear mapping near approximately linear mapping exist.

D. H. Hyers [6] was the first mathematician who presented the result concerning the stability of functional equations. He answered the question of Ulam, and proved that Cauchy equation is stable in Banach spaces. This result of Hyers was stated in [15]. The results of Hyers were generalized by Rassias [17]. The results of Hyers and Rassias can be stated in the following theorem.

Theorem1.1 [15]. Let $f: E_1 \to E_2$ be a function between Banach spaces such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \quad (1.1)$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \quad (1.2)$$

exists for each $x \in E_1$ *and* $A: E_1 \to E_2$ *is a unique additive function such that*

$\|f(x) - A(x)\| \leq \delta$

for every $x \in E_1$*. Moreover, if* $f(tx)$ *is continuous in t for each fixed* $x \in E_1$*, then the function A is linear*

Taking this famous result into consideration, the additive Cauchy equation

$f(x+y) = f(x) + f(y)$, if for every function $f : E_1 \to E_2$ satisfying the inequality (1.1) for some $\delta \geq 0$, is said to have the Hyers–Ulam stability on (E_1, E_2)

and for all $x, y \in E_1$, there exists a function $A : E_1 \to E_2$ such that $f - A$ is bounded on E_1.

The method which was provided by Hyers is the most important and powerful tool to study the stability of various functional equations.

Ten years after the publication of Hyers's theorem, D. G. Bourgin [5] extended the theorem of Hyers and stated it in his paper [5] without proof. Hamza and Yassen extended the work of Douglas, Gates and Heuer, and investigated Hyers-Ulam stability of abstract second order linear dynamic equations on time scales[3,4] Unfortunately, it seems that this result of Bourgin failed to receive attention from mathematicians at that time. No one has made use of this result for a long time.

In 1978, Th. M. Rassias [17] addressed the Hyers's stability theorem to weaken the condition for the boundedness of the norm of Cauchy difference and proved a considerably generalized result of Hyers.

Theorem 1.2 (Rassias) [15]. *Let $f : E_1 \to E_2$ be a function between Banach spaces.*

If f satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \qquad (1.3)$$

for some $\theta \geq 0, p$ with $0 \leq p < 1$ and for all $x, y \in E_1$, then there exists a unique function $A : E_1 \to E_2$ such that

$$\| f(x) - A(x)\| \leq \frac{2\theta}{2 - 2^p} \|x\|^p \qquad (1.4)$$

for each $x \in E_1$.

This exciting result of Rassias attracted a number of mathematicians who began to be stimulated to investigate the stability problems of functional equations. By regarding a large influence of S. M. Ulam, D. H. Hyers, and Th. M. Rassias on the study of stability problems of functional equations, the stability phenomenon proved by Th. M. Rassias is called the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability will be regarded as a special case of the Hyers–Ulam–Rassias stability.

For the last thirty years many results concerning the Hyers–Ulam–Rassias stability of various functional equations have been obtained, and a number of definitions of stability have been introduced. In [15] S.-M introduced the exact definition of the Hyers–Ulam–Rassias stability which is applicable to most functional equations.

Many papers were presented to discuss The Hyers-Ulam - Rasssias stability in different cases, for example, in [18] Yang-Lee presented a generalization of the Hyers-Ulam-Rassias stability of Jensen's Equation. In [7] Gwang Hui investigated a generalization of Hyers-Ulam-Rassias stability for a functional equations in n variables. Also Zhang Wanxlong [20] introduced Hyers-Ulam- Rassias stability for a multivalued iterative equation. Also, Takeshi Miura, Soon-Mo Jung [16] presented Hyers-Ulam–Rassias stability of the Banach space valued linear differential equations $y' = \lambda y$

In 2013 Li and Wang presented Hyers-Ulam-Rassias for semilinear differential equations. In [9] L.P. Castro presented a paper devoted to the study of Hyers, Ulam and Rassias types of stability for a class of nonlinear Volterra integral equations. In [19] Yasseen.N.A established the Hyers-Ulam-Rassias stability for the Volterra integral dynamic equation on time scales In [11]. By using a fixed point method, Mohamed Akkouchi presented the Hyers–Ulam stability and the Hyers–Ulam–Rassias stability for a general class of nonlinear Volterra integral equations in Banach spaces .In [8] Hamza, A.E.; Alghamdi, M.A.; Alharbi, M.S. introduced Hyers–Ulam and Hyers–Ulam–Rassias Stability of a Nonlinear Second-Order Dynamic.

2.Preliminaries [4,13]

the following definitions and notations will be used in proving our main results in Section 3.

Definition 2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ s.t $\sigma(t) - s > 0$.

Definition 2.2. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) = inf\{s \in \mathbb{T}: s > t\}$, and $\rho(t) = sup\{s \in \mathbb{T}: s < t\}$ are called the jump operators.

Definition 2.3. A point t ∈ $\mathbb{T}$ is said to be right–dense if $\sigma(t) = t$, right-scattered $if\ \sigma(t) > t$, left-dense $if\ \rho(t) = t$, left-scattered $if\ \rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$, and dense $if\ \rho(t) = t = \sigma(t)$.

Definition 2.4. Let t ∈ $\mathbb{T}$. The graininess function

$\mu: \mathbb{T} \to [0, \infty[$ is defined as $\mu(t) = \sigma(t) - t$.

Definition 2.5. A function $f: \mathbb{T} \to \mathbb{X}$ is called rd-continuous provided that

(i) f is continuous at every right –dense point;

(ii) $lim_{s \to t^-} f(s)$ exists (finite) at left-dense points in $\mathbb{T}$.

The set of rd-continuous functions $f: \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.6. (The Delta Derivative). A function $f: \mathbb{T} \to \mathbb{X}$ is called Δ-differentiable at $t \in \mathbb{T}^k$ if there exists an element $f^\Delta(t) \in \mathbb{X}$ such that for any $\varepsilon > 0$ there is $\delta > 0$ such that:

$\left\| \left[f(\sigma(t)) - f(s) \right] - f^\Delta(t)[\sigma(t) - s] \right\| \le \varepsilon |\sigma(t) - s|,\ s \in (t - \delta, t + \delta) \cap \mathbb{T}$.

In this case $f^\Delta(t)$ is called the delta derivative of *f* at *t*, provided it exists and we have

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s}. \quad (2.1)$$

If $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$, we say that f is delta differentiable on $\mathbb{T}^k$, where

$$\mathbb{T}^k = \begin{cases} \mathbb{T}\backslash(\rho(\sup \mathbb{T}), \sup \mathbb{T}] & if \sup \mathbb{T} < \infty \\ \mathbb{T} & if \sup \mathbb{T} = \infty. \end{cases}$$

The set of all functions $f: \mathbb{T} \to \mathbb{X}$ that are differentiable and whose derivatives are rd-continuous is denoted by $C^1_{rd}(\mathbb{T}, \mathbb{X})$.

Definition 2.7. A function $f: \mathbb{T} \to \mathbb{R}$ is called regulated provided that its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.8. Let $f: \mathbb{T} \to \mathbb{X}$ be regulated function. Any function *F* which satisfies $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$, is called a pre-antiderivative of f. We define the indefinite integral of a regulated function f by

$$\int f(t)\Delta t = F(t) + C, \tag{2.2}$$

where *C* is an arbitrary constant. We define the Cauchy integral of f by

$$\int_r^s f(t)\,\Delta t = F(s) - F(r), \quad r, \in \mathbb{T}. \tag{2.3}$$

Definition 2.9. We say that a function $p: \mathbb{T} \to \mathbb{R}$ is regressive provided

$1 + \mu(t)p(t) \neq 0$, $\quad$ *for all* $t \in \mathbb{T}$.

The set of all regressive functions $f: \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$.

Definition 2.10 (The Generalized Exponential Function).

If $p \in \mathcal{R}$, then we define the exponential function $e_p(t,s)$ by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(t)}\left(p(\tau)\right)\Delta\tau\right), \qquad for\ s, t \in \mathbb{T},$$

where

$$\xi_{\mu(s)}\big(p(s)\big) = \begin{cases} \dfrac{1}{\mu(s)} log(|1 + \mu(s)p(s)| + i\, Arg(1 + \mu(s)p(s))), & for\ \mu(s) > 0 \\ p(s) & \mu(s) = 0 \end{cases}$$

Definition 2.11 [12]. We say that the equation

$$x^{\Delta}(t) = F(t,x), \quad t \in \mathbb{T}. \qquad (2.4)$$

has Hyers-Ulam stability if for every $\varepsilon > 0$ and $u \in C_{rd}^{1}(\mathbb{T},\mathbb{X})$ satisfies

$$\left\| u^{\Delta}(t) - F(t,u(t)) \right\| < \varepsilon, \qquad t \in \mathbb{T},$$

and there exists a solution $x \in C_{rd}^{1}(\mathbb{T},\mathbb{X})$ *of* (2.4) such that:

$\|u(t) - x(t)\| < L\varepsilon, t \in \mathbb{T}$ for some $L > 0$. *where* L *is called a Hyers–Ulam stability constant.*

3.Main results

In this paper we establish the Hyers-Ulam-Rassias stability for the first and second order abstract linear dynamic equations on time scales.

First, we introduce the following definition.

Definition 3.1: Let $K \subseteq C_{rd}(\mathbb{T},\mathbb{R}^{+})$. We say that the n^{th}order dynamic equation

$$x^{\Delta^{n}}(t) = F\left(t, x^{\Delta^{n-1}}, .. x^{\Delta}, x\right), \quad t \in \mathbb{T}. \qquad (3.1)$$

has Hyers-Ulam-Rassias stability of type K *if for every* $\varphi \in K$ *and* $u \in C_{rd}^{n}(\mathbb{T},\mathbb{X})$ *which satisfies*

$$\left\| u^{\Delta^{n}}(t) - F(t, u^{\Delta^{n-1}}(t), .., u^{\Delta}(t), u) \right\| \le \varphi(t), \quad t \in \mathbb{T} \quad (3.2)$$

there exists a solution $x \in C_{rd}^{1}(\mathbb{T},\mathbb{X})$ *of* (3.1) *such that*:

$$\|u(t) - x(t)\| \le L\varphi(t), \qquad t \in \mathbb{T} \text{ for some } L > 0. \qquad (3.3)$$

where L is called a Hyers–Ulam stability constant.

If K is the family of all positive constant functions, then Hyers-Ulam-Rassias stability yields Hyers-Ulam stability.

Throughout the rest of the paper, we denote by

$$\boldsymbol{M} = \left\{ \boldsymbol{\varphi} \in \boldsymbol{C_{rd}}(\mathbb{T},\mathbb{R}^{+}) : \int_{t_0}^{t} \boldsymbol{\varphi}^{2}(\boldsymbol{s})\Delta \boldsymbol{s} \le \boldsymbol{\varphi}^{2}(\boldsymbol{t}) \right\}.$$

3.1 Hyers–Ulam-Rassias stability of abstract linear first order dynamic equations on time scales.

In this section we study the Hyers-Ulam-Rassias stability for the first order abstract linear dynamic equation on time scales of the form :

$$x^{\Delta}(t) - A(t)x(t) - f(t) = 0\,, \qquad t \in [a,b]_{\mathbb{T}}, t > t_0$$
$$x(t_0) = x_0 \in \mathbb{X},$$

This section contains two parts. We begin the first part by assuming that $A \in C_{rd}\big(\mathbb{T}, \mathrm{L}(\mathbb{X})\big), f \in C_{rd}(\mathbb{T}, \mathbb{X})$. In addition we assume that A satisfies the following conditions:

(1)$sup_t \|A(t)\| < \infty$

(2) A is regressive, *that is,* $(I + \mu(t)A(t))$ *is invertible for every* $t \in \mathbb{T}$, *where I is the identity operator.*

which implies that the following abstract initial value problem (I. V. P):

$$x^{\Delta}(t) - A(t)x(t) - f(t) = 0, \;\; t \in \mathbb{T}\,, \qquad t > t_{0,}$$
$$x(t_0) = x_0 \in \mathbb{X}, \qquad (3.1.1)$$

has the unique solution

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A\big(t,\sigma(s)\big)f(s)\Delta s. \qquad (3.1.2)$$

Theorem3.1.1 If the function

$$\boldsymbol{F(t) = \int_{t_0}^{t} \left\|e_A\big(t,\sigma(s)\big)\right\|^2 \Delta s, t_0 \in \mathbb{T}} \qquad (\mathbf{3.1.3})$$

is bounded, then equation (3.1.1) *has Hyers-Ulam-Rassias stability with type M, That is, whenever for every function* $\varphi \in M$ *and* $g \in C^1_{rd}(\mathbb{T}, \mathbb{X})$ *satisfies*:

$\left\|\boldsymbol{g}^{\Delta}(\boldsymbol{t})-\boldsymbol{A}(\boldsymbol{t})\boldsymbol{g}(\boldsymbol{t})-\boldsymbol{f}(\boldsymbol{t})\right\| \leq \boldsymbol{\varphi}(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{T},\ \boldsymbol{t} \geq \boldsymbol{t_0}$ **, there exists a solution** $\boldsymbol{v} \in \boldsymbol{C}_{\boldsymbol{rd}}^{\Delta}(\mathbb{T},\mathbb{X})$ $\boldsymbol{of}$ $(\mathbf{5.2.1})$ **such that:**

$$\|\boldsymbol{g}(\boldsymbol{t})-\boldsymbol{v}(\boldsymbol{t})\| \leq \boldsymbol{L}\boldsymbol{\varphi}(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{T}, \boldsymbol{t} \geq \boldsymbol{t_0}$$

for some constant $\boldsymbol{L} > \mathbf{0}$**.**

Proof. Let $\boldsymbol{\varphi} \in \boldsymbol{M}$**. Suppose there exists** $\boldsymbol{g} \in \boldsymbol{C}_{\boldsymbol{rd}}^{\mathbf{1}}(\mathbb{T},\mathbb{X})$ **that satisfies**

$$\left\|\boldsymbol{g}^{\Delta}(\boldsymbol{t})-\boldsymbol{A}(\boldsymbol{t})\boldsymbol{g}(\boldsymbol{t})-\boldsymbol{f}(\boldsymbol{t})\right\| \leq \boldsymbol{\varphi}(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{T}, \boldsymbol{t} \geq \boldsymbol{t_0} \qquad (\mathbf{3.1.4})$$

Set

$$h(t) := g^{\Delta}(t) - A(t)g(t) - f(t), \qquad g(t_0) = g_0 \in \mathbb{X}. \quad (3.1.5)$$

By [1], g is given by

$$g(t) = e_A(t,t_0)g_0 + \int_{t_0}^{t} e_A\big(t,\sigma(s)\big)[f(s)+h(s)]\Delta s. \qquad (3.1.6)$$

The unique solution of the initial value problem

$$v^{\Delta} - Av - f = 0, \ \ v(t_0) = g_0 \qquad (3.1.7)$$

is given by

$$v(t) = e_A(t,t_0)g_0 + \int_{t_0}^{t} e_A\big(t,\sigma(s)\big)f(s)\Delta s\,, t \in \mathbb{T}. \qquad (3.1.8)$$

Since $\|h(t)\| \leq \varphi(t), t \geq t_0,$ then

$$\|g(t) - v(t)\| = \left\|\int_{t_0}^{t} e_A\,(t,\sigma(s))\,h(s)\Delta s\right\|$$

$$\leq \int_{t_0}^{t} \|e_A\,(t,\sigma(s))\,h(s)\|\Delta s$$

$$\leq \sqrt{\left\{\int_{t_0}^{t}\|e_A\,(t,\sigma(s)\|^2\,\Delta s\right\}\left\{\int_{t_0}^{t}\|h(s)\|^2\,\Delta s\right\}}$$

$$\leq \sqrt{\left\{\int_{t_0}^{t} \|e_A\,(t,\sigma(s)\|^2\,\Delta s\right\}}\sqrt{\left\{\int_{t_0}^{t} (\varphi(s))^2\,\Delta s\right\}}$$

$$\leq \quad \sqrt{sup_{t\in\mathbb{T}} F(t)} \qquad \varphi(t).$$

This completes the proof.

As a direct consequence, in view of the boundedness of $F(t) = \int_{t_0}^{t} \|e_A(t,\sigma(s))\|^2 \Delta s$ on $[a,b]_{\mathbb{T}}$, we obtain the following result

Corollary 3.1.2. *The equation*

$$x^{\Delta}(t) = A(t)x(t) + f(t)\,, \qquad t \in [a,b]_{\mathbb{T}}, t > t_0\ , \qquad (3.1.9)$$

$$x(t_0) = x_0$$

has Hyers-Ulam-Rassias stability of type M.

In a treatment parallel to that of the previous part, it is possible to investigate the Hyers-Ulam-Rassias stability of first order dynamic equations when the linear operator A is non-regressive and time invariant. In this case we assume that A is the generator of a C_0–semigroup $\{T(t)\}_{t\in\mathbb{T}}$. Here $\mathbb{T} \subseteq \mathbb{R}^{\geq 0}$ is a time scale semigroup in the sense that, $a - b \in \mathbb{T}$, for all $a, b \in \mathbb{T}$ with $a > b$. We refer the reader to [2] . The solution of the I.V.P

$$x^{\Delta}(t) - Ax(t) - f(t) = 0, \;\; t \in \mathbb{T}\,, \qquad t > t_{0,}$$

$$x(t_0) = x_0 \in D(A), \qquad (3.1.10)$$

is given by

$$\boldsymbol{x(t) = T\,(t - t_0)\,x_0 + \int_{t_0}^{t} T\big(t - \sigma(s)\big)f(t)\Delta s.} \qquad (\mathbf{3.1.11})$$

Theorem3.1.3. Let A be the generator of a C_0–semigroup$\{T(t)\}_{t\in\mathbb{T}}$, $f \in C_{rd}(\mathbb{T},\mathbb{X})$ such that the function

$$F(t) = \int_{t_0}^{t} \|T(t-\sigma(s))\|^2 \Delta s, \;\; t_0 \in \mathbb{T} \qquad (3.1.12)$$

is bounded. Then the I.V.P$(3.1.10)$ has Hyers-Ulam-Rassias-stability of type M.

Proof. Let$\varphi \in M$. Suppose there exists $g \in C_{rd}^1(\mathbb{T},\mathbb{X})$ that satisfies

$$\|g^\Delta(t) - A(t)g(t) - f(t)\| \le \varphi(t)\,, \quad t \in \mathbb{T}, t \ge t_0.$$

Set

$$h(t) := g^\Delta(t) - A\,g(t) - f(t),\; t \in \mathbb{T},\; g(t_0) = g_0 \in D(A)$$

Since, the linear dynamic equation (3.1.10) including a non-regressive operator A, and despite of the exponential function does not exist, the dynamic equation has a solution given in terms of the semi-group generating A. see [2]. So, $g(t)$, according to relation (3.1.11), is given by,

$$g(t) = T\,(t-t_0)\,g_0 + \int_{t_0}^{t} T(t-\sigma(s))[f(s)+h(s)]\Delta s \qquad (3.1.13)$$

The unique solution of the initial value problem

$$v^\Delta - Av - f = 0, \qquad v(t_0) = g_0 \qquad (3.1.14)$$

is given by

$$v(t) = T(t-t_0)g_0 + \int_{t_0}^{t} T(t-\sigma(s))f(s)\Delta s. \qquad (3.1.15)$$

We conclude that

$$\|g(t) - v(t)\| = \left\| \int_{t_0}^{t} T(t - \sigma(s)) h(s) \Delta s \right\|$$

$$\leq \int_{t_0}^{t} \|T(t - \sigma(s)) h(s)\| \Delta s$$

$$\leq \int_{t_0}^{t} \|T(t - \sigma(s))\| \varphi(s) \Delta s$$

By Cauchy –Schwarz Inequality [10], we have

$$\|g(t) - v(t)\| \leq \sqrt{\left\{\int_{t_0}^{t} \|T(t - \sigma(s))\|^2 \Delta s\right\}\left\{\int_{t_0}^{t} (\varphi(s))^2 \Delta s\right\}}$$

$$\leq \sqrt{sup_{t \in \mathbb{T}} F(t)}\, \varphi(t).$$

This completes the proof.

As a direct consequence, in view of the boundedness of

$$F(t) = \int_{t_0}^{t} \|T\ (t - \sigma(s))\|^2 \Delta s \quad \text{on} [a, b]_{\mathbb{T}}, \qquad (3.1.16)$$

we obtain the following result

Corollary 3.1.4. The equation

$$x^{\Delta}(t) - A(t)x(t) - f(t) = 0\,, \qquad t \in [a, b]_{\mathbb{T}}, t > t_0$$

$$x(t_0) = x_0 \qquad (3.1.17)$$

has Hyers-Ulam-Rassias stability of type M.

3.2 The Hyers-Ulam-Rassias stability of the abstract linear second order dynamic equations

This section is devoted to investigating the Hyers-Ulam-Rassias stability of the abstract dynamic equation of the form

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) + R(t)x(t) = f(t), \quad t \in \mathbb{T} \qquad (3.2.1)$$

where $A, R: \mathbb{T} \to \mathrm{L}(\mathbb{X})$, the space of all bounded linear operators from a Banach space $\mathbb{X}$ into itself, and f is rd-continuous from a time scale $\mathbb{T}$ to $\mathbb{X}$. An example is provided to demonstrate the applicability of the main findings.

Theorem3.2.1 Assume there is a particular solution $z: \mathbb{T} \to L(\mathbb{X})$ of the corresponding Recatti equation

$$z^{\Delta}(t) + \big(A(t) - z^{\sigma}(t)\big)z(t) = R(t)\,, \qquad t \in \mathbb{T}\,, \qquad (3.2.2)$$

such that $D = z^{\sigma} - A,\ -z$ *are regressive* and $D \in C_{rd}\big(\mathbb{T}, \mathrm{L}(\mathbb{X})\big)$.

If the following functions:

$$F_1(t) = \int_{t_0}^{t} \|e_D(t, \sigma(s))\|^2 \Delta s, t_0 \in \mathbb{T} \qquad (3.2.3)$$

and

$$F_2(t) = \int_{t_0}^{t} \|e_{-z}(t, \sigma(s))\|^2 \Delta s, t_0 \in \mathbb{T} \qquad (3.2.4)$$

are bounded, then Equation (3.2.1) has Hyers-Ulam-Rassias stability of type M.

Proof: To show that equation (3.2.1) has Hyers-Ulam –Rassias stability, let $\varphi(t)$ be a positive function of type M and $y \in C_{rd}^2(\mathbb{T}, \mathbb{X})$ that satisfies

$$\|y^{\Delta\Delta}(t) + A(t)y^{\Delta}(t) + R(t)y(t) - f(t)\| \le \varphi(t), \quad t \in \mathbb{T}. \qquad (3.2.5)$$

Our aim is to find a solution $u: \mathbb{T} \to \mathbb{X}$ *of* (3.2.1) *such that*

$$\|y - u\| \le C\ \varphi(t) \text{ on } \mathbb{T} \text{ for some constant } C > 0.$$

Assume that z is a particular solution of Equation (3.2.2) that satisfies the hypothesis of the Theorem.

Set $g = y^{\Delta} + zy$. Then $g^{\Delta} = y^{\Delta\Delta} + z^{\sigma}y^{\Delta} + z^{\Delta}y$. Consequently, we have

$$\|g^{\Delta} - Dg - f\| =$$

$$= \|y^{\Delta\Delta} + z^{\sigma}y^{\Delta} + z^{\Delta}y(t) - (z^{\sigma} - A)(y^{\Delta} + zy) - f\|$$

$$= \|y^{\Delta\Delta} + Ay^{\Delta} + Ry - f\| \le \varphi(t) \text{on } \mathbb{T}, \text{ from } (3.2.5).$$

Since $F_1$is bounded, then by Theorem 2.1.1the Equation

$$x^{\Delta}(t) - D(t)x(t) = f(t), \quad t \in \mathbb{T} \qquad (3.2.6)$$

$$x(t_0) = x_0$$

has Hyers-Ulam-Rassias stability of type M, and there is a solution $v \in C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{X})$ of equation (3.2.6) such that

$\|g(t) - v(t)\| \le L\varphi(t), \quad t \in \mathbb{T}$, for some $L > 0$. It follows that $\left\|y^{\Delta}(t) + z(t)y(t) - v(t)\right\| \le L\varphi(t)$, . Again, in view of the boundedness of F_2 , there is a solution $u \in C^1_{rd}(\mathbb{T}, \mathbb{X})$ of the equation

$$x^{\Delta}(t) + z(t)x(t) - v(t) = 0, \quad t \in \mathbb{T}, \qquad (3.2.7)$$

$$x(t_0) = x_0$$

such that$\|y(t) - u(t)\| \le k\varphi(t), \ t \in \mathbb{T}$, for some $k > 0$.

Now, we can cheek that u is a solution of equation (3.2.1). Indeed, we have

$$u^{\Delta\Delta} + Au^{\Delta} + Ru - f = (v^{\Delta} - z^{\sigma}u^{\Delta} - z^{\Delta}u + Au^{\Delta} + Ru - f)$$

$$= (Dv + f) - (D + A)u^{\Delta} + (R - z^{\Delta})u + Au^{\Delta} - f$$

$$= D(v - u^{\Delta} - zu) = 0$$

Therefore, u is a solution of (3.2.1).

As a direct consequence, in view of the boundedness of $F_1(t) = \int_{t_0}^{t} \|e_D(t, \sigma(s))\|^2 \Delta s$, and

$$F_2(t) = \int_{t_0}^{t} \|e_{-z}(t, \sigma(s))\|^2 \Delta s \quad \text{on} [a, b]_{\mathbb{T}}$$

we obtain the following result, which depends on theorem 3.1 of [4].

Corollary 3.2.2. The equation

$$x^{\Delta\Delta}(t) + A(t)x^{\Delta}(t) + R(t)x(t) = f(t), \quad t \in [a, b]_{\mathbb{T}}, \qquad (3.2.8)$$

has Hyers-Ulam-Rassias stability of type M.

4. An Illustrative Numerical Example

In this section we give an illustrative example to show the effectiveness of the theoretical results.

Example.4.1

The linear dynamic equation

$$x^{\Delta}(t) = A(t)x(t), \qquad t \in \mathbb{T},\ \ t > t_0 \qquad (4.1)$$

$$x(0) = x_0 \ \in \ \mathbb{R}^2$$

has Hyres-Ulam- Rassias stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$ of type M where $A(t)$ is the 2x2 matrix defined by

$$A(t) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix},\ t \in \mathbb{T}$$

We have for $\mathbb{T} = \mathbb{R}^{\geq 0}$,

$$e_A(t, \sigma(s)) = e_A(t, s)$$

$$= I + \int_s^t A(y_1)\Delta y_1 + \int_s^t A(y_1) \int_s^{y_1} A(y_2)\Delta y_2 \Delta y_1 + \cdots$$

$$+ \int_s^t A(y_1) \int_s^{y_1} A(y_2) \ldots \int_s^{y_{i-1}} A(y_i)\Delta y_i \ldots \Delta y_1 + \cdots$$

$$= I - \int_s^t y_1 dy_1 I + \int_s^t y_1 \int_s^{y_1} y_2 dy_2\, dy_1\, I + \ldots + (-1)^i \int_s^t y_1 \int_s^{y_1} y_2 \ldots \int_s^{y_{i-1}} y_i dy_i \ldots dy_1 I + \ldots$$

$$= I - \frac{t^2 - s^2}{2} I + \frac{1}{2!}\left(\frac{t^2 - s^2}{2}\right)^2 I - \frac{1}{3!}\left(\frac{t^2 - s^2}{2}\right)^3 I + \ldots + (-1)^i \frac{1}{i!}\left(\frac{t^2 - s^2}{2}\right)^i I + \ldots \quad \text{And}$$

$$= e^{-\frac{t^2 - s^2}{2}} I$$

$$\|e_A(t,\sigma(s))\| = e^{\frac{-(t^2-s^2)}{2}}$$

$$= e^{\frac{-(t^2-s^2)}{2}} I$$

And

$$\|e_A(t,\sigma(s))\| = e^{\frac{-(t^2-s^2)}{2}}$$

Then, $\|e_A(t,\sigma(s))\|^2 = e^{-(t^2-s^2)}$

Since $F(t) = \int_0^t e^{-(t^2-s^2)}\, ds$ is bounded, then Equation (4.1) has Hyres-Ulam-Rassias stability on $\mathbb{T} = \mathbb{R}^{\geq 0}$ of type *M.*

It is clear that Equation (4.1) has also Hyer-Ulam stability.

References

[1] Alaa E. Hamza, Moneer Al-Qubaty, On The Exponential Operator Functions on Time Scales, Advances in Dynamical Systems and Applications, 7(1) (2012), 57-80.

[2] Alaa E. Hamza, Karima M. Oraby, Stability of Abstract Dynamic Equations on Time Scales, Advances in Difference Equations, (2012)(1) Doi:10.1186/1687.

[3] Alaa E. Hamza, Nesreen A.Yaseen, Hyers-Ulam stability of first order abstract linear dynamic equations on time scales, Advances in mathematics, 7(1) (2014), 1095- 1104.

[4] Alaa E. Hamza, Nesreen A.Yaseen. Hyers–Ulam stability of abstract second-order

linear dynamic equations on time scales. Int. J. Math.Anal. 2014, 8, 1421–1432.

[5] D. G. Bourgin, Approximate isometries, Bull. Amer. Math. Soc. 52 (1946), 704– 714.

[6] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat Acad. Sci. U.S.A, 27 (1941), 222-224

[7] Gwang.H.K, On the Hyers-Ulam-Rassias stability of func tional equations in n- variables, Journal of Mathematical Analysis and Applications, 299(2004) 375- 391.

[8] Hamza, A.E.; Alghamdi, M.A.; Alharbi, M.S. introduced Hyers–Ulam and Hyers–Ulam–Rassias Stability of a Nonlinear Second-Order Dynamic Equation on Time Scales Mathematics 2021, 9, 1507.

[9] L. P. Castro. and A. Ramos ,Hyers-Ulam-Rassias stability for a class of nonlinear Volterra integral equations, Ban- ach J. Math. Anal. 3 (1)(2009), 36–43.

[10] M. Bohner, A. Peterson, Dynamic Equations on Time Scale An introduction with Applications, Birkhauser An introduction with Applications, Birkhauser, Boston. Basel. Berlin, (2001).

[11] N. A.Yassen, On solutions of dynamic equations on time scales., Ph. D. Thesis, Ain Shams University, (2014)

[12] M. Al-Qubaty, On Theory of Dynamic equations on Time Scales, Ph. D. Thesis, Ain Shams University, (2011).

[13] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[14] S.-M, Jung, Hyres-Ulam stability of first order linear differential equations with constant coefficients, J. Math. Anal. Appl. 320 (2006), 549-561.

[15] S.-M Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Nonlinear Analysis, Springer Optimization and Its Applications, 48 (2011), 87-104

[16] T.MIURA, S.-M. Jung, S.-E. Takahasi, Hyers-Ulam-Rassias stability of the Banach space valued differential equation, Korean Math. Soc. 41 (2004) 995- 1005

[17] T. M. Rassias, On the stability of linear mapping in Banach spaces,

Proc.Amer. Math. Soc. 72(1978), 297-300.

[18] Yang. L., Kil-Woung Jun,A Generalization of the Hyers- Ulam-Rassias stability of Jensen"s equation, Journal of Mathematical Analysis and Applications. Vol 238(1999), 305-315.

[19]Yaseen.N.A Hyers-Ulam stability for Volterra Integral equation on time scales " Advanced in Computer Sciencce Vol.8 ,June (2015) 33-44

[20] Z. Wanxlong, Acta Mathematica Scientia, 28 (2008)