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Estimation of AR (2) Model with Dependent Errors for Bounded Stationary and Uncompleted Nonstationary Time Series

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Estimation of AR (2) Model with Dependent Errors for Bounded Stationary and Uncompleted Nonstationary Time Series

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Abstract

In this paper, the GLS and the ML estimators, the variance-covariance matrix, the unbiasedness for the GLS and the ML estimators of AR (2) model with dependent errors have been proved. A simulation study has been provided for bounded stationary (uncompleted nonstationary) under different conditions (five cases have been provided), for different sample sizes using MSE and Thiel's U as criteria for comparison.

Keywords: AR (2) Model; GLS Estimators; ML Estimators; Mean Squared Error; Thiel's Inequality Coefficient; Bounded Stationary Time Series; Bounded Uncompleted Nonstationary Time Series.

List of Abbreviations

Abbreviation	Meaning
AR (2)	**Autoregressive Models of Order (2)**
GLS	**Generalized Least Squares**
ML	**Maximum Lieklihood Method**
MSE	**Mean Squared Error**
Thiel's U	**Thiel's Inequality Coefficient**

1. Introduction

The classical regression model seeks to determine the relationship between the dependent variable and the independent variables. This regression model could be simple or multiple. However, in the linear regression model, certain assumptions are made on how a dataset will be produced by an underlying data-generating process. According to Greene (2002), these assumptions include linearity, homoscedasticity, normality, and no autocorrelation between the error terms. Moreover, the regression model describes the value of the dependent variable as the sum of two parts, a deterministic part, and a random part.

The error term is primarily a disturbance to an already stable relationship and can capture the remaining information in the dependent variable which could not be explained by the independent variables. Relating to the assumption on the error term, if the assumption of no correlation in the error term is violated, then, the underlying model would be rendered invalid with the standard errors of the parameters becoming biased. Moreover, if the errors are correlated, the least-squares estimators are inefficient and the estimated variances are not appropriate Granger and Newbold (1974) and Akpan, et al (2016).

By definition, autocorrelation is the lag correlation of a given series with itself, lagged by some time units Gujarati (2004). Thus, when applying regression models to economic/management data in the presence of autocorrelation, the ordinary least squares estimation method ceases to provide efficient estimators and appropriate variances.

The analysis of time series is very important and it is a rapidly evolving field, generally, time series is a sequence of values a specific variable has taken on over some time. The observations have a natural ordering in time. Usually, if a series of observations is referred to as a time series, then some regularity of the observation frequency are assumed. Of course, the observation frequency could be more frequent than yearly. For instance, observations may be available for each quarter, each month, or even each day of a particular period. Nowadays, time series of stock prices or other financial market variables are even available at a much higher frequency such as every few minutes or seconds, Lütkepohl and Krätzig (2004).

A model often specified for the generation of an economic time series is the stochastic difference equation with independently and identically distributed errors. In practice, the most common assumption is that the time series is stationary. However, there are situations in which the stationarity assumption is not appropriate. Two of such situations are testing the random walk hypothesis or unit root hypothesis and testing the first difference hypothesis, Evans and Savin (1981).

For the second-order autoregressive AR (2) model in the case of real roots, the stationarity conditions for an AR (2) processes are as follows:

$$\left.\begin{array}{l} 1)\ \ \rho_1 + \rho_2 < 1 \\ 2)\ \rho_2 - \rho_1 < 1 \\ 3)\ |\rho_2| < 1 \end{array}\right\} , \qquad ($$

where ρ_1 and ρ_2 are the autoregressive coefficients of the AR (2) model, David (2012).

Autoregressive time series with a unit root has been become the subject of much recent attention in the econometrics literature. In part, this is because the unit root hypothesis is of considerable interest in applications, not only with data from financial and commodity markets where it has a long history but also with aggregate time series, Phillips (1987).

In an attempt to overcome the weaknesses of the ordinary least squares estimation method in the presence of autocorrelation, this study seeks to apply the generalized least squares (GLS) estimation method on the AR (2) model since the least-squares estimation method does not make use of the information of the unexplained variance as captured by the error terms in the dependent variable, whereas the generalized least squares (GLS) takes such information, the unexplained variance into account explicitly and is accomplished, Akpan and Moffat (2018).

2. Model and Assumptions

The second-order autoregressive AR (2) with a constant model in case of dependent errors takes the following form:

$$y_t = \alpha + \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t, \qquad t = 1,\ldots,T, \tag{2}$$

where y_t is time series, T is the sample size, $y_0 = y_{-1} = 0$, u_t are dependent error terms ρ_1 and ρ_2 are the autoregressive coefficients, and α is the constant term.

Model (2) can be represented in matrix form as follows:

$$Y = X\beta + \mathbf{u}, \tag{3}$$

where:

$$\beta = \begin{bmatrix} \alpha \\ \rho_1 \\ \rho_2 \end{bmatrix}, \; X = \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix}, \; Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_T \end{bmatrix} \text{ and } \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_T \end{bmatrix} \tag{4}$$

By assuming that u_t are dependent error terms which can be generated by the first-order stationary autoregressive process AR (1) model then, it will be as follows:

$$u_t = \phi_1 u_{t-1} + e_t \ , \ |\phi_1| < 1 \ , \ t = 1 \ , \ . \ . \ . \ , \ T \ , \qquad (5)$$

where the error terms e_t are i.i.d. $N(0, \sigma_e^2)$ and achieved the following assumptions:

$$\left.\begin{array}{ll} 1)\ E(e_t) = 0 & \forall t \\ 2)\ \sigma_e^2 = E(e_t^2) = \sigma^2 & \forall t \\ 3)\ \sigma_{s\,t} = Cov(e_t, e_s) = E(e_t\, e_s) = 0 & \forall t \neq s \end{array}\right\} \qquad (6)$$

To the sample interval $t = 1, \ldots, T$ the variance-covariance matrix for the vector of error terms $\mathbf{u} = [u_1, u_2, \ . \ . \ . \ , u_T]$ for model (2) can be obtained by using the lag operator (L) as follows:

From Equation (5) $u_t = \phi_1 u_{t-1} + e_t$ then:

$$u_t - \phi_1 u_{t-1} = e_t$$

$$(1 - \phi_1 L) u_t = e_t$$

$$u_t = (1 - \phi_1 L)^{-1} e_t = \sum_{j=0}^{\infty} \phi_1^{\ j} L^j e_t = \sum_{j=0}^{\infty} \phi_1^{\ j} e_{t-j} \ , \ |\phi_1| < 1 \qquad (7)$$

Then $E(u_t), Var(u_t)$ and $Cov(u_t, u_{t-s})$ will be as follows:

$$\text{i) } E(u_t) = 0 \qquad (8)$$

$$\text{ii) } Var(u_t) = \frac{\sigma^2}{(1 - \phi_1^2)} \qquad (9)$$

$$\text{iii)}\, Cov(u_t, u_{t-s}) = \sigma^2 \phi_1^s \left(\frac{1}{1-\phi_1^2}\right),\ s = 1, 2, 3, \ldots \tag{10}$$

Then, by using Equations (9) and (10), the variance-covariance matrix for the vector of error terms $\mathbf{u}=[u_1, u_2, \ldots, u_T]$ will be:

$$Cov(\mathbf{u}) = E\left\{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}\begin{bmatrix} u_1 & u_2 & u_3 \cdots u_T \end{bmatrix}\right\} = \sigma^2 \Omega \tag{11}$$

Where:

$$\Omega = \frac{1}{(1-\phi_1^2)}\begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \tag{12}$$

Fox and Weisberg (2018)

3. GLS Estimation for AR (2) Model with Dependent Errors

In this section, the GLS estimators, unbiased of GLS estimators and the variance-covariance matrix for GLS estimators for the parameters of AR (2) model with constant in case of dependent errors will be derived.

Lemma (1): The GLS estimators for the parameters of AR (2) with a constant model in case of dependent errors as in Equation (3) under the assumptions of the model will be as follows:

$$\widetilde{\alpha}=\frac{K_{11}N_{11}+K_{12}N_{12}+K_{13}N_{13}}{G}$$

$$\widetilde{\rho}_1=\frac{K_{21}N_{11}+K_{22}N_{12}+K_{23}N_{13}}{G}$$

$$\widetilde{\rho}_2=\frac{K_{31}N_{11}+K_{32}N_{12}+K_{33}N_{13}}{G},$$

where:

$K_{11}=CF-E^2$, $K_{22}=\Delta F-B^2$, $K_{33}=\Delta C-A^2$, $K_{12}=K_{21}=BE-AF$

$K_{32}=K_{23}=AB-\Delta E$ and $K_{13}=K_{31}=AE-CB$

$G=\Delta CF-\Delta E^2-A^2F+2ABE-B^2C$

$\Delta=1-\phi_1+1-2\phi_1+\phi_1^2+1-2\phi_1+\phi_1^2+\cdots+1-\phi_1=T-2(T-1)\phi_1+(T-2)\phi_1^2$

$$\begin{aligned}A&=(1-\phi_1)y_0+(1-2\phi_1+\phi_1^2)y_1+(1-2\phi_1+\phi_1^2)y_2+\cdots+(1-\phi_1)y_{T-1}\\&=(1-\phi_1)y_0+(1-\phi_1)y_{T-1}+(1-2\phi_1+\phi_1^2)\textstyle\sum_{t=1}^{T-2}y_t\\&=[y_{-1}-\phi_1y_0]y_1+[y_{T-3}-\phi_1y_{T-2}]y_T+\textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t)+(1+\phi_1^2)y_{t-1}]y_{t+1}\end{aligned}$$

$$\begin{aligned}B&=(1-\phi_1)y_{-1}+(1-2\phi_1+\phi_1^2)y_0+(1-2\phi_1+\phi_1^2)y_1+\cdots+(1-\phi_1)y_{T-2}\\&=(1-\phi_1)y_{-1}+(1-\phi_1)y_{T-2}+(1-2\phi_1+\phi_1^2)\textstyle\sum_{t=1}^{T-2}y_{t-1}\end{aligned}$$

$$\begin{aligned}C&=(y_0-\phi_1y_1)y_0+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_1+[-\phi_1(y_1+y_3)+(1+\phi_1^2)y_2]y_2\\&\quad+\cdots+[y_{T-2}-\phi_1y_{T-1}]y_{T-1}\\&=(y_0-\phi_1y_1)y_0+[y_{T-2}-\phi_1y_{T-1}]y_{T-1}+\textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1})+(1+\phi_1^2)y_t]y_t\end{aligned}$$

$$\begin{aligned}E&=(y_0-\phi_1y_1)y_{-1}+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_0+[-\phi_1(y_1+y_3)+(1+\phi_1^2)y_2]y_1\\&\quad+\cdots+[y_{T-2}-\phi_1y_{T-1}]y_{T-2}\\&=(y_0-\phi_1y_1)y_{-1}+[y_{T-2}-\phi_1y_{T-1}]y_{T-2}+\textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1})+(1+\phi_1^2)y_t]y_{t-1}\end{aligned}$$

$$\begin{aligned}F&=(y_{-1}-\phi_1y_0)y_{-1}+[-\phi_1(y_{-1}+y_1)+(1+\phi_1^2)y_0]y_0+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_1\\&\quad+\cdots+[y_{T-3}-\phi_1y_{T-2}]y_{T-2}\\&=(y_{-1}-\phi_1y_0)y_{-1}+[y_{T-3}-\phi_1y_{T-2}]y_{T-2}+\textstyle\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t)+(1+\phi_1^2)y_{t-1}]y_{t-1}\end{aligned}$$

$$\begin{aligned}N_{11}&=(1-\phi_1)y_1+(1-2\phi_1+\phi_1^2)y_2+(1-2\phi_1+\phi_1^2)y_3+\cdots+(1-\phi_1)y_T\\&=(1-\phi_1)y_1+(1-\phi_1)y_T+(1-2\phi_1+\phi_1^2)\textstyle\sum_{t=1}^{T-2}y_{t+1}\end{aligned}$$

$$N_{12}=[y_0-\phi_1 y_1]y_1+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_2+[-\phi_1(y_1+y_3)+(1+\phi_1^2)y_2]y_3$$
$$+\cdots+[y_{T-2}-\phi_1 y_{T-1}]y_T$$
$$=[y_0-\phi_1 y_1]y_1+[y_{T-2}-\phi_1 y_{T-1}]y_T+\sum_{t=1}^{T-2}[-\phi_1(y_{t-1}+y_{t+1})+(1+\phi_1^2)y_t]y_{t+1}$$

$$N_{13}=[y_{-1}-\phi_1 y_0]y_1+[-\phi_1(y_{-1}+y_1)+(1+\phi_1^2)y_0]y_2+[-\phi_1(y_0+y_2)+(1+\phi_1^2)y_1]y_3$$
$$+\cdots+[y_{T-3}-\phi_1 y_{T-2}]y_T$$
$$=[y_{-1}-\phi_1 y_0]y_1+[y_{T-3}-\phi_1 y_{T-2}]y_T+\sum_{t=1}^{T-2}[-\phi_1(y_{t-2}+y_t)+(1+\phi_1^2)y_{t-1}]y_{t+1}$$

Proof:

The GLS Estimator $\tilde{\beta}$, in general, is defined as follows:

$$\tilde{\beta}=(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y \tag{13}$$

Fox and Weisberg (2018)

Where for the model in Equation (3),

$$\tilde{\beta}=\begin{bmatrix}\tilde{\alpha}\\ \tilde{\rho}_1\\ \tilde{\rho}_2\end{bmatrix},\ X=\begin{bmatrix}1 & y_0 & y_{-1}\\ 1 & y_1 & y_0\\ \vdots & \vdots & \vdots\\ 1 & y_{T-1} & y_{T-2}\end{bmatrix},\ Y=\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_T\end{bmatrix} \tag{14}$$

By substituting from Equations (12) and (14) in Equation (13) then the GLS estimator $\tilde{\beta}$ for the model in Equation (3) will be as follows:

$$\begin{bmatrix}\tilde{\alpha}\\ \tilde{\rho}_1\\ \tilde{\rho}_2\end{bmatrix}=\left\{\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ y_0 & y_1 & y_2 & \cdots & y_{T-1}\\ y_{-1} & y_0 & y_1 & \cdots & y_{T-2}\end{bmatrix}\left(\frac{1}{(1-\phi_1^2)}\begin{bmatrix}1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1}\\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2}\\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3}\\ \vdots & \vdots & \vdots & & \vdots\\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & y_0 & y_{-1}\\ 1 & y_1 & y_0\\ 1 & y_2 & y_1\\ \vdots & \vdots & \vdots\\ 1 & y_{T-1} & y_{T-2}\end{bmatrix}\right\}^{-1}$$

$$\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ y_0 & y_1 & y_2 & \cdots & y_{T-1}\\ y_{-1} & y_0 & y_1 & \cdots & y_{T-2}\end{bmatrix}\left(\frac{1}{(1-\phi_1^2)}\begin{bmatrix}1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1}\\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2}\\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3}\\ \vdots & \vdots & \vdots & & \vdots\\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}y_1\\ y_2\\ y_3\\ \vdots\\ y_T\end{bmatrix}$$

Then,

$$\begin{bmatrix}\tilde{\alpha}\\ \tilde{\rho}_1\\ \tilde{\rho}_2\end{bmatrix}=\begin{Bmatrix}\Delta & A & B\\ A & C & E\\ B & E & F\end{Bmatrix}^{-1}\begin{bmatrix}N_{11}\\ N_{12}\\ N_{13}\end{bmatrix}$$

$$\left.\begin{aligned}\tilde{\alpha}&=\frac{K_{11}N_{11}+K_{12}N_{12}+K_{13}N_{13}}{G}\\ \tilde{\rho}_1&=\frac{K_{21}N_{11}+K_{22}N_{12}+K_{23}N_{13}}{G}\\ \tilde{\rho}_2&=\frac{K_{31}N_{11}+K_{32}N_{12}+K_{33}N_{13}}{G}\end{aligned}\right\}, \qquad (15)$$

where:
$\Delta, A, B, C, E, F, G, N_{11}, N_{12}, N_{13}, K_{11}, K_{22}, K_{33}, K_{12}=K_{21}, K_{32}=K_{23}$ and $K_{13}=K_{31}$

are defined as in Lemma (1).

3.2 The Unbiasedness of GLS Estimators

From Equation (13) the GLS estimators for parameters of AR (2) model with constant in case of dependent errors is as follows:

$$\tilde{\beta}=(X'\Omega^{-1}X)^{-1}X'\Omega^{-1}Y$$

Then,

$$E(\tilde{\beta}) = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}E(Y)$$
$$= (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}X'\beta + E(\mathbf{u})$$

By using Equation (8) then, $E(\tilde{\beta}) = \beta$

3.3 Variance-Covariance Matrix for GLS Estimators

In this section, the variance-covariance matrix for GLS estimators $\tilde{\beta}$ of parameters of AR (2) model with constant in case of dependent errors will be obtained according to the following Lemma:

Lemma (2): The variance-covariance matrix for GLS estimators $\tilde{\beta}$ will be as follows:

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix} = \frac{\sigma^2}{G}\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix},$$

where:

$\Delta, A, B, C, E, F, G, N_{11}, N_{12}, N_{13}, K_{11}, K_{22}, K_{33}, K_{12} = K_{21}, K_{32} = K_{23}$ and $K_{13} = K_{31}$

are defined as in Lemma (1).

Proof:

The variance-covariance matrix for GLS estimators, in general, is as follows:

$$V(\tilde{\beta}) = \sigma^2 (X'\Omega^{-1}X)^{-1} \tag{16}$$

Fox and Weisberg (2018)

Where for the model in Equation (3),

$$V(\tilde{\beta}) = \begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix},\ X = \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ \vdots & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix},$$

$$\Omega = \frac{1}{(1-\phi_1^2)} \begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \tag{17}$$

By substituting from Equation (17) in Equation (16) then the $V(\tilde{\beta})$ for the model in Equation (3) will be as follows:

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix} =$$

$$\sigma^2 \left\{ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ y_0 & y_1 & y_2 & \cdots & y_{T-1} \\ y_{-1} & y_0 & y_1 & \cdots & y_{T-2} \end{bmatrix} \left(\frac{1}{(1-\phi_1^2)} \begin{bmatrix} 1 & \phi_1 & \phi_1^2 & \cdots & \phi_1^{T-1} \\ \phi_1 & 1 & \phi_1 & \cdots & \phi_1^{T-2} \\ \phi_1^2 & \phi_1 & 1 & \cdots & \phi_1^{T-3} \\ \vdots & \vdots & \vdots & & \vdots \\ \phi_1^{T-1} & \phi_1^{T-2} & \phi_1^{T-3} & \cdots & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & y_0 & y_{-1} \\ 1 & y_1 & y_0 \\ 1 & y_2 & y_1 \\ 1 & \vdots & \vdots \\ 1 & y_{T-1} & y_{T-2} \end{bmatrix} \right\}^{-1}$$

Then,

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix} = \sigma^2 \begin{Bmatrix} \Delta & A & B \\ A & C & E \\ B & E & F \end{Bmatrix}^{-1}$$

$$\begin{bmatrix} V(\tilde{\alpha}) & Cov(\tilde{\alpha},\tilde{\rho}_1) & Cov(\tilde{\alpha},\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_1) & V(\tilde{\rho}_1) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) \\ Cov(\tilde{\alpha},\tilde{\rho}_2) & Cov(\tilde{\rho}_1,\tilde{\rho}_2) & V(\tilde{\rho}_2) \end{bmatrix} = \frac{\sigma^2}{G} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

4. The ML Estimation for AR (2) Model in Case of Dependent Errors

In this section, the ML estimators, unbiasedness of ML estimators and the variance-covariance matrix for ML estimators for the parameters of AR (2) model with constant in case of dependent errors will be derived.

Lemma (3): The ML estimators for the parameters of AR (2) with a constant model in case of dependent errors as in Equation (3) under the assumptions of the model will be as the GLS estimators as in Lemma (1).

Proof:

It is easy to verify that ML estimators of $\hat{\beta}$ will be the same as GLS estimators as in Equation (13). So, the ML estimators will be also unbiased.

Lemma (4): The variance-covariance matrix for ML estimators $V(\hat{\beta})$ will be the same as the variance-covariance matrix for GLS estimators as in Lemma (2), omitted the proof.

5. Simulation Study

In this section, a simulation study by using R package is used to obtain MSE, Thiel's U for GLS estimates of AR (2) with a constant model in case of dependent errors in two cases: the first case for bounded stationary time series and the second case for bounded uncompleted nonstationary time series, as follows:

5.1. Bounded Stationary Time Series

A simulation study by using R package is used to obtain MSE and Thiel’s U for GLS estimates of AR (2) model with constant (initial value of $\alpha = 0.1$) in case of dependent errors which obtained in Equation (15) for stationary time series by using initial values $\rho_1 = 0.95$ and $\rho_2 = 0.03$ for which the three conditions for stationarity in Equation (1) exists, when y_t is bounded time series with fixed bounds with lower bound at $\underline{b}$ and upper bound at $\bar{b}$, $y_t \in [\underline{b}, \bar{b}]$, and $\underline{b} = \underline{c}\, T^{1/2}[1-\phi_1]^{-1}$, $\bar{b} = \bar{c}\, T^{1/2}[1-\phi_1]^{-1}$, for four boundaries value $\bar{c} = -\underline{c} = 0.3, 0.5, 0.7$ and 0.9, in case of five samples size T = 30, 50, 100, 200 and 500 and in case of ten values for the coefficient of dependent errors $\phi_1 = \pm 0.5, \pm 0.4, \pm 0.3, \pm 0.2$ and ± 0.1 by 5000 replications and the following results can be discussed for the next five cases:

Case (1): T = 30

It can be noticed, from Table (1) given in the Appendix of tables, that for sample size T = 30 and for values of $\bar{c} = -\underline{c} = 0.9, 0.7, 0.5$ and 0.3, in general, the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, the values of Thiel's U are ascending with the descending values of ϕ_1, for sample size T = 30 and $\bar{c} = -\underline{c} = 0.5$ the values of MSE are the smallest values for all cases of $\bar{c} = -\underline{c}$ and for sample size T = 30 and for $\bar{c} = -\underline{c} = 0.9$ the values of Thiel's U are the smallest values for all cases of $\bar{c} = -\underline{c}$.

Case (2): T = 50

It can be noticed, from Table (1) given in the Appendix of tables, that for sample size T = 50 and for values of $\bar{c} = -\underline{c} = 0.9, 0.7, 0.5$ and 0.3, in general, the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, the values of Thiel's U are ascending with the descending values of ϕ_1, for sample size T = 50, $\bar{c}=-\underline{c}=0.3$ and for negative values of ϕ_1 the values of MSE are the smallest values for all cases of $\bar{c}=-\underline{c}$ and for sample size T = 50, $\bar{c}=-\underline{c}=0.9$ and for positive values of ϕ_1 the values of Thiel's U are the smallest values for all cases of $\bar{c}=-\underline{c}$.

Case (3): T = 100

It can be noticed, from Table (1) given in the Appendix of tables, that for sample size T = 100 and for values of $\bar{c}=-\underline{c}=0.9, 0.7, 0.5$ and 0.3, in general, the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, the values of Thiel's U are ascending with the descending values of ϕ_1, for sample size T = 100, $\bar{c}=-\underline{c}=0.3$ and negative values of ϕ_1 the values of MSE are the smallest values for all cases of $\bar{c}=-\underline{c}$, for sample size T = 100, $\bar{c}=-\underline{c}=0.9$ and positive values of ϕ_1 the values of Thiel's U are the smallest values for all cases of $\bar{c}=-\underline{c}$.

Case (4): T = 200

It can be noticed, from Table (1) given in the Appendix of tables, that for sample size T = 200 and for values of $\bar{c}=-\underline{c}=0.9, 0.7, 0.5$ and 0.3, in general, the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, the values of Thiel's U are ascending with the descending values of ϕ_1, for sample size T = 200, $\bar{c}=-\underline{c}=0.3$ and negative values of ϕ_1 the values of MSE are the smallest values for all cases of $\bar{c}=-\underline{c}$ and for sample size T = 200 and for $\bar{c}=-\underline{c}=0.9$ and for positive values of ϕ_1 the values of Thiel's U are the smallest values for all cases of $\bar{c}=-\underline{c}$.

Case (5): T = 500

It can be noticed, from Table (1) given in the Appendix of tables, that for sample size T = 500 and for values of $\bar{c}=-\underline{c}=0.9, 0.7, 0.5$ and 0.3, in general, the values of MSE for positive values of ϕ_1 are more than the values of MSE for negative values of ϕ_1, the values of Thiel's U for positive values of ϕ_1 are less than the values of Thiel's U for negative values of ϕ_1, the values of MSE fluctuated with the descending values of ϕ_1, the values of Thiel's U are ascending with the descending values of ϕ_1, for sample size T = 500, $\bar{c}=-\underline{c}=0.3$ and negative values of ϕ_1 the values of MSE are the smallest values for all cases of $\bar{c}=-\underline{c}$ and for sample size T = 500, $\bar{c}=-\underline{c}=0.9$ and positive values of ϕ_1 the values of Thiel's U are the smallest values for all cases of $\bar{c}=-\underline{c}$.

5.2. Bounded Uncompleted Nonstationary Time Series

A simulation study by using R package is used to obtain MSE and Thiel's U for GLS estimators of AR (2) model with constant in case of dependent errors which obtained in Equation (15) for uncompleted nonstationary time series by using five pairs of initial values for ρ_1 and ρ_2 which one or two conditions of the three conditions for stationarity in Equation (1) exist so the nonstationary is uncompleted in various ways, when y_t is bounded time series with fixed bounds with lower bound at $\underline{b}$ and upper bound at $\bar{b}$, $y_t \in [\underline{b}, \bar{b}]$, and $\underline{b} = \underline{c}\ T^{1/2}[1-\phi_1]^{-1}$, $\bar{b} = \bar{c}\ T^{1/2}[1-\phi_1]^{-1}$, for four boundaries value $\bar{c} = -\underline{c} = 0.3, 0.5, 0.7$ and 0.9, in case of five samples size T = 30, 50, 100, 200 and 500 and ten values for the coefficient of dependent errors $\phi_1 = \pm 0.5, \pm 0.4, \pm 0.3, \pm 0.2$ and ± 0.1 by 5000 replications and the results can be discussed for the next five cases:

Case 1: The First Condition Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = -2$ and $\rho_2 = 1.1$ which make the first condition of stationarity conditions for AR (2) model in Equation (1) exists. It can be noticed, from Table (2) given in the Appendix of tables, that for all T and for all values of $\bar{c} = -\underline{c}$, MSE for positive values of ϕ_1 is greater than the corresponding values of negative values of ϕ_1. For all T and all values of $\bar{c} = -\underline{c}$, MSE is descending as ϕ_1 descending. For all T and for all values of $\bar{c} = -\underline{c}$ the values of Thiel's U are very closed to each other for the whole range of ϕ_1.

Case 2: The Second Condition Exists

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1 = 2.3$ and $\rho_2 = 1.2$ which make the second condition of stationarity conditions for AR (2) model in Equation (1) exists. It can be noticed, from Table (3) given in the Appendix of tables, that for all T and for all values of $\bar{c} = -\underline{c}$, MSE for positive values of ϕ_1 is greater than the corresponding values of negative values of ϕ_1. For all T and all values of $\bar{c}=-\underline{c}$, MSE, in general, is descending as ϕ_1 descending. For all T and all values of $\bar{c}=-\underline{c}$ the values of Thiel's U are very closed to each other for the whole range of ϕ_1.

Case 3: The First and the Second Conditions Exist

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1=1.3$ and $\rho_2=-1.5$ which make the first and the second conditions of stationarity conditions for AR (2) model in Equation (1) exists. It can be noticed, from Table (4) given in the Appendix of tables, that for all T and all values of $\bar{c}=-\underline{c}$, both MSE and Thiel's U are increasing as ϕ_1 increase.

Case 4: The First and the Third Conditions Exist

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1=-1.7$ and $\rho_2=0.8$ which make the first and the third conditions of stationarity conditions for AR (2) model in Equation (1) exists. It can be noticed, from Table (5) given in the Appendix of tables, that as ϕ_1 increases, MSE is increasing for all T and all values of $\bar{c}=-\underline{c}$. The same way can be noticed to Thiel's U except for T = 30, Thiel's U is decreasing as ϕ_1 increases.

Case 5: The Second and the Third Conditions Exist

In this case, the initial values for ρ_1 and ρ_2 are $\rho_1=1.7$ and $\rho_2=0.1$ which make the second and the third conditions of stationarity conditions for AR (2) model in Equation (1) exists. It can be noticed, from Table (6) given in the Appendix of tables, for T = 30, 50 and for all values of $\bar{c}=-\underline{c}$, MSE and Thiel's U are fluctuated to the values of ϕ_1 while, for sample size T = 100, 200 and 500 and for all values of $\bar{c}=-\underline{c}$, MSE is increasing as are ϕ_1 increases, while to Thiel's U it can be said that it decreases as ϕ_1 increases.

6. Conclusion

For bounded stationary time series, MSE has fluctuated values especially for the small size of time series through the values of ϕ_1 and it can be noticed that it has the least values especially for negative values of ϕ_1. On the other hand, Thiel's U has an inverse relationship to the values of ϕ_1 . For bounded nonstationary time series, five cases have been presented under different conditions. It can be said that MSE is non-stable and sometimes has inflation values, especially for small sample sizes while Thiel's U can be considered as a good criterion for comparison whatever the sample sizes and the assumptions of the model.

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Appendix of Tables …… Table (1): MSE and Thiel's U for GLS Estimators of AR (2) Model for Bounded Stationary Time Series

	$\bar{c}=-\underline{c}=0.3$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	95.03	0.165	356.3	0.118	1.484	0.074	0.163	0.050	0.063	0.031
0.4	25.84	0.174	27.04	0.123	1.137	0.079	0.151	0.056	0.054	0.035
0.3	57.05	0.175	3.173	0.131	0.278	0.087	0.095	0.062	0.049	0.040
0.2	16.83	0.188	77.96	0.139	0.088	0.094	0.044	0.069	0.046	0.044
0.1	1.567	0.197	271.1	0.148	0.043	0.103	0.042	0.076	0.044	0.049
- 0.1	5.280	0.223	0.052	0.177	0.040	0.124	0.041	0.092	0.043	0.060
- 0.2	0.615	0.240	0.048	0.192	0.041	0.136	0.042	0.101	0.043	0.065
- 0.3	0.111	0.257	0.224	0.208	0.042	0.150	0.043	0.111	0.044	0.071
- 0.4	0.063	0.281	1.624	0.227	0.045	0.165	0.046	0.123	0.047	0.079
- 0.5	0.050	0.303	0.048	0.252	0.049	0.183	0.050	0.134	0.051	0.088

	$\bar{c}=-\underline{c}=0.5$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	65.974	0.154	12.076	0.112	38.881	0.069	0.916	0.045	0.172	0.028
0.4	18.455	0.159	163.99	0.113	1.624	0.072	0.252	0.049	0.136	0.031
0.3	73.087	0.160	17.006	0.119	6.365	0.078	0.128	0.054	0.121	0.035
0.2	10.465	0.170	23.405	0.125	1.272	0.084	0.106	0.060	0.112	0.038
0.1	41.616	0.183	1.669	0.134	0.526	0.092	0.158	0.066	0.106	0.042
- 0.1	2.781	0.201	0.137	0.155	0.099	0.110	0.097	0.080	0.102	0.051
- 0.2	12.916	0.215	0.141	0.168	0.098	0.121	0.099	0.088	0.103	0.056
- 0.3	0.220	0.235	0.301	0.184	0.102	0.133	0.103	0.097	0.107	0.062
- 0.4	1.534	0.257	0.158	0.202	0.109	0.148	0.110	0.108	0.113	0.069
- 0.5	0.140	0.280	0.126	0.227	0.120	0.165	0.121	0.121	0.124	0.077

	$\bar{c}=-\underline{c}=0.7$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	48.508	0.147	179.6	0.101	139.06	0.063	10.98	0.041	1.725	0.026
0.4	117.72	0.152	100.0	0.106	206.41	0.066	2.966	0.045	0.294	0.028
0.3	133.94	0.153	179.4	0.108	3960.8	0.070	0.749	0.048	0.241	0.031
0.2	284.19	0.158	7.209	0.116	2.131	0.075	0.767	0.053	0.214	0.034
0.1	49.755	0.162	15.44	0.121	1.274	0.081	0.190	0.058	0.201	0.038
- 0.1	8.439	0.184	0.327	0.140	0.630	0.097	0.180	0.069	0.191	0.045
- 0.2	10.33	0.193	0.390	0.153	0.187	0.106	0.184	0.076	0.193	0.050
- 0.3	2.200	0.211	0.866	0.165	0.203	0.117	0.191	0.085	0.199	0.055
- 0.4	0.423	0.231	0.228	0.181	0.204	0.130	0.205	0.094	0.212	0.061
- 0.5	0.275	0.248	0.258	0.203	0.225	0.146	0.227	0.106	0.234	0.069

	$\bar{c}=-\underline{c}=0.9$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	2356.3	0.136	308.8	0.097	153.13	0.060	9.557	0.038	1.783	0.023
0.4	2544.5	0.138	30.62	0.098	39.119	0.061	4.601	0.039	0.491	0.025
0.3	33.503	0.139	10.82	0.100	30.043	0.064	4.176	0.042	0.437	0.027
0.2	6077.4	0.143	38.78	0.104	24.468	0.067	0.860	0.046	0.362	0.030
0.1	461.82	0.150	18.85	0.107	1.114	0.071	0.311	0.050	0.335	0.032
- 0.1	87.822	0.160	1.834	0. 122	0.343	0.084	0.288	0.060	0.313	0.039
- 0.2	17.242	0.172	4.544	0.133	0.826	0.093	0.293	0.066	0.315	0.043
- 0.3	3.574	0.187	0.922	0.145	0.306	0.102	0.305	0.073	0.325	0.047
- 0.4	58.042	0.205	0.671	0.159	0.325	0.114	0.328	0.082	0.345	0.052
- 0.5	1.284	0.220	2.209	0.178	0.361	0.128	0.364	0.092	0.380	0.059

Table (2): Case 1: MSE and Thiel's U for GLS Estimators of AR (2) Model for Bounded Uncompleted Nonstationary Time Series.

	$\bar{c}=-\underline{c}=0.3$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	4.6604	0.8345	7.6291	0.9228	17.9855	0.9812	40.1476	0.9959	106.9454	0.9987
0.4	2.7515	0.8384	5.3425	0.9223	12.6106	0.9793	28.0278	0.9947	74.4377	0.9982
0.3	2.2504	0.8413	4.0078	0.9212	9.3440	0.9770	20.6820	0.9932	54.7918	0.9975
0.2	1.5560	0.8436	3.1189	0.9193	7.2077	0.9743	15.8980	0.9914	42.0209	0.9966
0.1	1.2689	0.8447	2.5041	0.9170	5.7376	0.9711	12.6110	0.9892	33.2562	0.9956
- 0.1	0.9154	0.8458	1.7393	0.9116	3.9072	0.9633	8.5166	0.9833	22.3417	0.9927
- 0.2	0.7961	0.8461	1.4963	0.9085	3.3189	0.9586	7.1952	0.9795	18.8135	0.9906
- 0.3	0.7156	0.8468	1.3126	0.9053	2.8662	0.9533	6.1715	0.9750	16.0722	0.9881
- 0.4	0.6600	0.8478	1.1749	0.9018	2.5147	0.9471	5.3667	0.9694	13.9043	0.9848
- 0.5	0.6275	0.8495	1.0760	0.8984	2.2429	0.9401	4.7292	0.9626	12.1669	0.9804
	$\bar{c}=-\underline{c}=0.5$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	27.787	0.8511	24.126	0.9379	57.7885	0.9881	132.583	0.9970	347.951	0.9990
0.4	10.194	0.8576	16.813	0.9383	40.5577	0.9866	92.5421	0.9961	242.198	0.9986
0.3	7.3760	0.8613	12.652	0.9373	30.0413	0.9847	68.2679	0.9950	178.276	0.9981
0.2	5.1933	0.8643	9.7266	0.9358	23.1658	0.9825	52.4558	0.9936	136.718	0.9975
0.1	4.2173	0.8659	7.7893	0.9340	18.4310	0.9799	41.5891	0.9918	108.191	0.9967
- 0.1	2.9522	0.8682	5.3831	0.9293	12.5264	0.9733	28.0454	0.9872	72.6583	0.9944
- 0.2	2.5709	0.8689	4.6137	0.9265	10.6233	0.9692	23.6692	0.9841	61.1665	0.9928
- 0.3	2.3848	0.8697	4.0284	0.9234	9.1542	0.9644	20.2744	0.9804	52.2328	0.9908
- 0.4	2.0876	0.8706	3.5840	0.9201	8.0075	0.9588	17.5990	0.9757	45.1615	0.9882
- 0.5	1.9526	0.8720	3.2563	0.9164	7.1121	0.9521	15.4698	0.9698	39.4854	0.9846
	$\bar{c}=-\underline{c}=0.7$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	71.910	0.8906	64.394	0.9618	143.6172	0.9923	321.811	0.9978	857.357	0.9993
0.4	26.692	0.8966	45.191	0.9617	100.7422	0.9911	224.604	0.9971	596.634	0.9990
0.3	18.397	0.8995	32.536	0.9608	74.5657	0.9897	165.654	0.9963	439.055	0.9986
0.2	39.371	0.9018	25.230	0.9594	57.4463	0.9879	127.245	0.9952	336.610	0.9981
0.1	5899.9	0.9033	20.191	0.9574	45.6519	0.9859	100.841	0.9939	266.293	0.9976
- 0.1	7.4191	0.9034	13.856	0.9524	30.9265	0.9805	67.9140	0.9903	178.697	0.9959
- 0.2	6.3804	0.9028	11.815	0.9493	26.1689	0.9771	57.2633	0.9879	150.359	0.9948
- 0.3	5.6121	0.9021	10.247	0.9459	22.4852	0.9730	48.9908	0.9849	128.320	0.9933
- 0.4	5.0378	0.9014	9.0362	0.9420	19.5942	0.9681	42.4571	0.9811	110.862	0.9913
- 0.5	4.6282	0.9007	8.1112	0.9376	17.3136	0.9621	37.2363	0.9763	96.8264	0.9887
	$\bar{c}=-\underline{c}=0.9$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	146.20	0.9241	153.12	0.9771	330.8508	0.9957	699.225	0.9985	1888.38	0.9995
0.4	69.338	0.9278	106.68	0.9771	230.4835	0.9951	488.052	0.9980	1314.01	0.9993
0.3	43.075	0.9305	77.984	0.9766	170.2820	0.9941	359.889	0.9974	966.860	0.9990
0.2	137.59	0.9321	57.391	0.9754	131.1325	0.9928	276.356	0.9967	741.167	0.9987
0.1	24.026	0.9325	45.735	0.9736	104.1211	0.9912	218.920	0.9957	586.245	0.9983
- 0.1	18.223	0.9310	31.179\	0.9690	70.3474	0.9871	147.262	0.9932	393.231	0.9972
- 0.2	14.157	0.9294	26.461	0.9660	59.4096	0.9844	124.063	0.9914	330.770	0.9964
- 0.3	13.964	0.9276	22.816	0.9625	50.9170	0.9811	106.025	0.9892	282.174	0.9953
- 0.4	10.924	0.9256	19.969	0.9585	44.2203	0.9771	91.7518	0.9864	243.650	0.9939
- 0.5	9.9045	0.9234	17.746	0.9538	38.8896	0.9720	80.3073	0.9826	212.640	0.9921

Table (3): Case 2: MSE and Thiel's U for GLS Estimators of AR (2) Model for Bounded Uncompleted Nonstationary Time Series.

	$\bar{c}=-\underline{c}=0.3$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	423.638	0.5087	350.3571	0.5276	77.4829	0.5422	94.8204	0.5485	196.211	0.5526
0.4	2922.51	0.5109	179.8456	0.5279	3853372	0.5412	81.6940	0.5482	135.958	0.5523
0.3	22.0869	0.5104	67308.60	0.5270	34.3120	0.5407	41.4411	0.5478	99.9890	0.5522
0.2	16.3232	0.5110	25.7777	0.5251	22.1273	0.5399	31.0894	0.5473	76.6214	0.5521
0.1	44.8257	0.5122	32.2150	0.5256	12.5224	0.5389	24.5674	0.5470	60.5878	0.5520
- 0.1	8.3044	0.5091	4.5174	0.5235	8.2599	0.5380	16.4845	0.5466	40.6156	0.5519
- 0.2	3.1661	0.5087	3.5359	0.5228	6.9529	0.5379	13.8715	0.5465	34.1523	0.5518
- 0.3	8.0441	0.5088	3.1526	0.5230	5.9417	0.5379	11.8393	0.5465	29.1226	0.5518
- 0.4	1.5885	0.5102	4.6123	0.5239	5.1434	0.5383	10.2295	0.5466	25.1337	0.5519
- 0.5	13.2065	0.5132	2.3004	0.5255	4.5044	0.5390	8.9354	0.5469	21.9199	0.5520

	$\bar{c}=-\underline{c}=0.5$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	551.045	0.5083	13056.28	0.5272	683.763	0.5421	268.813	0.5492	644.651	0.5526
0.4	28550.1	0.5104	437.0074	0.5284	115.709	0.5417	195.159	0.5486	438.816	0.5524
0.3	238.987	0.5094	13778.57	0.5282	65.2628	0.5407	149.559	0.5480	322.718	0.5522
0.2	69782.1	0.5108	912.1559	0.5266	52.5709	0.5401	106.654	0.5476	247.199	0.5521
0.1	471.214	0.5106	273.9795	0.5254	38.7766	0.5396	79.7399	0.5472	195.438	0.5520
- 0.1	12.3923	0.5086	17.2481	0.5237	25.8374	0.5385	53.4956	0.5468	130.974	0.5519
- 0.2	19.8982	0.5078	13.0436	0.5233	21.7436	0.5383	45.0072	0.5467	110.113	0.5518
- 0.3	13.1506	0.5078	9.2137	0.5230	18.5706	0.5382	38.4034	0.5467	93.8783	0.5518
- 0.4	4.8465	0.5084	7.8714	0.5236	16.0627	0.5384	33.1693	0.5468	81.0014	0.5518
- 0.5	4.2574	0.5109	6.9028	0.5249	14.0522	0.5390	28.9578	0.5470	70.6237	0.5519

	$\bar{c}=-\underline{c}=0.7$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	488.441	0.5082	1204.527	0.5300	3345.92	0.5428	764.714	0.5490	1814.81	0.5529
0.4	3287.71	0.5120	467.2807	0.5299	227.054	0.5427	1054.71	0.5486	1072.75	0.5526
0.3	2520.62	0.5135	4673.904	0.5301	304.476	0.5425	321.661	0.5484	788.825	0.5525
0.2	331.709	0.5149	142.4217	0.5292	174.300	0.5416	255.041	0.5479	604.378	0.5524
0.1	2200.87	0.5146	57.7601	0.5285	94.8244	0.5405	189.513	0.5476	477.843	0.5523
- 0.1	222.879	0.5118	72.7786	0.5268	62.1599	0.5395	127.057	0.5472	320.235	0.5522
- 0.2	26.9470	0.5108	35.1918	0.5259	52.2304	0.5392	106.881	0.5471	269.227	0.5521
- 0.3	20.0224	0.5103	22.0148	0.5253	44.5748	0.5391	91.1825	0.5471	229.524	0.5521
- 0.4	16.5346	0.5108	19.0392	0.5256	38.5242	0.5392	78.7343	0.5471	198.026	0.5521
- 0.5	10.1568	0.5122	16.6531	0.5264	33.6679	0.5396	68.7099	0.5472	172.632	0.5522

	$\bar{c}=-\underline{c}=0.9$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	5891.75	0.5091	22608.70	0.5301	2987.51	0.5446	1391.67	0.5496	3430.01	0.5533
0.4	4831.27	0.5106	968.7085	0.5298	308521	0.5444	937.450	0.5493	2339.03	0.5529
0.3	1198.10	0.5140	1111.284	0.5317	1063.30	0.5440	943.937	0.5493	1719.57	0.5528
0.2	1635.14	0.5173	482.8183	0.5319	401.723	0.5435	513.579	0.5487	1317.92	0.5527
0.1	5490.08	0.5165	482.1744	0.5309	223.579	0.5425	404.496	0.5484	1041.71	0.5527
- 0.1	240763	0.5137	68.5108	0.5278	136.417	0.5413	271.019	0.5480	697.984	0.5526
- 0.2	64.1734	0.5125	60.7706	0.5275	114.672	0.5410	227.940	0.5479	586.751	0.5525
- 0.3	48.3563	0.5126	47.7168	0.5269	97.8672	0.5409	194.416	0.5478	500.168	0.5525
- 0.4	29.7566	0.5124	41.0534	0.5267	84.5489	0.5409	167.825	0.5478	431.468	0.5525
- 0.5	21.1371	0.5127	35.9912	0.5273	73.8412	0.5411	146.396	0.5479	376.065	0.5525

Table (4): Case 3: MSE and Thiel's U for GLS Estimators of AR (2) Model for Bounded Uncompleted Nonstationary Time Series.

	$\bar{c} = -\underline{c} = 0.3$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	0.9481	0.6842	1.2526	0.7704	3.2635	0.8612	8.4033	0.9196	28.160	0.9636
0.4	0.5071	0.6805	0.9185	0.7612	2.3128	0.8491	5.8791	0.9098	19.601	0.9579
0.3	0.3966	0.6765	0.7057	0.7520	1.7286	0.8380	4.3477	0.9007	14.430	0.9525
0.2	0.3084	0.6719	0.5614	0.7430	1.3436	0.8277	3.3486	0.8922	11.069	0.9473
0.1	0.2588	0.6668	0.4588	0.7342	1.0762	0.8179	2.6604	0.8840	8.7609	0.9423
- 0.1	0.2588	0.6668	0.3256	0.7164	0.7380	0.7990	1.7991	0.8682	5.8835	0.9324
- 0.2	0.1687	0.6470	0.2807	0.7071	0.6270	0.7895	1.5189	0.8604	4.9512	0.9275
- 0.3	0.1498	0.6387	0.2450	0.6972	0.5398	0.7797	1.3001	0.8523	4.2249	0.9224
- 0.4	0.1342	0.6291	0.2161	0.6863	0.4701	0.7692	1.1259	0.8438	3.6481	0.9171
- 0.5	0.1212	0.6179	0.1924	0.6740	0.4134	0.7574	0.9850	0.8343	3.1824	0.9113
	$\bar{c} = -\underline{c} = 0.5$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	2.5285	0.6971	4.1335	0.7853	11.0754	0.8720	29.517	0.9265	95.6010	0.9674
0.4	1.5701	0.6962	3.0319	0.7777	7.8457	0.8615	20.640	0.9181	66.5522	0.9627
0.3	1.2391	0.6942	2.3253	0.7703	5.8594	0.8521	15.253	0.9104	48.9970	0.9582
0.2	1.0091	0.6914	1.8451	0.7631	4.5498	0.8435	11.738	0.9033	37.5821	0.9539
0.1	0.8409	0.6879	1.5036	0.7560	3.6400	0.8355	9.3170	0.8966	29.7439	0.9498
- 0.1	0.6150	0.6784	1.0599	0.7416	2.4895	0.8201	6.2878	0.8840	19.9706	0.9420
- 0.2	0.5365	0.6724	0.9106	0.7339	2.1118	0.8125	5.3028	0.8777	16.8038	0.9381
- 0.3	0.4731	0.6654	0.7919	0.7254	1.8153	0.8044	4.5340	0.8713	14.3367	0.9342
- 0.4	0.4211	0.6570	0.6959	0.7160	1.5783	0.7957	3.9222	0.8644	12.3772	0.9300
- 0.5	0.3780	0.6468	0.6172	0.7048	1.3857	0.7857	3.4275	0.8568	10.7949	0.9255
	$\bar{c} = -\underline{c} = 0.7$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	5.4466	0.7074	11.6533	0.8028	30.2284	0.8831	77.8573	0.9329	249.296	0.9713
0.4	4.0637	0.7086	8.4863	0.7956	21.3542	0.8737	54.4212	0.9256	173.482	0.9674
0.3	3.2050	0.7081	6.4670	0.7888	15.9037	0.8654	40.1925	0.9190	127.670	0.9637
0.2	2.5882	0.7064	5.1013	0.7824	12.3154	0.8579	30.9078	0.9130	97.8862	0.9603
0.1	2.1481	0.7039	4.1335	0.7762	9.8264	0.8509	24.5140	0.9074	77.4375	0.9570
- 0.1	1.5553	0.6964	2.8827	0.7638	6.6864	0.8379	16.5152	0.8969	51.9463	0.9509
- 0.2	1.3495	0.6914	2.4639	0.7573	5.6579	0.8314	13.9154	0.8918	43.6887	0.9478
- 0.3	1.1834	0.6855	2.1320	0.7502	4.8519	0.8247	11.8868	0.8866	37.2569	0.9448
- 0.4	1.0473	0.6783	1.8644	0.7423	4.2084	0.8175	10.2735	0.8811	32.1496	0.9416
- 0.5	0.9346	0.6693	1.6455	0.7329	3.6864	0.8092	8.9692	0.8749	28.0265	0.9381
	$\bar{c} = -\underline{c} = 0.9$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	33.962	0.7243	47.4211	0.8190	76.5290	0.8969	185.067	0.9413	590.018	0.9753
0.4	9.8532	0.7247	20.9422	0.8123	53.9669	0.8889	129.313	0.9353	410.556	0.9722
0.3	8.5296	0.7242	15.9039	0.8062	40.1133	0.8820	95.4550	0.9300	302.101	0.9693
0.2	6.1683	0.7224	12.4863	0.8003	30.9989	0.8757	73.3608	0.9251	231.587	0.9667
0.1	5.0825	0.7204	10.0727	0.7948	24.6824	0.8699	58.1475	0.9206	183.177	0.9642
- 0.1	3.6279	0.7141	6.9633	0.7840	16.7255	0.8591	39.1217	0.9123	122.834	0.9595
- 0.2	3.1268	0.7099	5.9267	0.7784	14.1238	0.8539	32.9413	0.9083	103.289	0.9573
- 0.3	2.7242	0.7049	5.1073	0.7725	12.0875	0.8485	28.1207	0.9043	88.0676	0.9550
- 0.4	2.3962	0.6989	4.4484	0.7659	10.4638	0.8428	24.2882	0.9000	75.9816	0.9526
- 0.5	2.1257	0.6914	3.9108	0.7582	9.1482	0.8364	21.1910	0.8954	66.2255	0.9501

Table (5): Case 4: MSE and Thiel's U for GLS Estimators of AR (2) Model for Bounded Uncompleted Nonstationary Time Series.

	$\bar{c}=-\underline{c}=0.3$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	392.193	0.8661	7.9692	0.9435	18.5562	0.9879	40.8991	0.9972	107.841	0.9991
0.4	50.0059	0.8694	5.6536	0.9432	13.0684	0.9863	28.5522	0.9963	75.0583	0.9987
0.3	2.6778	0.8717	4.1981	0.9419	9.6377	0.9843	21.0662	0.9951	55.2447	0.9981
0.2	1.6397	0.8728	3.2648	0.9397	7.4312	0.9820	16.1895	0.9936	42.3638	0.9975
0.1	1.3504	0.8730	2.6167	0.9370	5.9113	0.9792	12.8377	0.9918	33.5227	0.9967
- 0.1	0.9447	0.8712	1.8068	0.9305	4.0156	0.9721	8.6598	0.9870	22.5109	0.9943
- 0.2	0.8218	0.8699	1.5473	0.9267	3.4045	0.9677	7.3098	0.9838	18.9497	0.9927
- 0.3	0.7294	0.8688	1.3494	0.9226	2.9325	0.9626	6.2624	0.9799	16.1814	0.9906
- 0.4	0.6635	0.8678	1.1985	0.9182	2.5639	0.9567	5.4371	0.9751	13.9903	0.9879
- 0.5	0.6199	0.8674	1.0862	0.9135	2.2757	0.9498	4.7802	0.9690	12.2315	0.9843
	$\bar{c}=-\underline{c}=0.5$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	16.6963	0.8783	25.552	0.9553	59.3289	0.9926	134.662	0.9980	350.382	0.9993
0.4	9.5824	0.8844	17.502	0.9554	41.6665	0.9913	93.9955	0.9973	243.882	0.9990
0.3	16.0932	0.8884	13.066	0.9547	30.8604	0.9898	69.3339	0.9964	179.505	0.9986
0.2	5.4232	0.8903	10.152	0.9532	23.7894	0.9880	53.2654	0.9953	137.649	0.9981
0.1	5.2398	0.8914	8.1063	0.9512	18.9163	0.9858	42.2195	0.9940	108.916	0.9975
- 0.1	3.0604	0.8918	5.5763	0.9461	12.8309	0.9801	28.4455	0.9902	73.1191	0.9958
- 0.2	2.6460	0.8914	4.7618	0.9428	10.8650	0.9764	23.9907	0.9877	61.5384	0.9945
- 0.3	2.3328	0.8908	4.1375	0.9392	9.3433	0.9720	20.5312	0.9845	52.5321	0.9929
- 0.4	2.1077	0.8902	3.6574	0.9351	8.1502	0.9668	17.7999	0.9806	45.3987	0.9908
- 0.5	1.9419	0.8899	3.2940	0.9306	7.2107	0.9605	15.6189	0.9755	39.6657	0.9879
	$\bar{c}=-\underline{c}=0.7$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	726.310	0.9123	69.737	0.9734	146.799	0.9952	325.643	0.9986	861.979	0.9995
0.4	175.016	0.9178	45.647	0.9732	102.943	0.9943	227.282	0.9980	599.837	0.9993
0.3	698.408	0.9213	34.900	0.9726	76.1926	0.9931	167.620	0.9974	441.393	0.9990
0.2	15.9970	0.9225	29.539	0.9713	58.6870	0.9917	128.738	0.9965	338.382	0.9987
0.1	11.1964	0.9238	20.832	0.9695	46.6197	0.9900	102.005	0.9955	267.672	0.9982
- 0.1	7.6341	0.9233	14.255	0.9646	31.5392	0.9856	68.6543	0.9927	179.577	0.9970
- 0.2	6.5614	0.9220	12.126	0.9615	26.6591	0.9826	57.8595	0.9907	151.072	0.9961
- 0.3	5.7524	0.9205	10.483	0.9580	22.8736	0.9790	49.4688	0.9883	128.897	0.9949
- 0.4	5.1104	0.9188	9.2042	0.9540	19.8936	0.9747	42.8332	0.9852	111.323	0.9934
- 0.5	4.6455	0.9170	8.2127	0.9492	17.5294	0.9693	37.5186	0.9811	97.1823	0.9913
	$\bar{c}=-\underline{c}=0.9$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	1946.72	0.9407	186.74	0.9840	343.874	0.9973	705.247	0.9990	1895.21	0.9997
0.4	397.635	0.9444	110.67	0.9841	233.656	0.9968	492.617	0.9987	1318.74	0.9995
0.3	92.6417	0.9464	76.763	0.9839	172.951	0.9959	362.982	0.9982	970.310	0.9993
0.2	87.1593	0.9480	58.928	0.9827	133.100	0.9950	278.708	0.9976	743.781	0.9991
0.1	25.0989	0.9484	46.888	0.9814	105.661	0.9937	220.755	0.9969	588.280	0.9988
- 0.1	41.4462	0.9467	31.942	0.9773	71.3295	0.9904	148.434	0.9949	394.529	0.9979
- 0.2	16.1124	0.9450	27.046	0.9746	60.1995	0.9882	125.018	0.9935	331.822	0.9973
- 0.3	12.6585	0.9428	23.271	0.9714	51.5482	0.9854	106.789	0.9918	283.025	0.9965
- 0.4	11.1207	0.9403	20.311	0.9676	44.7135	0.9820	92.3586	0.9895	244.331	0.9954
- 0.5	9.9689	0.9375	17.980	0.9630	39.2548	0.9776	80.7705	0.9864	213.167	0.9939

Table (6): Case 5: MSE and Thiel's U for GLS Estimators of AR (2) Model for Bounded Uncompleted Nonstationary Time Series.

	$\bar{c}=-\underline{c}=0.3$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	62.2447	0.3070	90.962	0.2972	117.607	0.2904	146.941	0.2860	26.8865	0.2853
0.4	181.248	0.3040	24.64	0.2952	26.3725	0.2872	6.6863	0.2849	13.9756	0.2851
0.3	1664445	0.3028	37.724	0.2916	15.3291	0.2857	4.2877	0.2848	10.2753	0.2852
0.2	15.8018	0.3001	2.7934	0.2893	2.4654	0.2855	3.2916	0.2848	7.8786	0.2853
0.1	1335.75	0.2982	1.1802	0.2879	1.3591	0.2852	2.5449	0.2850	6.2353	0.2855
- 0.1	1.3311	0.2982	1.0416	0.2907	0.8828	0.2876	1.7168	0.2866	4.1906	0.2862
- 0.2	0.5458	0.3019	0.4396	0.2936	0.7455	0.2898	1.4509	0.2878	3.5303	0.2868
- 0.3	0.8584	0.3076	0.3544	0.2985	0.6435	0.2931	1.2455	0.2896	3.0177	0.2876
- 0.4	1.2998	0.3165	0.3066	0.3053	0.5650	0.2975	1.0844	0.2921	2.6128	0.2887
- 0.5	6.5487	0.3285	0.2781	0.3149	0.5046	0.3036	0.9574	0.2956	2.2891	0.2903
	$\bar{c}=-\underline{c}=0.5$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	609.6899	0.3071	124.11	0.2970	37449.5	0.2913	93.060	0.2867	65.008	0.2850
0.4	321.068	0.3049	66.107	0.2927	1280.86	0.2883	20.043	0.2852	44.985	0.2848
0.3	382.356	0.3014	7.7131	0.2896	92.4176	0.2855	13.701	0.2843	33.084	0.2848
0.2	57.8768	0.2955	34.558	0.2874	5.8339	0.2840	10.454	0.2842	25.357	0.2849
0.1	162.012	0.2943	10.921	0.2859	4.3956	0.2837	8.2315	0.2844	20.058	0.2850
- 0.1	18.3241	0.2939	175.16	0.2871	3.1483	0.2854	5.5434	0.2855	13.463	0.2855
- 0.2	1.4150	0.2959	9.4759	0.2899	2.2980	0.2871	4.6782	0.2865	11.332	0.2859
- 0.3	0.8538	0.3004	1.0675	0.2939	1.9773	0.2897	4.0086	0.2879	9.6776	0.2865
- 0.4	0.6013	0.3075	0.8987	0.2998	1.7295	0.2934	3.4823	0.2899	8.3699	0.2873
- 0.5	0.5374	0.3182	0.8987	0.2998	1.5377	0.2985	3.0655	0.2928	7.3226	0.2886
	$\bar{c}=-\underline{c}=0.7$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	544.75	0.3100	31058	0.2988	251.81	0.2906	16949	0.2858	160.24	0.2851
0.4	249.492	0.3056	928.70	0.2943	454.850	0.2871	53.866	0.2847	110.11	0.2849
0.3	402.116	0.3002	57.760	0.2903	132.476	0.2857	787.31	0.2840	80.958	0.2848
0.2	40.1925	0.2953	87.612	0.2870	13.8782	0.2836	25.338	0.2840	62.043	0.2849
0.1	64.7329	0.2922	22.003	0.2862	9.8293	0.2830	19.601	0.2839	49.073	0.2850
- 0.1	38.7894	0.2917	4.9810	0.2850	6.5857	0.2843	13.141	0.2847	32.930	0.2853
- 0.2	4.3473	0.2923	3.5878	0.2869	5.4941	0.2855	11.079	0.2855	27.713	0.2857
- 0.3	3.4093	0.2958	2.5121	0.2897	4.7118	0.2875	9.4816	0.2866	23.658	0.2861
- 0.4	1.7883	0.3008	2.1208	0.2942	4.1060	0.2904	8.2236	0.2881	20.449	0.2868
- 0.5	1.6631	0.3094	1.8944	0.3010	3.6330	0.2946	7.2228	0.2904	17.874	0.2877
	$\bar{c}=-\underline{c}=0.9$									
	T = 30		T = 50		T = 100		T = 200		T = 500	
ϕ_1	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U	MSE	Thiel's U
0.5	363.631	0.3053	1531.2	0.2956	810.440	0.2916	953.38	0.2869	347.04	0.2851
0.4	1268.23	0.3022	372.60	0.2908	155.118	0.2875	113.46	0.2849	239.90	0.2848
0.3	30882.2	0.2977	993.53	0.2892	66.0124	0.2854	171.04	0.2843	176.37	0.2848
0.2	1071.02	0.2937	149.84	0.2870	114.662	0.2843	70.878	0.2839	135.14	0.2848
0.1	224.857	0.2898	48.650	0.2840	257.740	0.2833	41.636	0.2836	106.87	0.2849
- 0.1	18.4343	0.2867	9.9624	0.2823	14.8093	0.2835	27.967	0.2842	71.672	0.2851
- 0.2	16.2645	0.2868	13.880	0.2837	12.0173	0.2843	23.561	0.2847	60.291	0.2853
- 0.3	4.0775	0.2888	6.1884	0.2860	10.2815	0.2857	20.142	0.2855	51.442	0.2857
- 0.4	10.5568	0.2939	5.1787	0.2895	8.9345	0.2879	17.443	0.2867	44.434	0.2861
- 0.5	3.6037	0.3007	4.0119	0.2944	7.8719	0.2910	15.288	0.2884	38.801	0.2868