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# Evaluation of the Elliptic Integrals

***M.A. Sharaf*[1] *and F.A.Alrawjih*[2]**
[1]*Department of Astronomy, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia*
[2]*Department of Mathematics, Faculty of Science, University of Dammam, Dammam Saudi Arabia*

**Abstract:**

*The present paper is devoted for establishing accurate computational algorithms for the incomplete and complete elliptic integrals (EI) of the first, second and third kind. For these goals, we first derived some properties of EI that could be used to check the validity and the accuracy of the algorithms; in addition, particular continued fraction expansion of the ratio of the complete elliptic integrals of the second and first kinds is also derived. Secondly, we established the trigonometric series expansions of EI, together with the recurrence formulae of their coefficients so as to facilitate the computations. Also, Gautschi's algorithm of the top-down continued fraction evaluation is described. Numerical applications are performed for: (a) the incomplete elliptic integrals using their trigonometric series expansions, (b) the complete elliptic integrals of the second kind from the complete elliptic integrals of the first kind using Gautschi's algorithm. Finally the numerical results were checked by two ways:*

- *i- by satisfying the conditions given by properties of EI.*
- *ii- by comparing their values with those list in slandered tables.*

*In this respect, the numerical results show excellent arguments with these ways, a fact which proves the validity, accuracy and the effeteness of our algorithms*

***Keywords:***
*Elliptic integrals, recursive computations algorithms, continued fraction, trigonometric series expansions.*

## 1. Introduction

It can be shown that, if R(x,y) is a rational function of x and y and g(x) is a polynomial in x of degree 3 or 4, then the integral

$$\int R\{x, \sqrt{g(x)}]\; dx,$$

can be expressed as a linear combination of terms, each of which is either an elementary function, or an elliptic integral of the first, second, or third kind. Accordingly, numerical evaluation of a few integrals permits precise evaluation of a broad class of integrals. On the other hand the elementary problem in analytical mechanics – the motion of the simple pendulum – cannot be accurately described without resorting to elliptic integrals. Moreover, elliptic integrals play an important role in many aspect of Astrodynamics, e.g. the gravitational attraction of a solid homogeneous ellipsoid upon an exterior particle is represented in terms of elliptic integrals. Again the period of rectilinear oscillation [1] of an infinitesimal mass along the straight line through the center of mass and perpendicular to the plane of rotation of the finite masses is a linear combination of complete elliptic integrals of the first, second, and third kinds. Also the planetary orbit determination in Einstein's gravitational theory, leads to an elliptic integral [2]. In fact there are many applications of the elliptic integral e.g. [3].

Due to their importance as mentioned briefly in the above, the present paper is devoted for establishing accurate computational algorithms for the incomplete and complete elliptic integrals (EI) of the first, second and third

## 2. Basic Formulations

### *2.1 Definitions*

The normal forms of the elliptic integrals F and E of the first and second kind, respecti are $$F(k,\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}=\int_0^{\sin\phi}\frac{dx}{\sqrt{(1-x^2)(1-x^2k^2)}},\qquad(1)$$

$$E(k,\phi)=\int_0^{\phi}\sqrt{1-k^2\sin^2\phi}\;d\phi=\int_0^{\sin\phi}\sqrt{\frac{1-x^2k^2}{1-x^2}}dx,\qquad(2)$$

which are functions of the amplitude $\phi$, where $0 < \phi \le \pi/2$ and of the modulus $k$, where $0 \le k \le 1$, (The symbol $m$, called the parameter, is also used in place of the square of the modulus, i.e. $m \equiv k^2$). The first form of these special functions is called Legendre's form. The alternate, obtained from the first by setting $x = \sin\phi$ is called Jacobi's form.

When the amplitude $\phi = \pi/2$, the integrals are then complete elliptic integrals and denoted by $K(k)$ and $E(k)$ or simply $K$ and $E$. Thus

$$F(k,\pi/2)\equiv K(k)\equiv K \quad\text{and}\quad E(k,\pi/2)\equiv E(k)\equiv E.\qquad(3)$$

Otherwise, they are referred to as incomplete elliptic integrals.

The elliptic integral of the third kind is

$$\Pi(\upsilon,k,\phi)=\int_0^{\phi}\frac{d\phi}{(1-\upsilon\sin^2\phi)\sqrt{1-k^2\sin^2\phi}}=\int_0^{\sin\phi}\frac{dx}{(1-\upsilon x^2)\sqrt{(1-x^2)(1-x^2k^2)}},\qquad(4)$$

where $\upsilon$ is the characteristic. The properties of the integral depend on the location of the characteristic.

***2.2 Some basic properties of the hypergeometric function***

In the subsequent analysis we need some basic properties of the hypergeometric function $y = F_1(\alpha,\beta,\gamma;x)$ which is defined as

$$F_1(\alpha,\beta,\gamma;x)=\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}\frac{x^k}{k!},\qquad(5)$$

where

$$(\eta_0)=1;\qquad(\eta)_k=\eta\,(\eta+1)\,(\eta+2)\dots(\eta+k-1).\qquad(6)$$

Among the properties of $y = F_1(\alpha,\beta,\gamma;x)$, we need the following basic three properties:

**1-**Gauss's differential equation for the hypergeometric function $y = F_1(\alpha,\beta,\gamma;x)$ is:

$$x(1-x)\frac{d^2y}{dx^2}+\left[\gamma-(\alpha+\beta+1)x\right]\frac{dy}{dx}-\alpha\,\beta\,y=0.\qquad(7)$$

**2-** Gauss's identity of the hypergeometric function $y = F_1(\alpha,\beta,\gamma;x)$ is:

$$\gamma(1-x)F_1(\alpha,\beta,\gamma;x)=\gamma F_1(\alpha-1,\beta,\gamma;x)-(\gamma-\beta)\,x\,F_1(\alpha,\beta,\gamma+1;x).\qquad(8)$$

**3-** The ratio $F_1(\alpha,\beta+1,\gamma+1;x)/F_1(\alpha,\beta,\gamma;x)$ is given as a continued fraction of the form

$$\frac{F_1(\alpha,\beta+1,\gamma+1;x)}{F_1(\alpha,\beta,\gamma;x)}=\cfrac{1}{1-\cfrac{\delta_1x}{1-\cfrac{\delta_2x}{1-\cfrac{\delta_3x}{1-\dots}}}},\qquad(9)$$

where

$$\delta_{2n+1}=\frac{(\alpha+n)(\gamma-\beta+n)}{(\gamma+2n)(\gamma+2n+1)},\qquad\delta_{2n}=\frac{(\beta+n)(\gamma-\alpha+n)}{(\gamma+2n-1)(\gamma+2n)}.\qquad(10)$$

***2.3 Some properties of the elliptic integrals***

In what follows some properties of the elliptic integrals will be listed, while their proofs are given in Appendix A. These properties could be used to check the validity and the accuracy of the computational algorithms which used for the valuations of IE, as

**1-** For fixed amplitude, we have

$$\phi\le F(k,\phi)\le\ln(\tan\phi+\sec\phi),\qquad(11)$$

$$\sin\phi \le E(k,\phi) \le \phi. \tag{12}$$

**2-** $F(k,\phi)$ and $E(k,\phi)$ are both odd functions of $\phi$, i.e.

$$F(k,-\phi) = -F(k,\phi), \tag{13}$$

$$E(k,-\phi) = -E(k,\phi). \tag{14}$$

**3-** $F(k,\phi)$ and $E(k,\phi)$ satisfy the identities

$$F(k, n\pi+\phi) = 2n\,K(k) + F(k,\phi), \tag{15}$$

$$E(k, n\pi+\phi) = 2n\,E(k) + E(k,\phi), \tag{16}$$

where $n$ is any positive integer.

**4-** The special case of the elliptic integral of the third kind for which the characteristic $\upsilon$ is equal to the parameter $m = k^2$ is given by

$$(1-k^2)\Pi(k^2,k,\phi) = E(k,\phi) - \frac{k^2 \sin 2\phi}{2\sqrt{1-k^2\sin^2\phi}}. \tag{17}$$

**5-** The complete elliptic integrals $K(m)$ and $E(m)$ of the first and second kind, satisfy the following differential equations

$$2m(1-m)\frac{dK}{dm} = E-(1-m)K, \tag{18}$$

$$2m\frac{dE}{dm} = E-K, \tag{19}$$

$$m(1-m)\frac{d^2K}{dm^2} + (1-2m)\frac{dK}{dm} - \frac{1}{4}K = 0, \tag{20}$$

$$m(1-m)\frac{d^2E}{dm^2} + (1-m)\frac{dE}{dm} + \frac{1}{4}E = 0. \tag{21}$$

**6-** The complete elliptic integrals $K(m)$ and $E(m)$ could be expressed in terms of the hypergeometric functions $F_1$ as

$$K(m) = \frac{\pi}{2}F_1\left(\frac{1}{2},\frac{1}{2},1;m\right), \tag{22}$$

$$E(m) = \frac{\pi}{2}F_1\left(-\frac{1}{2},\frac{1}{2},1;m\right), \tag{23}$$

where $m \equiv k^2$.

**7-** The complete elliptic integrals $K(m)$ and $E(m)$ satisfy the identity

$$(1-m)\frac{K(m)}{E(m)} = 1 - \frac{m}{2}\frac{F_1\left(\frac{1}{2},\frac{1}{2},2;m\right)}{F_1\left(\frac{1}{2},-\frac{1}{2},1;m\right)}, \tag{24}$$

**8-** The complete elliptic integral $E(m)$ of the second kind could be expressed in terms of the complete elliptic integral $K(m)$ of the first kind by the continued fraction expansion as

$$E(m) = \cfrac{(1-m)K(m)}{1-\cfrac{m}{2-\cfrac{3m}{4-\cfrac{m}{2-\cfrac{5m}{8-\cfrac{3m}{2-\cfrac{7m}{12-\cfrac{5m}{2-\ddots}}}}}}}}. \tag{25}$$

## 3. Trigonometric Series Expansions of the Incomplete Elliptic Integrals

In this section, the trigonometric series expansions of the elliptic integrals of the first second and third kind will be established.

### *3.1 The incomplete elliptic integral of the first kind*

Since

$$F(k,\phi)=\int_0^{\phi}(1-k^2\,sin^2\phi)^{-1/2}\,d\phi\,, \tag{26}$$

then by using the binomial theorem we get

$$F(k,\phi)=\sum_{n=0}^{\infty}(-1)^n\begin{pmatrix}-\frac{1}{2}\\ n\end{pmatrix}k^{2n}\,s_n\,, \tag{27}$$

where $s_n=\int_0^{\phi}sin^{2n}\phi\,d\phi$. Expressing $sin^{2n}\varphi$ as a series of cosine multiples of $\phi$, we get

$$s_n=2^{-2n}\left\{\begin{pmatrix}2n\\ n\end{pmatrix}\phi+\sum_{m=1}^{n}(-1)^m\begin{pmatrix}2n\\ n-m\end{pmatrix}\frac{sin2m\phi}{m}\right\}. \tag{28}$$

### *3.2 The incomplete elliptic integral of the second kind*

By the same way as in the above section we get

$$E(k,\phi)=\sum_{n=0}^{\infty}(-1)^n\begin{pmatrix}\frac{1}{2}\\ n\end{pmatrix}k^{2n}\,s_n\,, \tag{29}$$

where $s_n$ is given by Equation (28).

### *3.3 The incomplete elliptic integral of the third kind*

Recalling the definition of the elliptic integral of the third kind as

$$\Pi(\upsilon,k,\phi)=\int_0^{\phi}(1-\upsilon\sin^2\phi)^{-1}\,(1-k^2\sin^2\phi)^{-1/2}\,d\phi\,. \tag{30}$$

Expand each term of the integrand using the binomial theorem, perform the product of the resulting power series of $sin^{2q}\phi$, then expressing $sin^{2q}\phi$ as a series of cosine multiples of $\phi$, we get

$$\Pi(\upsilon,k,\phi)=\sum_{n=0}^{\infty}\upsilon^n\,s_n\sum_{m=0}^{n}(-1)^m\begin{pmatrix}-\frac{1}{2}\\ m\end{pmatrix}\left(\frac{k^2}{\upsilon}\right)^m. \tag{31}$$

## 4. Computational Developments

The computational developments for the EI will be considered through the following points.

### *4.1 Computation forms of the equations*

For the computational purposes the above infinite series should be truncated up to (N+1) terms. So, we have

$$F(k,\phi)=\sum_{n=0}^{N}T_n\,s_n\,, \tag{32}$$

$$F(k,\phi)=\sum_{n=0}^{N}H_n\,s_n\,, \tag{33}$$

$$\Pi(\upsilon,k,\phi)=\sum_{n=0}^{N}G_n\,s_n\,, \tag{34}$$

where

$$T_n = (-1)^n \binom{-\frac{1}{2}}{n}, \tag{35}$$

$$H_n = (-1)^n \binom{\frac{1}{2}}{n}, \tag{36}$$

$$G_n = \upsilon^n \sum_{m=0}^{n} T_m \left(\frac{k^2}{\upsilon}\right)^m . \tag{37}$$

***4.2 Recurrence formulae***

In what follows, some recurrence formulae for the coefficients will be established to facilitate their computations.

From equations (35) and (36) we get for the *T's* and *H's* coefficients the recurrence formulae

$$T_{n+1} = \frac{1}{2}(2n+1)T_n ; \quad T_0 = 1, \tag{38}$$

$$H_{n+1} = \frac{1}{2}(2n+1)H_n ; \quad H_0 = 1. \tag{39}$$

From Equation (37) we have

$$\begin{aligned} G_{n+1} &= \upsilon^{n+1} \sum_{m=0}^{n+1} T_m \left(\frac{k^2}{\upsilon}\right)^m \\ &= \upsilon^{n+1} \left\{ T_{n+1} \left(\frac{k^2}{\upsilon}\right)^{n+1} + \sum_{m=0}^{n} T_m \left(\frac{k^2}{\upsilon}\right)^m \right\} \\ &= \upsilon^{n+1} T_{n+1} \left(\frac{k^2}{\upsilon}\right)^{n+1} + \upsilon G_n , \end{aligned}$$

consequently

$$\frac{G_{n+1} - \upsilon G_n}{G_n - \upsilon G_{n-1}} = \upsilon^n \left(\frac{T_{n+1}}{T_n}\right)\left(\frac{k^2}{\upsilon}\right) \xrightarrow{U\sin g\, Equ.(38)} = \frac{1}{2}(2n+1)k^2 ,$$

from which we get the recurrence formula

$$G_{n+1} = \frac{1}{2} G_n \left\{ 2\upsilon + (2n+1)k^2 \right\} - \frac{1}{2} k^2 \upsilon (2n+1) G_{n-1} , \tag{40}$$

while its initial values are obtained from Equations (35) and (37) and we get

$$G_0 = 1 ; \quad G_1 = \upsilon + \frac{1}{2}k^2 . \tag{41}$$

Finally, a recurrence formula for the s function could be obtained as follows.
Write $s_n$ as

$$s_n = \int_0^{\phi} sin^{2n-1}\phi \ \ sin\phi \ \ d\phi ,$$

integrating by parts we deduce that

$$(2n-1)\,s_{n-1} - 2n\,s_n = cos\phi \ sin^{2n-1}\phi ,$$

also

$$(2n+1)\,s_n - 2(n+1)\,s_{n+1} = cos\phi \ sin^{2n+1}\phi .$$

From these two equations we get the recurrence formula

$$s_{n+1} = \frac{1}{2(n+1)}\left\{\left[2n + 2n\,sin^2\phi + 1\right]s_n - (2n-1)\,sin^2\phi\, s_{n-1}\right\}, \qquad (42)$$

while its initial values are

$$s_0 = \phi; \qquad s_1 = \frac{1}{2}\phi - \frac{1}{4}sin2\phi . \qquad (43)$$

***4.3 Continued fraction method***

In fact, continued fraction expansions are generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than the series.

***4.3.1 Top-Down Continued Fraction Evaluation***

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the $n^{th}$ convergent were accumulated separately with three-term recurrence formulae. The draw back of the first method is obviously, having to decide far down the fraction to being in order to ensure convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top-down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi [4] proposed very concise algorithm to evaluate continued fraction from the top-down and may be summarized as follows. If the continued fraction is written as

$$c = \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \ddots}}} ,$$

then initialize the following parameters

$$a_1 = 1, \quad b_1 = n_1/d_1, \quad c_1 = n_1/d_1;$$

and iterate ($k = 1, 2, \ldots$) according to

$$a_{k+1} = \frac{1}{1 + \left[\dfrac{n_{k+1}}{d_k\, d_{k+1}}\right] a_k}, \qquad b_{k+1} = \left[a_{k+1} - 1\right] b_k , \qquad c_{k+1} = c_k + b_{k+1} .$$

In the limit, the *c* sequence converges to the value of the continued fraction.

Continued fraction method was used in many problems in astrophysics [e.g. 5 & 6] as well as in the special functions of Astrodynamics [e.g. 7 & 8].

### *4.4 Numerical applications*

1. Consider the 30 random values of $\upsilon$, $\phi$ (radian) and $m$ $(\equiv k^2)$ listed in Table (1) of Appendix B. Numerical applications of the analytical formulations of subsections 4.1 and 4.2 yield for the incomplete elliptic integrals EI1, EI2 and EI3 of the first, second and third kind respectively the values listed in Table (2) of Appendix B.
2. The complete elliptic $E(m)$ $(m \equiv k^2)$ of the second kind could be computed in terms of the complete elliptic integral $K(m)$ of the first kind by the continued fraction expansion of Equation (25). For this purpose we used Gautschi's algorithm with

$n_1 = (1 - m)\, K$, $n_2 = -m$, $n_i$ ($i$ odd) $= -i\, m$, $n_i$ ($i$ even) $= -(i - 3)\, m$,

$d_1 = 1$, $d_j$ ($j$ odd) $= 2j - 2$, and $d_j$ ($j$ even) $= 2$,

so, we get the values of $E(m)$ for some m as listed in Table (3) of Appendix B.

Finally the numerical results were checked by two ways:

i- by satisfying the conditions given by properties of EI.

ii- by comparing their values with those list in slandered tables.

In this respect, the numerical results show excellent arguments with these ways, a fact which proves the validity, accuracy and the effeteness of our algorithms.

### *Appendix A*
### *The Proofs of the Formulae of Subsection 2.3*

*Formulae (11) & (12):*

Since $F(k,\phi)$ and $E(k,\phi)$ are continuous functions, then

$$Min[F(k,\phi)] \le F(k,\phi) \le Max[F(k,\phi)],$$

and

$$Min[E(k,\phi)] \le E(k,\phi) \le Max[E(k,\phi)].$$

From Equations (1) & (2) we have

$$Min[F(k,\phi] = \int_0^\phi d\phi = \phi\,,$$

$$Max[F(k,\phi] = \int_0^\phi \frac{d\phi}{\sqrt{1 - sin^2\phi}} = \int_0^\phi sec\phi\; d\phi = ln(tan\phi + sec\phi)\,;$$

$$Max[E(k,\phi] = \int_0^\phi d\phi = \phi\,,$$

$$Min[E(k,\phi] = \int_0^\phi \sqrt{1 - sin^2\phi}\; d\phi = sin\phi\,;$$

which prove the inequalities (11), (12).

*Formulae (13) & (14):*

Equations (13) and (14) follow directly from the definitions of $F(k,\phi)$ and $E(k,\phi)$.

*Formulae (15) & (16):*

We can write Equations (1) and (2) as

$$F(k,\phi = \int_0^{\pi/2} (1 - k^2\, sin^2\psi)^{-1/2}\; d\psi + \int_{\pi/2}^{\pi} (1 - k^2\, sin^2\psi)^{-1/2}\; d\psi - \int_\phi^\pi (1 - k^2\, sin^2\psi)^{-1/2}\; d\psi\,,$$

$$E(k,\phi = \int_0^{\pi/2} (1 - k^2\, sin^2\psi)^{1/2}\; d\psi + \int_{\pi/2}^{\pi} (1 - k^2\, sin^2\psi)^{1/2}\; d\psi - \int_\phi^\pi (1 - k^2\, sin^2\psi)^{1/2}\; d\psi\,.$$

In the second and third integrals, we let $\theta = \pi - \psi$, then $\phi = -\phi$, so we get by using Equations (13) and (14), that

$$F(k,\pi + \phi) = 2\, K(k) + F(k,\phi),$$
$$E(k,\pi + \phi) = 2\, E(k) + E(k,\phi).$$

These are Equations (15) and (16) which are true for $n = 1$.

In the above equations we let $\phi = \phi + \pi$, we get

$$F(k,2\pi + \phi) = 2\, K(k) + F(k,\pi + \phi) = 4\, K(k) + F(k,\phi),$$
$$E(k,2\pi + \phi) = 2\, E(k) + E(k,\pi + \phi) = 4\, E(k) + E(k,\phi).$$

These are Equations (15) and (16) which are true for $n = 2$.

Assume Equations (15) and (16) which are true for $n$, so

$$F(k, n\pi+\phi) = 2n\, K(k) + F(k,\phi),$$

$$E(k, n\pi+\phi) = 2n\, E(k) + E(k,\phi);$$

then let $\phi = \phi + \pi$, in these two equations we get

$$F\{k,(n+1)\pi+\phi\} = 2n\, K(k) + F(k,\pi+\phi) = 2(n+1)\, K(k) + F(k,\phi),$$

$$E\{k,(n+1)\pi+\phi\} = 2n\, E(k) + E(k,\pi+\phi) = 2(n+1)\, E(k) + E(k,\phi).$$

These are Equations (15) and (16) which are true for $n = n + 1$, since they are true for $n = 1$, $n = 2$, …, consequently they are for any $n$ positive integer.

*Formula (17):*

Let $$y = \sqrt{1-k^2\, sin^2\phi}\,,$$

then $$y' = \frac{-k^2\, sin2\phi}{2\,\sqrt{1-k^2\, sin^2\phi}}\,,$$

and $$\frac{dy^2}{dx^2} = y'' = -T_1 - T_2\,, \qquad \text{(A-1)}$$

where $$T_1 = \frac{k^2\, cos2\phi}{\sqrt{1-k^2\, sin^2\phi}} = \frac{k^2}{\sqrt{1-k^2\, sin^2\phi}} + 2\sqrt{1-k^2\, sin^2\phi} - \frac{2}{\sqrt{1-k^2\, sin^2\phi}}\,,$$

$$T_2 = \frac{k^4\, sin^2 2\phi}{4(1-k^2\, sin^2\phi)^{3/2}} = \frac{-k^2\, cos^2\phi}{(1-k^2\, sin^2\phi)^{1/2}} + \frac{k^2\, cos^2\phi}{(1-k^2\, sin^2\phi)^{3/2}}\,,$$

that is $$T_2 = \frac{-k^2}{(1-k^2\, sin^2\phi)^{1/2}} - (1-k^2\, sin^2\phi)^{1/2} + \frac{2}{(1-k^2\, sin^2\phi)^{1/2}} + \frac{k^2-1}{(1-k^2\, sin^2\phi)^{3/2}}\,.$$

Using the above expressions for $T_1$ and $T_2$ into Equation (A-1), and then integrating the resulting equation, so we get

$$(1-k^2)\,\Pi(k^2,k,\phi) - E(k,\phi) = \frac{k^2\, sin2\phi}{2\sqrt{1-k^2\, sin^2\phi}}\,,$$

which is Equation (17).

*Formulae (18):*

Since $$K(m) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-m\, sin^2\phi}}\,,$$

differentiating with respect to $m$ we get

$$\frac{dK}{dm} = -\frac{1}{2m}K(m) + \frac{1}{2m}\Pi(m,m,\pi/2)\,,$$

then using Equation (17) with $\phi = \pi/2$, we get Equation (18).

*Formulae (19)* : Since $$E(m) = \int_0^{\pi/2} \sqrt{1-m\, sin^2\phi}\; d\phi\,,$$

differentiating with respect to $m$ we get

$$\frac{dE}{dm} = \frac{1}{2m}\int_0^{\pi/2} \frac{1-m\, sin^2\phi - 1}{\sqrt{1-m\, sin^2\phi}}\; d\phi = \frac{1}{2m}(E-K)\,,$$

then is Equation (19).

*Formulae (20):*

Differentiating Equation (18) and then using Equation (19) for *dE/dm* and Equation (18) for *(K – E)/2m* we get Equation (20).

*Formulae (21):*

Differentiating Equation (19) and then using Equation (18) for *dK/dm* and Equation(19) for *K*, we get Equation (21).

*Formulae (22):*

Comparing Equation (20) with Gauss's Equation (7) for hypergeometric function we get, when

$$y \equiv K,\ x \equiv m,\ \gamma = 1 \Rightarrow \alpha + \beta = 1,\ \alpha\beta = ¼ \Rightarrow \alpha = ½,\ \beta = ½,$$

consequently $K(m) = C_1\, F_1\left(\frac{1}{2},\frac{1}{2},1;m\right).$

Satisfies Gauss's equation. Since K(0) = π/2 and $F_1\left(\frac{1}{2},\frac{1}{2},1;0\right)=1$, then $C_1 = \pi/2$ and Equation (22) is proved.

*Formulae (23):*

Comparing Equation (21) with Gauss's Equation (7) for hypergeometric function we get when

$y \equiv E,\ x \equiv m,\ \gamma = 1 \Rightarrow \alpha + \beta = 0,\ \alpha\beta = -\tfrac{1}{4} \Rightarrow \alpha = -\tfrac{1}{2},\ \beta = \tfrac{1}{2},$

consequently

$$K(m) = C_2\, F_1\left(-\frac{1}{2},\frac{1}{2},1;m\right).$$

That satisfies Gauss's equation. Since $E(0) = \pi/2$ and $F_1\left(-\frac{1}{2},\frac{1}{2},1;0\right)=1$, then $C_2 = \pi/2$ and Equation (23) is proved.

*Formulae (24):*

Let $\gamma = 1,\ \alpha = \tfrac{1}{2},\ \beta = \tfrac{1}{2}$ and $x \equiv m$

in Equation (8), and then divide both sides of the resulting equation by $F_1\left(-\frac{1}{2},\frac{1}{2},1;m\right)$ and using Equations (22) & (23), then Equation (24) follows immediately.

*Formulae (25):*

Let $\gamma = 1,\ \alpha = \tfrac{1}{2},\ \beta = -\tfrac{1}{2}$ and $x \equiv m$

in Equation (9) an using Equation (10) for the $\delta$'s we get

$$Q = \frac{F_1\left(\frac{1}{2},\frac{1}{2},2;m\right)}{F_1\left(\frac{1}{2},-\frac{1}{2},1;m\right)} = \cfrac{1}{1-\cfrac{3m/8}{1-\cfrac{m/8}{1-\cfrac{5m/16}{1-\ddots}}}}.$$

Using Equation (24) for *E(m)*, Equation (25) yields directly.

***Appendix B***
***Numerical Results***

Table (1): Some values of $\upsilon$, $\phi$ (radian) and $m$ ($\equiv k^2$).

| **No.** | $\upsilon$ | $\phi$ | ***m*** | **No.** | $\upsilon$ | $\phi$ | ***m*** |
|---|---|---|---|---|---|---|---|
| 1 | 0.886722 | 0.866325 | 0.594413 | 16 | 0.371839 | 1.077740 | 0.617374 |
| 2 | 0.619108 | 0.918401 | 0.427062 | 17 | 0.152632 | 0.203947 | 0.264464 |
| 3 | 0.355766 | 1.019590 | 0.539380 | 18 | 0.196058 | 0.852554 | 0.619489 |
| 4 | 0.402134 | 0.978971 | 0.218435 | 19 | 0.453673 | 0.955426 | 0.805967 |
| 5 | 0.104672 | 0.422304 | 0.501638 | 20 | 0.867869 | 0.386516 | 0.316444 |
| 6 | 0.722051 | 1.058850 | 0.310770 | 21 | 0.235311 | 0.883901 | 0.431224 |
| 7 | 0.648157 | 0.318158 | 0.682142 | 22 | 0.321819 | 0.291290 | 0.892084 |
| 8 | 0.712991 | 0.414826 | 0.518002 | 23 | 0.661250 | 0.757539 | 0.299885 |
| 9 | 0.375125 | 0.361961 | 0.527260 | 24 | 0.810326 | 0.550609 | 0.625692 |
| 10 | 0.141408 | 0.685071 | 0.388569 | 25 | 0.248589 | 0.117576 | 0.736812 |
| 11 | 0.422924 | 0.245789 | 0.615998 | 26 | 0.602711 | 0.572889 | 0.665022 |
| 12 | 0.797856 | 0.282745 | 0.739104 | 27 | 0.405484 | 0.937949 | 0.660505 |
| 13 | 0.610132 | 0.975660 | 0.682898 | 28 | 0.508192 | 0.771637 | 0.607257 |
| 14 | 0.510288 | 0.729252 | 0.206240 | 29 | 0.243918 | 0.895272 | 0.435174 |
| 15 | 0.143886 | 1.159740 | 0.817152 | 30 | 0.780660 | 0.714040 | 0.554252 |

Table (2): The values of the three incomplete integrals for the input values of Table (1).

| **No.** | ***EI1*** | ***EI2*** | ***EI3*** | **No.** | ***EI1*** | ***EI2*** | ***EI3*** |
|---|---|---|---|---|---|---|---|
| 1 | 0.933299 | 0.807440 | 1.224670 | 16 | 1.214580 | 0.965883 | 1.401850 |
| 2 | 0.972148 | 0.869631 | 1.174260 | 17 | 0.204313 | 0.203569 | 0.204744 |
| 3 | 1.116500 | 0.936493 | 1.260500 | 18 | 0.919637 | 0.793721 | 0.962597 |
| 4 | 1.009400 | 0.950075 | 1.143340 | 19 | 1.091690 | 1.847205 | 1.265270 |
| 5 | 0.428625 | 0.416148 | 0.431257 | 20 | 0.389534 | 0.383540 | 0.407324 |
| 6 | 1.114580 | 1.007690 | 1.520750 | 21 | 0.932513 | 0.839556 | 0.988356 |
| 7 | 0.321859 | 0.314533 | 0.329100 | 22 | 0.295029 | 0.287636 | 0.297739 |
| 8 | 0.421020 | 0.408793 | 0.439145 | 23 | 0.778307 | 0.737722 | 0.890671 |
| 9 | 0.366147 | 0.357859 | 0.372217 | 24 | 0.568427 | 0.533769 | 0.620661 |
| 10 | 0.705534 | 0.665635 | 0.720589 | 25 | 0.117776 | 0.117377 | 0.117911 |
| 11 | 0.247321 | 0.244275 | 0.249444 | 26 | 0.594392 | 0.552746 | 0.636896 |
| 12 | 0.285559 | 0.279981 | 0.291818 | 27 | 1.035650 | 0.855554 | 1.170130 |
| 13 | 1.091190 | 0.880298 | 1.356850 | 28 | 0.819890 | 0.728231 | 1.911368 |
| 14 | 0.741775 | 0.717093 | 0.813214 | 29 | 0.946269 | 0.848894 | 1.006710 |
| 15 | 1.426770 | 0.971483 | 1.514440 | 30 | 0.748335 | 0.682450 | 0.867226 |

Table (3): *E(m)* for some *m* as computed from Gautschi's algorithm.

| *m* | *E(m)* | *m* | *E(m)* |
|---|---|---|---|
| 0.807939 | 1.17311 | 0.168876 | 1.50222 |
| 0.863506 | 1.13337 | 0.226277 | 1.47776 |
| 0.651545 | 1.26983 | 0.539360 | 1.33055 |
| 0.458234 | 1.37138 | 0.408748 | 1.39525 |
| 0.314211 | 1.43898 | 0.714159 | 1.23317 |
| 0.565508 | 1.31689 | 0.581473 | 1.30841 |
| 0.314999 | 1.43862 | 0.888529 | 1.11403 |
| 0.274959 | 1.45650 | 0.691199 | 1.24689 |
| 0.939370 | 1.10154 | 0.910660 | 1.09592 |
| 0.983505 | 1.02436 | 0.773534 | 1.19600 |
| 0.707370 | 1.23726 | 0.125024 | 1.52048 |
| 0.419361 | 1.39019 | 0.139654 | 1.51443 |
| 0.958540 | 1.05199 | 0.552426 | 1.32376 |
| 0.861890 | 1.13458 | 0.559323 | 1.32014 |
| 0.938420 | 1.07145 | 0.559516 | 1.32004 |

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