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AN INTRODUCTION TO INFINITE SERIES

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1. INFINITE SERIES

1.1 INTRODUCTION TO INFINITE SERIES:

Perhaps the most widely used technique in the physicist's toolbox is the use of infinite series (i.e. sums consisting formally of an infinite number of terms) to represent functions, to bring them to forms facilitating further analysis, or even as a prelude to numerical evaluation. The acquisition of skills in creating and manipulating series expansions is therefore an absolutely essential part of the training of one who seeks competence in the mathematical methods of physics. An important part of this skill set is the ability to recognize the functions represented by commonly encountered expansions, and it is also of importance to understand issues related to the convergence of infinite series.

1.2. FUNDAMENTAL CONCEPTS

The usual way of assigning a meaning to the sum of an infinite number of terms is by introducing the notion of partial sums. If we have an infinite sequence of terms $u_1, u_2, u_3 , u_4 , u_5, ..,$ we defined the *i*- th partial sum as

$$S_i = \sum_{n=1}^{i} u_n \qquad \text{(i)}$$

This is a finite summation and offers no difficulties. If the partial sums S_i converge to a finite limi as $i \rightarrow \infty$

$$\lim_{i\to\infty} S_i = S \qquad \text{(ii)}$$

Examples:

(1) $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots$

(2) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

1.3 CONVERGENT AND DIVERGENT SERIES

The infinite series $\sum_{n=1}^{i} u_n$ is said to be convergent and to have the value S. Not

define the infinite series as equal to S and that a necessary condition

for convergence to

a limit is that $\lim_{i\to\infty} u_n$

$= 0$, This conditon, however, is not sufficient to guarantee

convergence.

Sometimes it is convenient to apply the condition in (ii) in a form

called the Cauchy criterion, $\forall\, \varepsilon > 0, \exists\, N \in \aleph, s.t. \left|s_j - s_i\right| <$

$\varepsilon, \forall\, i, j > N$. This means that the partial sum must cluster together as we move far out in the sequence.

Some series diverge, meaning that the, sequence of partial sums approaches $\pm\infty$; others may have partial sums that oscillate between two values, as for example

$$\sum_{n=1}^{\infty} u_n = 1 - 1 + 1 - 1 + 1 - \cdots - (-1)^n + \cdots$$

This series does not converge to a limit, and can be called oscillatory, Often the terms divergent is extended to include oscillatory series as well. It is important to be able to determine whether, or under what conditions, a series we would like to use is convergent.

<u>Examples:</u>

(1) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \; divergent.$

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} \; convergent.$

(3) $\sum_{n=1}^{\infty} 1 \; divergent.$

(4) $\sum_{n=1}^{\infty} \frac{1}{2^n + 5n + 6} \; divergent.$

1.4 THE GEOMETRIC SERIES

The geometric series, starting with $u_0 = 1$ and with a ratio of successive terms r = $\frac{u_{n+1}}{u_n}$, has the form

1 + r+ r^2 + r^3 + …….. + r^{n-1} + ……

Its n- th partial sum S_n (that of the first n terms) is

$$S_n = \frac{1- r^n}{1-r}$$

Restricting attention to $|r| \prec 1$, so that for large n, r^n approaches zero, S_n possesses the limit

$$\lim_{n\to\infty} S_n = \frac{1}{1-r}$$

Showing that for $|r| \prec 1$, the geometric series converges. It clearly diverges (or is oscillatory) for $|r| \geq 1$, as the individual terms do not then approach zero at large *n*.

Examples:

(1) $\sum_{n=0}^{\infty} \frac{5}{2^n}, \quad r = \frac{1}{2} \prec 1 \; Convergent$

(2) $\sum_{n=0}^{\infty} (3)^{2n} (4)^{1-n}$, $r = \frac{9}{4} \geq 1$ *divergent*

(3) $\sum_{n=0}^{\infty} x^{n}$, *if* $|r| = |x| < 1$ *convergent*

if $|r| = |x| \geq 1$ *divergent* , $S = \frac{1}{1-x}$

* THE HARMONIC SERIES

We consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \ \ldots .$$

The terms approach zero for large n, i.e. $\lim_{n\to\infty} \frac{1}{n}$

$= 0$, but this is not sufficient

to guarantee convergence. If we group the terms (without changing their order)as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{16}\right) + \ \ldots.$$

each pair of parentheses encloses *p* terms of the form

$$\frac{1}{p+1} + \frac{1}{p+2} + \ldots + \frac{1}{p+p} \succ \frac{p}{2p} = \frac{1}{2}$$

Forming partial sums by adding the parenthetical groups one by one, we obtain

$$s_1 = 1, \; s_2 = \frac{3}{2}, \quad s_3 \succ \frac{4}{2}, \quad s_4 \succ \frac{5}{2}, \ldots\ldots \; s_n \succ \frac{n+1}{2}$$

This show that $s_n \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent.

Therefore the harmonic series diverges.

1.5 COMPARISON TEST

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent.

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is convergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example:

$$\sum_{n=1}^{\infty} \frac{5}{2\,n^2 + 4n + 3}$$

$$a_n = \frac{5}{2\,n^2+4n+3} < b_n = \frac{5}{2\,n^2}$$

$$\sum b_n = \sum \frac{5}{2\,n^2} \text{ , convergent. So, } \sum a_n \text{ convergent}$$

1.6 THE LIMIT COMPARISON TEST

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$, where c is a finite number $c > 0$, then either both series converge or both diverge.

Example:

$$\sum_{n=1}^{\infty} \frac{5}{2^n - 1}$$

$$a_n = \frac{1}{2^n - 1} \qquad , \quad \text{take} \quad b_n = \frac{1}{2^n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0 \ , \ \sum b_n = \sum \left(\frac{1}{2}\right)^n \text{ convergence}$$

So, $\sum a_n$ is also convergence.

TEST FOE DIVERGENCE

$$\text{If} \lim_{n\to\infty} a_n \text{ does not exist or If} \lim_{n\to\infty} a_n \neq 0, \text{then the series} \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

Examples:

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

But as before $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent . (Harmonic series).

$$(2)\sum_{n=1}^{\infty}\frac{n}{n+1} \quad \text{divergent,}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n+1} \neq 0$$

1.7 ALTERNATING SERIES

An alternating series is a series whose terms are alternately positive and negative.

* Alternating series test:

If the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n-1} \quad b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \cdots b_n$$

> 0 satisfies

(i)$b_{n+1} \leq b_n$ for all n

(ii)$\lim\limits_{n\to\infty} b_n = 0$

Then the series is convergent.

Example:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n+1} = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \cdots$$

(i)$a_n = \frac{1}{n}$, $a_n + 1 \prec a_n$

(ii)$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n} = 0$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ is convergent.}$$

* ABSOLUTE AND CONDITIONAL CONVERGENCE

An infinite series is absolutely convergent if the absolute values of its terms form a convergent series. If it converges, but not absolutely, it is termed conditionally convergent.

If a series $\sum a_n$ is absolutely convergent, then it is convergent.

Example:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

It is absolutely convergent and convergent.

* RATIO TEST

An alternate statement of the ratio test is in the form of a limit:

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\begin{cases} < \mathrm{I}, \text{then } \sum_{n=1}^{\infty} a_n \text{ is convergent} \\ > \text{or} = \infty, \text{then } \sum_{n=1}^{\infty} a_n \text{ is divergent} \\ = \mathrm{I}, \text{indeterminate.} \end{cases}$$

Example:

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

So, divergent.

* THE ROOT TEST

(i)If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely

convergent (and therefore convergent).

(ii)If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, $or \lim_{n\to\infty} \sqrt[n]{|a_n|}$

$$= \infty \text{ then the series } \sum_{n=1}^{\infty} a_n \text{ is divergent.}$$

(iii)If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root test is inconclusive.

Example:

$$\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$$

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left[\left(\frac{2n+3}{3n+2}\right)^n\right]^{\frac{1}{n}} = \frac{2}{3} \prec 1$$

$$\therefore \sum_{n=1}^{\infty} a_n \quad \text{convergent.}$$

* THE INTEGRAL TEST

Suppose f is continuous, positive, decreasing function on $[1,\infty)$

and let $a_n = f\,(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and

only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent.

In other words:

(i)if $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) if $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example:

$a_n \frac{1}{n^2+1}$, take $f(x) = \frac{1}{x^2+1}$

$$\int_1^{\infty} f(x)dx = \frac{\pi}{4} \text{ convergent}$$

So, $\sum_{n=1}^{\infty} a_n$ is convergent.

* p-SERIES

The $p-$series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Examples:

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $p = 2 > 1$, convergent.

(2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$, $p = \frac{1}{5} < 1$, divergent.

2. OPERATIONS ON SERIES THEOREM:

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series

$\sum c_n$ (where c is a constant), $\sum (a_n$

$+ b_n)$ and $\sum (a_n - b_n)$, and

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

2.1 POWER SERIES

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \dots$$

Where x is a variable and the c_n 's are constants called the coefficients of the series.

The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \dots + c_n x^n + \dots$$

Whose domain is the set of all *x* for which the series converges.

Notice that *f* resembles a polynomial. The only difference is that *f* has infinitely many terms.

If we take $c_n = 1$ for all *n*, the power series becomes the geometric series.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \dots + x^n + \cdots$$

Which converges when $-1 < x < 1$ and diverges when $|a| \geq 1$.

More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

$$= c_0 + c_1 (x-a) + c_2 (x-a)^2$$

$$+ c_3 (x-a)^4 \dots$$

is called a power series in (x – a) or a power series centered at a or a power series about a.

2.2 THEOREM:

For a given power series $\sum_{n=0}^{\infty} c_n (x - a)^n$, there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. (The number R is called the radius of convergence of the power series).

$$a - R < x < a + R$$

2.3 REPRESENTATIONS OF FUNCRIONS AS POWER SERIES

We know from Geometric series that

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \ldots \quad |x| < 1$$

when $a = 1$ and $r = x$

$$\frac{a}{1-x} = \frac{1}{1-x}$$

If we take $f(x) = \frac{1}{1-x}$, we can write

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + \ldots, \ |x| < 1$$

This called representation of $f(x)$ as power series.

3. TAYLOR AND MACLAURIN SERIES

3.1 THEOREM:

If f has a power series representation (expansion) at a,

that is,

$$\text{if} \qquad f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n back into the series, we see that

if f has a power series expansion at a, then it must be of the

following form:

$$c_n = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \qquad (*)$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

The series in Equation (*) is called the Taylor series of the

function f at a

For the special case $a = 0$, the Taylor series becomes

$$f\ (x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\ (a)}{n!}$$

$$= f(0) + \frac{f'\ (a)}{1!}\ x + \frac{f''\ (a)}{2!}\ x^2 + \ldots$$

This case arises frequently enough that it is given the special name Maclaurin series.

3.2 SOME IMPORTANT SERIES

There are a few series that arises so often all physicists should recognize them.

Here is a short list that is worth committing in memory.

$$\text{(i)}\ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \ldots - \infty < x < \infty$$

$$\text{(ii)}\ \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\ x^{2n+1}}{(2n+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots - \infty < x < \infty$$

$$(iii)\ \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots - \infty < x < \infty$$

$$(iv)\ \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

$$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots - \infty < x < \infty$$

$$(v)\ \cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots - \infty$$

$$< x < \infty$$

$$(vi)\ \frac{1}{1-x} = \sum_{n=0}^{\infty} 1 + x + x^2 + x^3 + x^4 + \ldots - 1 \leq x$$

$$< 1$$

(vii)In$(1+x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\, x^n}{n!} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \;\dots -1$$

$$< x \le 1$$

$$\text{(viii)}(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = \sum_{n=0}^{\infty} \frac{(p-n+1)_n}{n!} x^n \quad -1$$

$$< x < 1$$

3.3 THE BINOMIAL SERIES

If k is any real number $|x| < 1$, then

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k\,(k-1)}{2!} x^2$$

$$+ \frac{k\,(k-1)(k-2)}{3!} x^3 + \dots$$

The binomial series always converges when $|x| < 1$.

3.4 EXAMPLE:

Find the Maclaurin series for $f(x) = (1+\ x)^k$, where k is any

real number

Solution:

$f(x) = (1+x)^k$	$f(0) = 1$
$f'(x) = k(1+x)^{k-1}$	$f'(0) = k$
$f''(x) = k(k-1)(1+x)^{k-2}$	$f''(0) = k(k-1)$
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$	$f'''(0) = k(k-1)(k-2)$
$f^{(n)}(x) = k(k-1)\ldots(k-n+1)(1+x)^{k-n}$	$f^{(n)}(0) = k(k-1)\ldots(k-n+1)$

Therefore the Maclaurin series of $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\ldots(k-n+1)}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

This if the binomial series.

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