Benha Journal of Applied Sciences (BJAS) **print : ISSN 2356–9751**
Vol.(5) Issue(7) Part (2) (2020), (183-187) **online : ISSN 2356–976x**
http://bjas.bu.edu.eg

New Oscillation Criteria for Second-Order Neutral Delay Dynamic Equations with a Nonpositive Neutral Term on Time Scales

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Abstract

In this paper, we investigate the oscillatory behavior of solutions of a class of second order nonlinear neutral delay dynamic equations with nonpositive coefficient

$$[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+q(t)x^{\beta}(\delta(t))=0,$$

where $\alpha\geq\beta$ are ratios of odd positive integers and $z(t):=x(t)-p(t)x(\tau(t))$. The obtained result not only new but also answered the question that introduced by Li [12].

Keywords: Second order, Nonlinear dynamic equations, Oscillation, Riccati transformation.

1. Introduction

Differential, difference equations, and dynamic equations on time scales have an enormous potential for applications in biology, engineering, economics, physics, neural networks, social sciences, etc. Recently, significant attention has been devoted to the oscillation theory of various classes of equations; see, e.g., [1-19]. The aim of this work is to study the oscillation of a class of second-order nonlinear delay dynamic equations

$$[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}+q(t)x^{\beta}(\delta(t))=0, \qquad (1.1)$$

where $\alpha\geq\beta$ are ratios of odd positive integers and $z(t):=x(t)-p(t)x(\tau(t))$.

Under the following assumptions

$(I1)$ $r\in C_{rd}([t_0,\infty)_{\mathbb{T}},(0,\infty))$.

$(I2)$ $p,q\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $0\leq p(t)\leq p_0\leq 1$, $q(t)\geq 0$ and $q(t)$ is not identically zero for large t.

$(I3)$ $\tau,\delta\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T}),\delta^{\Delta}\geq 0$, $\tau(t)\leq t$, $\delta(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\delta(t)=\infty$.

$(I4)$ $h(t)=\tau^{-1}(\delta(t))$.

By a solution of (1.1), we mean a function $x\in C_{rd}[T_x,\infty)_{\mathbb{T}}$, $T_x\in[t_0,\infty)_{\mathbb{T}}$ which has the property $r(z^{\Delta})^{\alpha}\in C^1_{rd}[T_x,\infty)_{\mathbb{T}}$ and satisfies (1.1) on $[T_x,\infty)_{\mathbb{T}}$. We consider only those solutions x of (1.1) which satisfy $\sup|x(t)|:t\in[T_x,\infty)_{\mathbb{T}}>0$ for all $T\in[T_x,\infty)_{\mathbb{T}}$.

A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is termed nonoscillatory.

In previous years, many papers studied the oscillatory behavior for different classes of dynamic equations on time scale. Note that if $\alpha=\beta$ then (1.1) called half-linear, and called sub-half-linear when $\alpha>\beta$.

many studies have been devoted to the oscillatory behavior of solutions to different classes of equations with nonnegative neutral coefficients; see, e.g [1, 2, 16, 17] and the references cited therein. However, for equations with nonpositive neutral coefficients, there are relatively fewer results in the literature; see [3, 4, 11-15, 19]

For instance, Ming *et al*. [18] investigate oscillatory behavior of solutions to a class of second-order nonlinear neutral delay dynamic equations with nonpositive neutral coefficients of the form

$$\left[r(t)\left(z^{\Delta}(t)\right)^{\alpha}\right]^{\Delta}+q(t)f\left(x\left(\delta(t)\right)\right)=0,\quad t\in[t_0,\infty)_{\mathrm{T}}, \qquad (1.2)$$

where $\alpha\geq 1$ is a ratio of odd integers and $z(t)=x(t)-p(t)x(\tau(t))$. with $\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)\Delta s=\infty$, and present new oscillation criteria for

$$[r(t)(z'(t))^{\alpha}]'+q(t)f(x(\delta(t)))=0,\quad t\geq t_0,$$

under the assumption

$$\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)\mathrm{d}s<\infty,$$

for T = Z, in [?] Seghar *et al.*. discussed the difference equation

$$\Delta\big(a_n\Delta(x_n-p_nx_{n-k})\big)+q_nf(x_{n-l})=0,\quad n\geq n_0, \qquad (1.3)$$

where $0\leq p_n\leq p<1, q_n>0$, and k,l are positive integers, and they obtained several oscillation criteria for (1.4) assuming that $\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty$. Grace *et al*. [9] present some new oscillation criteria for second order nonlinear difference equations with a nonlinear nonpositive neutral term of the form

$$\Delta\left(a(t)\left(\Delta\big(x(t)-p(t)x(t-k)\big)\right)^{\alpha}\right)+q(t)x^{\beta}(t+1-m)=0, \qquad (1.4)$$

where α,β are ratio of positive odd integers, also in [7] Grace investigate the oscillation criteria of

$$\Delta\left(a(t)\left(\left(x(t)-p(t)x\big(\sigma(t)\big)\right)\right)^{\alpha}\right)+q(t)x^{\beta}(\tau(t))=0, \qquad (1.5)$$

where α,β are ratio of positive odd integers $\alpha\geq\beta$.

More exactly existing literature does not provide any criteria which ensure oscillation of all solutions of (1.1). In view of the above motivation. Our aim here is to present sufficient conditions which ensure that all solutions of (1.1) are oscillatory and unify the study that presented in [7, 9].

We consider the following

$$R(v,u)=\int_u^v \frac{1}{r^{1/\alpha}(s)}\Delta s, \qquad v\geq u\geq t_0.$$

and assume that

$$R(t,t_0)\to\infty \quad as \ t\to\infty. \tag{1.6}$$

The following improper integral plays an important role in our study

$$\int_{t_0}^{\infty} q(s)\Delta s. \tag{1.7}$$

In the convergent case we define

$$Q(t)=\int_t^{\infty} q(s)\Delta s, \quad t\geq t_0. \tag{1.8}$$

2. Auxiliary results

Theorem 2.1. *Assume that* $v:\mathbb{T}\to\mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}}:=v(\mathbb{T})$ *is a time scale. Let* $w:\widetilde{\mathbb{T}}\to\mathbb{R}$. *If* $v^{\Delta}(t)$ *and* $w^{\widetilde{\Delta}}(v(t))$ *exist for* $t\in\mathbb{T}^{\kappa}$, *then*

$$(w\circ v)^{\Delta}=(w^{\widetilde{\Delta}}\circ v)v^{\Delta}.$$

Where $\widetilde{\Delta}$ denotes to the derivative on $\widetilde{\mathbb{T}}$. *[5].*

Lemma 2.1 *Let x be a positive solution of (1.1). Then we have the following two cases:*

(I) $\quad z(t)>0,\ \ z^{\Delta}(t)>0,\ \ (r(t)(z^{\Delta}(t))^{\alpha})^{\Delta}\leq 0,$

(II) $\ z(t)<0,\ \ z^{\Delta}(t)>0,\ \ (r(t)(z^{\Delta}(t))^{\alpha})^{\Delta}\leq 0,$

for $t\in[t_1,\infty)_{\mathbb{T}}$, where $t_1\in[t_0,\infty)_{\mathbb{T}}$, is sufficiently large.

3. Main Results

For simplicity, we consider that

$$K(t)=\begin{cases} M, & \alpha=\beta,\\ MR^{\frac{\beta-\alpha}{\alpha}}(t,t_2), & \alpha>\beta, \end{cases} \ \text{forsome } M>0.$$

Theorem 3.1 *Assume that (1.6) holds and* $\delta^{\Delta}(t)>0$. *If*

$$\limsup_{t\to\infty}\left[\varphi(t)Q(t)+\int_{t_2}^{t}\Big(q(s)\varphi(s)-\frac{\alpha^{\alpha}\beta^{-\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\delta(s))(\varphi^{\Delta}(s))^{\alpha+1}}{(\delta^{\Delta}(s))^{\alpha}\varphi^{\alpha}(s)K^{\alpha}(\delta(\sigma(s)))}\Big)\Delta s\right]=\infty, \tag{3.1}$$

and

$$\limsup_{t\to\infty}\left[r^{\frac{1-\alpha}{\alpha}}(h(t))\int_{h(t)}^{t}R(h(t),h(s))\frac{q(s)}{p^{\beta}(h(s))}\Delta s\right]>1, \tag{3.2}$$

for some $M>0$, then (1.1) oscillatory.

Proof. Assume that x is a nonoscillatory solution of (1.1) such that $x(t)>0$, $x(\tau(t))>0$ and $x(\delta(t))>0$, for $t\in[t_1,\infty)_{\mathbb{T}}$.

From lemma 2.1 we have that $z(t)$ satisfies either (I) or (II) for $t\in[t_1,\infty)_{\mathbb{T}}$.

Case 1. First we suppose that $z(t)$ satisfies (I). From definition of $z(t)$,

$$x(t)=z(t)+p(t)x(\tau(t))\geq z(t), \tag{3.3}$$

also, from (1.1) we can write

$$[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}\leq -q(t)z^{\beta}(\delta(t)). \tag{3.4}$$

Defining the Riccati Transformation

$$\omega(t)=\varphi(t)\frac{r(t)(z^{\Delta}(t))^{\alpha}}{z^{\beta}(\delta(t))}. \tag{3.5}$$

It is clear that $\omega(t)>0$ and

$$\omega^{\Delta}(t)=[r(t)(z^{\Delta}(t))^{\alpha}]^{\Delta}\frac{\varphi(t)}{z^{\beta}(\delta(t))}+ r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}\left(\frac{\varphi(t)}{z^{\beta}(\delta(t))}\right)^{\Delta}, \tag{3.6}$$

Applying the Pötzsche chain rule and Theorem 2.1, we get

$$\begin{aligned}(z^{\beta}(\delta(t)))^{\Delta}&=\beta\left\{\int_0^1[z(\delta(t))+h\mu[z(\delta(t))]^{\Delta}]^{\beta-1}dh\right\}z^{\Delta}(\delta(t))\delta^{\Delta}(t)\\ &=\beta\left\{\int_0^1[(1-h)z(\delta(t))+h\mu z(\delta^{\sigma}(t))]^{\beta-1}dh\right\}z^{\Delta}(\delta(t))\delta^{\Delta}(t)\\ &\geq\beta z^{\beta-1}(\delta^{\sigma}(t))z^{\Delta}(\delta(t))\delta^{\Delta}(t).\end{aligned}$$

This with (3.6) and taking into account the conditions $z^{\Delta}(t)>0$, $\delta^{\Delta}(t)\geq 0$, leads to

$$\omega^{\Delta}(t)\leq q(t)\varphi(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) -\beta\delta^{\Delta}(t)\varphi(t)\frac{r(\sigma(t))(z^{\Delta}(\sigma(t)))^{\alpha}}{z^{\beta}(\delta(\sigma(t)))}\frac{z^{\Delta}(\delta(t))}{z(\delta(\sigma(t)))}. \tag{3.7}$$

From definition of $\omega(t)$, we get that

$$\omega^{\Delta}(t)\leq -q(t)\varphi(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) -\beta\delta^{\Delta}(t)\frac{\varphi(t)z^{\Delta}(\delta(t))}{\varphi(\sigma(t))z(\delta(\sigma(t)))}\omega(\sigma(t)). \tag{3.8}$$

From definition of $\omega(t)$ and since $[r(t)(z^{\Delta})^{\alpha}]$ non-increasing, we get that

$$z^{\Delta}(\delta(t))\geq\frac{z^{\frac{\beta}{\alpha}}(\delta(\sigma(t)))}{r^{\frac{1}{\alpha}}(\delta(t))\varphi^{\frac{1}{\alpha}}(\sigma(t))}\omega^{\frac{1}{\alpha}}(\sigma(t)). \tag{3.9}$$

From (3.8) and (3.9), we conclude that

$$\omega^{\Delta}(t)\leq -q(t)\varphi(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) -\beta\delta^{\Delta}(t)\varphi(t)\frac{z^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)))}{\varphi^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \tag{3.10}$$

Since $[r(t)(z^{\Delta}(t))^{\alpha}]$ Non-increasing, so there exist $C>0$ such that $r(t)(z^{\Delta}(t))^{\alpha}<C$ for some $t\geq t_2$, which leads to

$$z^{\Delta}(t)<C^{\frac{1}{\alpha}}r^{\frac{-1}{\alpha}}(t). \tag{3.11}$$

By Integrating (3.11) from t to t_2, we get

$$z(t)<C^{\frac{1}{\alpha}}R(t,t_2). \tag{3.12}$$

This leads to, $z^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)))=1$ for $\beta=\alpha$.

Also for $\beta<\alpha$

$$z^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)))>MR^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)),t_2), \quad forsome\ \ M>0. \tag{3.13}$$

Combining (3.10) and (3.13), we get

$$\omega^{\Delta}(t)\leq -q(t)\varphi(t)+\frac{\varphi^{\Delta}(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) -\beta M\delta^{\Delta}(t)\varphi(t)\frac{R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)),t_2)}{\varphi^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)),$$

$$\omega^\Delta(t) \le -q(t)\varphi(t) + \frac{\varphi^\Delta(t)}{\varphi(\sigma(t))}\omega(\sigma(t)) -\beta\delta^\Delta(t)\varphi(t)\frac{K(\delta(\sigma(t)))}{\varphi^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}\omega^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \quad (3.14)$$

Employing the inequality

$$B\omega - A\omega^{\frac{\alpha+1}{\alpha}} \le \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^\alpha},$$

with $B = \frac{\varphi^\Delta(t)}{\varphi(\sigma(t))}$, $A = \beta\delta^\Delta(t)\varphi(t)\frac{K(\delta(\sigma(t)))}{\varphi^{\frac{\alpha+1}{\alpha}}(\sigma(t))r^{\frac{1}{\alpha}}(\delta(t))}$,

we get that

$$\omega^\Delta(t) \le -q(t)\varphi(t) + \frac{\alpha^\alpha\beta^{-\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\delta(t))(\varphi^\Delta(t))^{\alpha+1}}{(\delta^\Delta(t))^\alpha\varphi^\alpha(t)K^\alpha(\delta(\sigma(t)))}. \quad (3.15)$$

By integrating (3.15) from t_2 to t

$$\omega(t) \le \omega(t_2) - \int_{t_2}^t (q(t)\varphi(t) - \frac{\alpha^\alpha\beta^{-\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\delta(t))(\varphi^\Delta(t))^{\alpha+1}}{(\delta^\Delta(t))^\alpha\varphi^\alpha(t)K^\alpha(\delta(\sigma(t)))})\Delta s. \quad (3.16)$$

On the other hand, we can get from (3.14)

$$\omega^\Delta(t) \le -q(t)\varphi(t) + \frac{\varphi^\Delta(t)}{\varphi(\sigma(t))}\omega(\sigma(t)), \quad (3.17)$$

which equivalent to

$$\left(\frac{\omega(t)}{\varphi(t)}\right)^\Delta \le -q(t). \quad (3.18)$$

By integrating (3.18) from t to u, we get

$$\frac{\omega(u)}{\varphi(u)} \le \frac{\omega(t)}{\varphi(t)} - \int_t^u q(t)\Delta s. \quad (3.19)$$

By letting $u \to \infty$, we get

$$\omega(t) \ge \varphi(t)Q(t). \quad (3.20)$$

From (3.16) and (3.20)

$$\varphi(t)Q(t) + \int_{t_2}^t (q(t)\varphi(t) - \frac{\alpha^\alpha\beta^{-\alpha}}{(\alpha+1)^{\alpha+1}}\frac{r(\delta(t))(\varphi^\Delta(t))^{\alpha+1}}{(\delta^\Delta(t))^\alpha\varphi^\alpha(t)K^\alpha(\delta(\sigma(t)))})\Delta s \le \omega(t_2), \quad (3.21)$$

which contradiction with (3.1).

Case 2. Now, we suppose that $z(t)$ satisfies (II). If we put $y = -z$, which with (1.1) leads to

$$[r(t)(y^\Delta(t))^\alpha]^\Delta \ge q(t)x^\beta(\delta(t)). \quad (3.22)$$

By using $y(t) \le p(t)x(\tau(t))$, we get

$$y(h(t)) \le p(h(t))x(\delta(t)). \quad (3.23)$$

This with (3.22) leads to

$$[r(t)(y^\Delta(t))^\alpha]^\Delta \ge \frac{q(t)y^\beta(h(t))}{p^\beta(h(t))}. \quad (3.24)$$

Also for $t_2 \le u \le v$, we can write

$$y(u) - y(v) = \int_u^v \frac{1}{r^{1/\alpha}(s)}(-r(s)(y^\Delta(s))^\alpha)^{1/\alpha}\Delta s, \quad (3.25)$$

$$y(u) \ge R(v,u)(-r(v)(y^\Delta(v))^\alpha)^{1/\alpha}. \quad (3.26)$$

Setting $u = h(s)$ and $v = h(t)$, we get

$$y(h(s)) \ge R(h(t),h(s))(-r(h(t))(y^\Delta(h(t)))^\alpha)^{1/\alpha}. \quad (3.27)$$

By Integrating (3.24) from $h(t)$ to t in view of (3.27), we get

$$-r(h(t))y^\Delta(h(t)) \ge -r^{1/\alpha}(h(t))y^\Delta(h(t))\int_{h(t)}^t \frac{q(s)}{p^\beta(h(s))}R(h(t),h(s))\Delta s, \quad (3.28)$$

which leads to

$$1 \ge r^{\frac{1-\alpha}{\alpha}}(h(t))\int_{h(t)}^t \frac{q(s)}{p^\beta(h(s))}R(h(t),h(s))\Delta s, \quad (3.29)$$

which contradiction with (3.2). Note that $y^\Delta(t) \to 0$ as $t \to \infty$ is used when $\alpha > \beta$.

Theorem 3.2. *Assume that (1.6) holds and $\delta^\Delta(t) > 0$. If*

$$\liminf_{t\to\infty}\left(\frac{1}{Q(t)}\int_t^\infty \delta^\Delta(s)r^{\frac{-1}{\alpha}}(\delta(s)) R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(s)),t_2)Q^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Delta s.\right) > \frac{\alpha}{\beta M(\alpha+1)^{(\alpha+1)/\alpha}}, \quad (3.30)$$

and (3.2) for some $M > 0$, then (1.1) oscillatory.

Proof. Let x be a nonoscillatory solution of (1.1) such that $x(t) > 0, x(\tau(t)) > 0$ and $x(\delta(t)) > 0$, for $t \in [t_1,\infty)_{\mathbb{T}}$.

From lemma 2.1 we have that $z(t)$ satisfies either (I) or (II) for $t \in [t_1,\infty)_{\mathbb{T}}$.

Case 1. First we suppose that $z(t)$ satisfies (I). From definition of $z(t)$,

$$x(t) = z(t) + p(t)x(\tau(t)) \ge z(t), \quad (3.31)$$

also, from (1.1) we can write

$$[r(t)(z^\Delta(t))^\alpha]^\Delta \le -q(t)z^\beta(\delta(t)). \quad (3.32)$$

Defining the Riccati Transformation

$$v(t) = \frac{r(t)(z^\Delta(t))^\alpha}{z^\beta(\delta(t))}. \quad (3.33)$$

It is clear that $v(t) > 0$ and

$$v^\Delta(t) = \frac{[r(t)(z^\Delta(t))^\alpha]^\Delta}{z^\beta(\delta(t))} + r(\sigma(t))(z^\Delta(\sigma(t)))^\alpha\left(\frac{1}{z^\beta(\delta(t))}\right)^\Delta. \quad (3.34)$$

Applying the Pötzsche chain rule and Theorem 2.1, we get

$$\begin{aligned}(z^\beta(\delta(t)))^\Delta &= \beta\left\{\int_0^1 [z(\delta(t)) + h\mu[z(\delta(t))]^\Delta]^{\beta-1}dh\right\}z^\Delta(\delta(t))\delta^\Delta(t)\\ &= \beta\left\{\int_0^1 [(1-h)z(\delta(t)) + h\mu z(\delta^\sigma(t))]^{\beta-1}dh\right\}z^\Delta(\delta(t))\delta^\Delta(t)\\ &\ge \beta z^{\beta-1}(\delta^\sigma(t))z^\Delta(\delta(t))\delta^\Delta(t).\end{aligned}$$

This with (3.34), leads to

$$v^\Delta(t) \le -q(t) - \beta\delta^\Delta(t)\frac{z^\Delta(\delta(t))}{z(\delta(\sigma(t)))}v(\sigma(t)) \quad (3.35)$$

From definition of $v(t)$ and since $[r(t)(z^\Delta)^\alpha]$ non-increasing, we get that

$$\frac{z^\Delta(\delta(t))}{z^{\frac{\beta}{\alpha}}(\delta(\sigma(t)))} \ge \frac{v^{\frac{1}{\alpha}}(\sigma(t))}{r^{\frac{1}{\alpha}}(\delta(t))}. \quad (3.36)$$

From (3.35) and (3.36), we conclude that

$$v^{\Delta}(t) \leq -q(t) -\beta\delta^{\Delta}(t)r^{\frac{-1}{\alpha}}(\delta(t))z^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)))v^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \quad (3.37)$$

Since $[r(t)(z^{\Delta}(t))^{\alpha}]$ Non-increasing, so there exist $C > 0$,such that $r(t)(z^{\Delta}(t))^{\alpha} < C$ for some $t \geq t_2$, which leads to

$$z^{\Delta}(t) < C^{\frac{1}{\alpha}}r^{\frac{-1}{\alpha}}(t). \quad (3.38)$$

By Integrating (3.38) from t to t_2, we get

$$z(t) < C^{\frac{1}{\alpha}}R(t,t_2). \quad (3.39)$$

This leads to, $z^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t))) = 1$ for $\beta = \alpha$.
Also for $\beta < \alpha$

$$z^{\frac{\beta-\alpha}{\alpha}}\left(\delta(\sigma(t))\right) > MR^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)),t_2), \quad forsome \quad M > 0. \quad (3.40)$$

Combining (3.37) and (3.40), we get

$$v^{\Delta}(t) \leq -q(t) -\beta M\delta^{\Delta}(t)r^{\frac{-1}{\alpha}}(\delta(t))R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(t)),t_2)v^{\frac{\alpha+1}{\alpha}}(\sigma(t)). \quad (3.41)$$

By integrating (3.41) from t to u, we get that

$$v(u) - v(t) \leq -\int_t^u q(s)\Delta s -\beta M\int_t^u \delta^{\Delta}(s)r^{\frac{-1}{\alpha}}(\delta(s))R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(s)),t_2)v^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Delta s. \quad (3.42)$$

Letting $u \to \infty$, we obtain

$$v(t) \geq Q(t) +\beta M\int_t^{\infty} \delta^{\Delta}(s)r^{\frac{-1}{\alpha}}(\delta(s))R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(s)),t_2)v^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Delta s. \quad (3.43)$$

Which yields to

$$\frac{v(t)}{Q(t)} \geq 1 + \beta M\frac{1}{Q(t)}\int_t^{\infty} \delta^{\Delta}(s)r^{\frac{-1}{\alpha}}(\delta(s))R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(s)),t_2) Q^{\frac{\alpha+1}{\alpha}}(\sigma(s))\left(\frac{v(\sigma(s))}{Q(\sigma(s))}\right)^{\frac{\alpha+1}{\alpha}}\Delta s. \quad (3.44)$$

Let $\lambda = \inf_{t\in[t_1,\infty)}(\frac{v(t)}{Q(t)})$. Then it is easy to see that $\lambda \geq 1$ and $\lambda \geq 1 + \frac{\alpha}{(\alpha+1)^{(\alpha+1)/\alpha}}\lambda^{(\alpha+1)/\alpha}$. Then by the simple calculation we get

$$\frac{\alpha}{\beta M(\alpha+1)^{(\alpha+1)/\alpha}} \geq \frac{1}{Q(t)}\int_t^{\infty} \delta^{\Delta}(s)r^{\frac{-1}{\alpha}}(\delta(s))R^{\frac{\beta-\alpha}{\alpha}}(\delta(\sigma(s)),t_2)Q^{\frac{\alpha+1}{\alpha}}(\sigma(s))\Delta s,$$

which contradiction with the possible value of λ and α.

Case 2. Now, we consider that $z(t)$ satisfies (II). We can get a contradiction to (3.2), the proof is similar to the proof of Theorem3.2. The proof is complete.

Example 3.1 *Assume $\mathbb{T} = \mathbb{R}$. Consider the second order neutral delay differential equation*

$$\left(x(t) - \tfrac{1}{2}x\left(t - \tfrac{\pi}{2}\right)\right) + 8x(t-\pi) = 0, \quad t \geq 1. \quad (3.45)$$

Where $\alpha = \beta = 1$, $p(t) = \frac{1}{2}$, $q(t) = 8$, $\tau(t) = t - \frac{\pi}{2}$, $\delta(t) = t - \pi$ and $h(t) = t - \frac{\pi}{2}$. Choose $M = 1 > 0$. Then, condition (3.1) becomes

$$\limsup_{t\to\infty}\left[\varphi(t)Q(t) + \int_{t_2}^{t}(8\varphi(s) - \frac{1}{4}\frac{(\varphi'(s))^2}{\varphi(s)})ds\right] = \infty,$$

and condition (3.2) becomes

$$\limsup_{t\to\infty}\left[4\int_{t-\frac{\pi}{2}}^{t}(t-s)ds\right] = \frac{\pi^2}{2} > 1.$$

So it is Clearly that (3.45) satisfied all conditions of Theorem 3.1, then equation (3.45) is oscillatory.

Remark 3.1 *Theorem 3.1. of [18] can be applied to (3.45) which yields that every solution of equation (3.45) is almost oscillatory. However Theorem 3.1 implies that every solution of (3.45) is oscillatory.*

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