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A New Closed Form Equation to Design Laminated Composite Tubes with Specified Torsional Stiffness

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Abstract: A New theoretical formula is derived to calculate the torsional stiffness of both thin and thick-walled laminated composite tubes of [θ/-θ]n configuration. This equation can also be used to design the composite tube with a specified value of torsional stiffness accurately and simply. A numerical example is presented to illustrate the design procedure. The method is validated using the results obtained from an accurate method based on three-dimensional elasticity theory from the literature.

1. Introduction

The use of composite materials is ever increasing in commercial, defense, and industrial applications. Composite tubes are often used as idealized models for structures such as aircraft fuselages, missile bodies, drive shafts, and storage tanks. Important parameters such as axial stiffness, bending stiffness, and torsional stiffness decide the properties of the composite tube. Torsional stiffness is a key factor judging the performance of cylindrical structural members that are used as torsion bars or drive shafts. There are some research studies concerned the analysis of the twisting behavior of composite tubes. A number of these analytical works on the torsional stiffness of composite tubes, are presented in next lines.

Based on 3D elasticity theory, Jolicoeur, C. and A. Cardou [1] obtained a general analytical solution for an elastic body consisting of an assembly of coaxial hollow circular cylinders made of orthotropic material, and subjected to axisymmetric loadings. The analysis was based on the three-dimensional elasticity theory. The torsional stiffness (GJ) of the whole assembly was calculated. Another theoretical solution was presented by E. E. Elsoaly and R. M. Gadelrab[2] to determine twisting angle of filament winding composite tubes under pure torque, based on the classical lamination theory. Average properties of these layers are considered for the multilayered composite tubes, which were manufactured by filament winding angles of [θ/-θ]. Mo Yang and Ingoing Zhang [3] derived a new theoretical solution for the torsional stiffness of thin-walled composite tube based on the classical lamination theory. Experimental work was done on a tube made of carbon/Epoxy with different of winding angles. M. I. El-Geuchy and S. V. Hoa [4] studied the flexural behavior of thin and thick walled composite tubes and demonstrated the effect of the interaction between the tube layers on enhancing the tube flexural stiffness.

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In the present work, a simple new closed form equation is derived to design and calculate torsional stiffness of both thin and thick-walled composite tubes. The derivation is based on the interaction effects between the tube layers of $[\theta/-\theta]_n$ configuration presented in [4]. The numerical results are compared with those of Jolicoeur, C. and A. Cardou [1] to validate the derived formula.

2. Derivation of the Theoretical Formula

Figure (1) shows a hollow thick-walled tube, with internal and external diameters D_i and D_o made of a homogeneous orthotropic material. The tube is of thickness t, made of *n* thin layers with stacking sequence [θ/-θ] and subjected to pure torsional loading T.

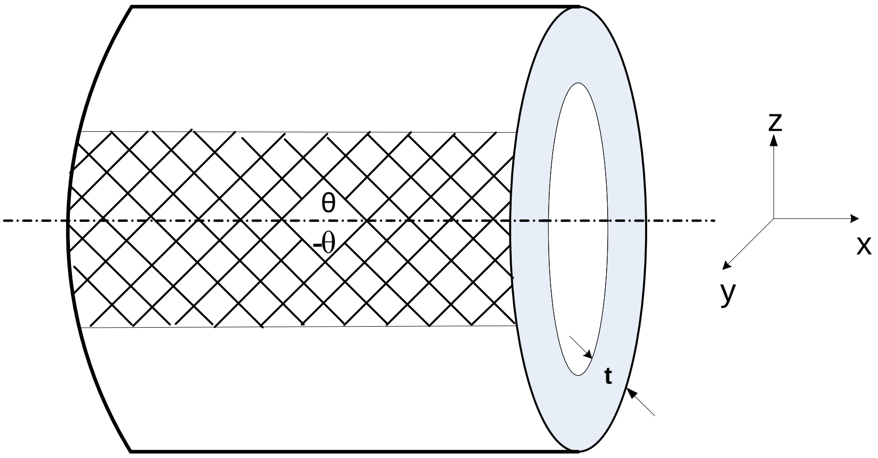

Fig. (1) A laminated composite tube with $[\theta/-\theta]_n$ configuration

The constitutive equations for one lamina with θ orientation can be written in the following matrix equation:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & \bar{S}_{16} \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & \bar{S}_{26} \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{44} & \bar{S}_{45} & 0 \\ 0 & 0 & 0 & \bar{S}_{45} & \bar{S}_{55} & 0 \\ \bar{S}_{16} & \bar{S}_{26} & \bar{S}_{36} & 0 & 0 & \bar{S}_{66} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{1}$$

where $\bar{S}_{ij}$ are the coefficients of off-axis compliance. There are different coupling parameters in the composite lamina at orientation [θ]. These parameters make the composite lamina to deform in one direction due to a generated deformation in the direction of loading. One can observe the first coupling parameters are the Poisson's ratios (υ_{xy}) and (υ_{yx}) which relate the axial strain (ε_x) and the hoop strain (ε_y) as shown in the following equations:

$$\upsilon_{xy} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\bar{S}_{12}}{\bar{S}_{11}} \tag{2}$$

$$\upsilon_{yx} = -\frac{\varepsilon_x}{\varepsilon_y} = -\frac{\bar{S}_{12}}{\bar{S}_{22}} \tag{3}$$

These coupling parameters exist in both isotropic and anisotropic materials. Other coupling parameters that exist only in anisotropic materials are the mutual influences of both first and second kinds. These parameters relate the normal strains to shear strains. The mutual influences of first kind are stated in the following equations:

$$\eta_{x,xy} = \frac{\varepsilon_x}{\gamma_{xy}} \qquad \eta_{y,xy} = \frac{\varepsilon_y}{\gamma_{xy}} \tag{4}$$

These parameters can be written in terms of transformed compliances as follows:

$$\eta_{x,xy} = \frac{\bar{S}_{16}}{\bar{S}_{66}} \qquad \eta_{y,xy} = \frac{\bar{S}_{26}}{\bar{S}_{66}} \tag{5}$$

While, the coefficient of mutual influence of the second kind ($\eta_{xy,x}$) and ($\eta_{xy,y}$) are used to calculate the generated shear strains as a cause of axial strains when the lamina is subjected to normal stresses such that:

$$\eta_{xy,x} = \frac{\gamma_{xy}}{\varepsilon_x} \qquad \eta_{xy,y} = \frac{\gamma_{xy}}{\varepsilon_y} \tag{6}$$

Similarly, these coefficients can be written in terms of transformed compliances,

$$\eta_{xy,x} = \frac{\bar{S}_{16}}{\bar{S}_{11}} \qquad \eta_{xy,y} = \frac{\bar{S}_{26}}{\bar{S}_{22}} \tag{7}$$

For carbon-epoxy composite material, of properties provided in Table (1), Figures (2-7) illustrate the variation of these coupling coefficients of interaction with orientation angle θ.

Table (1) Mechanical properties of (Carbon/Epoxy) [6]

E_1(GPa)	E_2(GPa)	E_3(GPa)	ν_{12}	ν_{23}	ν_{13}	G_{12}(GPa)	G_{23}(GPa)	G_{13}(GPa)
155	12.1	12.1	0.248	0.458	0.248	4.4	3.2	4.4

From these presented figures, one can observe the following:

$$\nu_{xy}(\theta) = \nu_{xy}(-\theta) \qquad \nu_{yx}(\theta) = \nu_{yx}(-\theta) \tag{8}$$

While

$$\eta_{xy,x}(\theta) = -\eta_{xy,x}(-\theta) \qquad \eta_{xy,y}(\theta) = -\eta_{xy,y}(-\theta) \tag{9}$$

$$\eta_{x,xy}(\theta) = -\eta_{x,xy}(-\theta) \qquad \eta_{y,xy}(\theta) = -\eta_{y,xy}(-\theta) \tag{10}$$

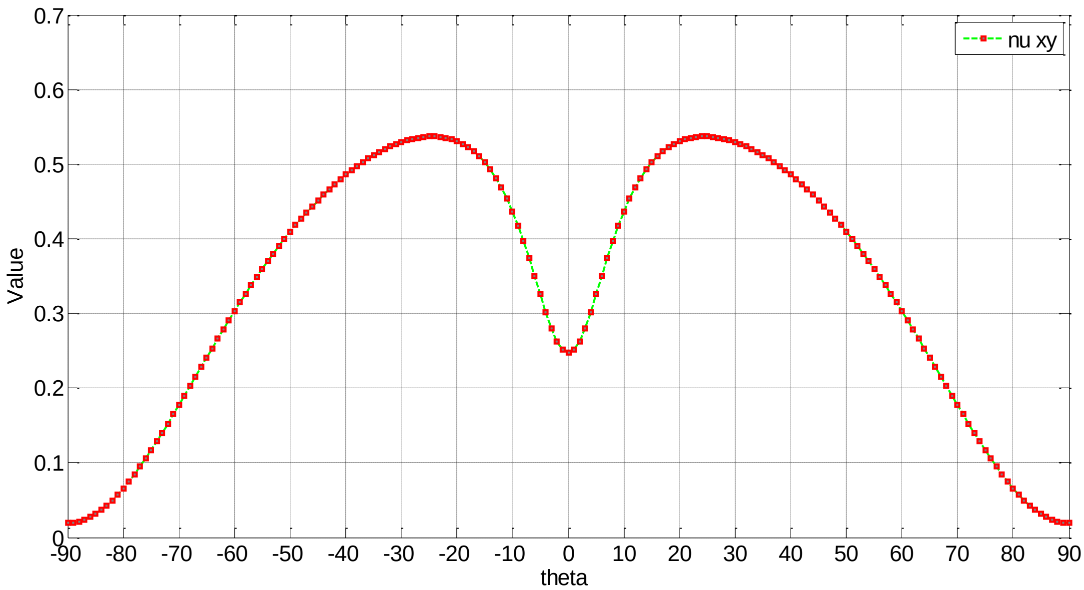

Fig. (2) υ_{xy} variation with θ for carbon/epoxy

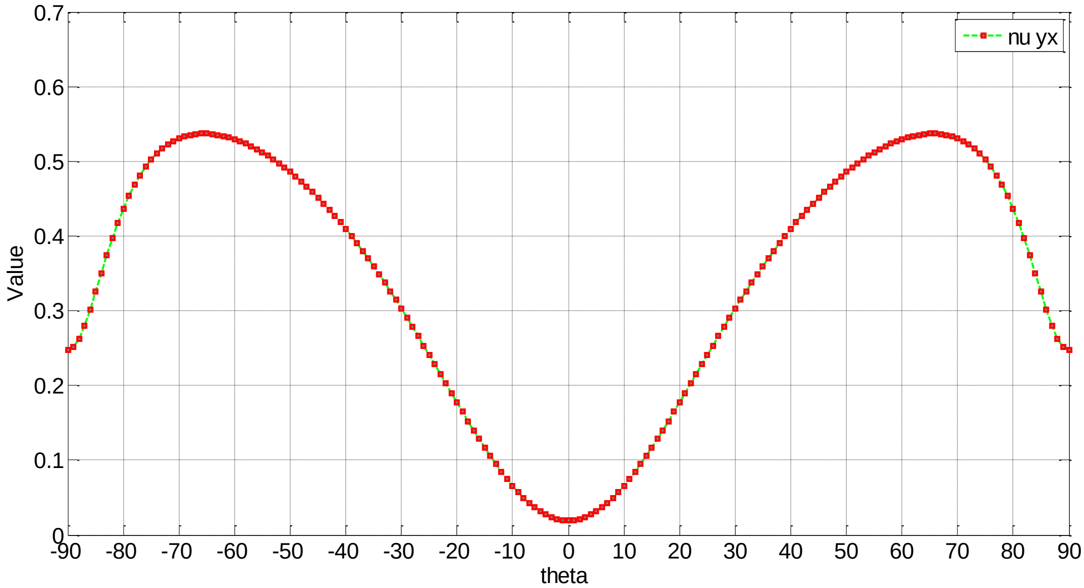

Fig. (3) υ_{yx} variation with θ for carbon/epoxy

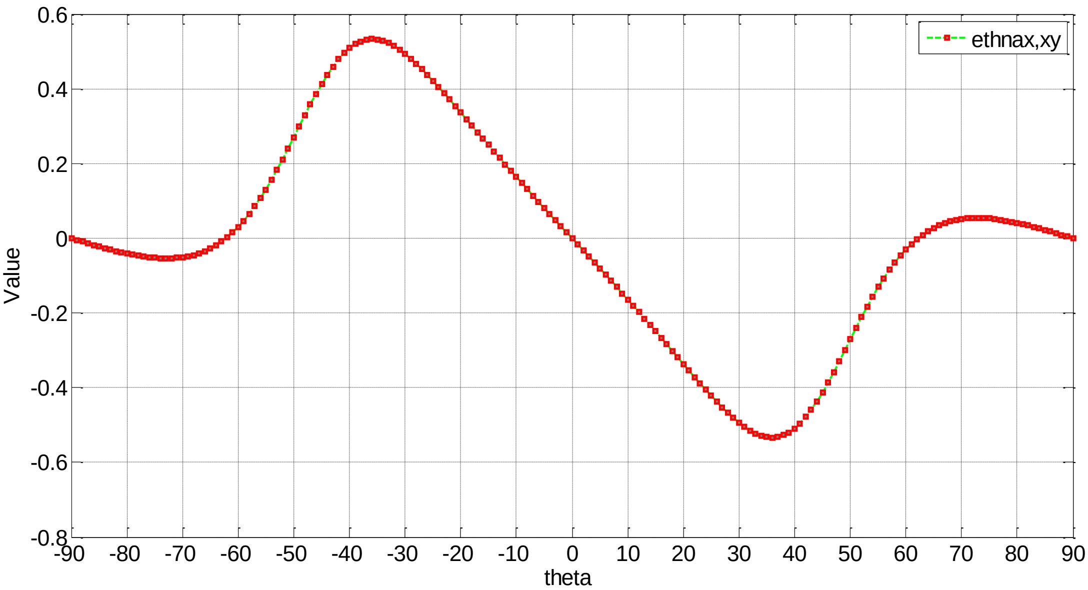

Fig. (4) $\eta_{x,xy}$ variation with θ for carbon/epoxy

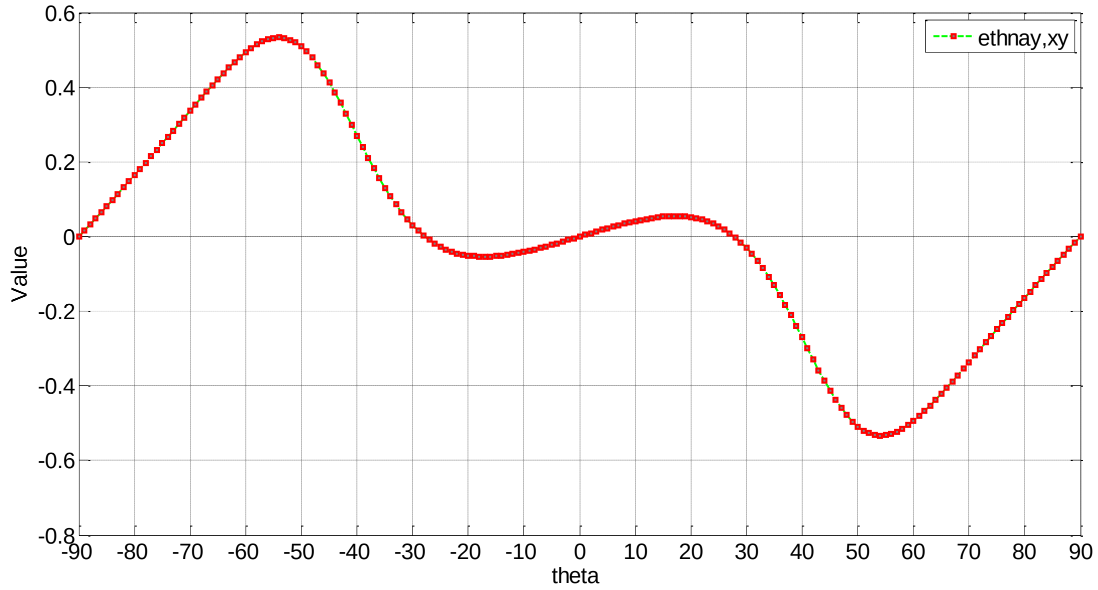

Fig. (5) $\eta_{y,xy}$ variation with θ for carbon/epoxy

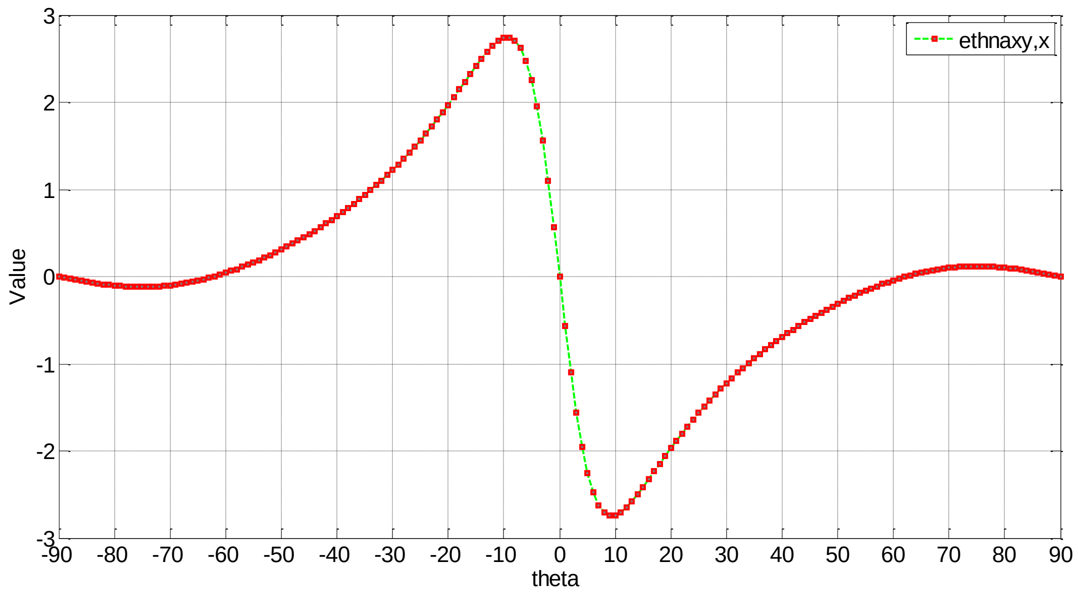

Fig. (6) $\eta_{xy,x}$ variation with θ for carbon/epoxy

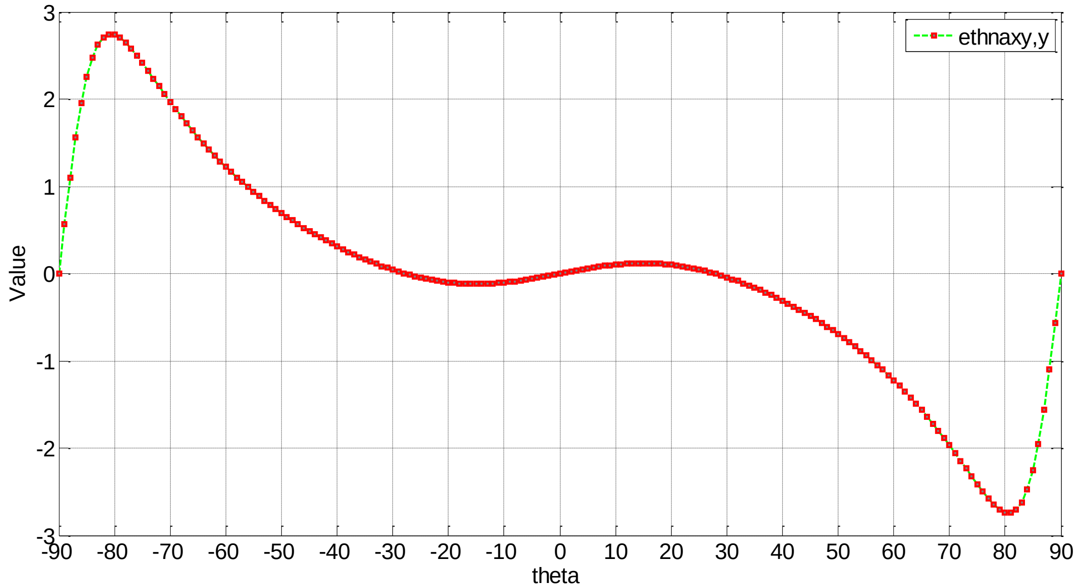

Fig. (7) $\eta_{xy,y}$ variation with θ for carbon/epoxy

In case of a composite tube composed of a layer with one lamina of θ orientation and subjected to a torsional load, axial strains (ε_x, ε_y) are generated in the x and y directions due to the generated shear strain (γ_{xy}). While in a layer made of two laminas [θ,-θ], and taking into considerations equation (10). One can observe the following:

$$\eta_{x,xy}{}_{Layer1} = -\eta_{x,xy}{}_{Layer2} \quad \text{and} \quad \eta_{y,xy}{}_{Layer1} = -\eta_{y,xy}{}_{Layer2} \tag{11}$$

So, the generated axial strains (ε_x and ε_y) in the lamina of orientation (+θ) are opposing the generated axial strains in the lamina of orientation (-θ), such that at the interface between the two laminae

$$\varepsilon_x = \varepsilon_y = 0 \tag{12}$$

This interaction at the interface generates axial stresses (σ_x and σ_y). This situation leads to enhance the rigidity of the composite lamina at the interface such that it will have a higher effective shear modulus than G_{xy} for a layer composed of two laminae of only [θ/θ] orientation. This situation is explained in details at the following paragraphs.

By substituting equation (12) in the constitutive matrix equation, we get the following equations:

$$\varepsilon_x = \bar{S}_{11}\sigma_x + \bar{S}_{12}\sigma_y + \bar{S}_{16}\tau_{xy} = 0 \tag{13}$$

$$\varepsilon_y = \bar{S}_{12}\sigma_x + \bar{S}_{22}\sigma_y + \bar{S}_{26}\tau_{xy} = 0 \tag{14}$$

Form equation (13), using equations (2), (7)

$$\sigma_x = \frac{-1}{\bar{S}_{11}}(\bar{S}_{12}\sigma_y + \bar{S}_{16}\tau_{xy}) \tag{15}$$

$$\sigma_x = \upsilon_{xy}\sigma_y - \eta_{xy,x}\tau_{xy} \tag{16}$$

And from equation (14), using equations (3), (7)

$$\sigma_y = \frac{-1}{\bar{S}_{22}}(\bar{S}_{12}\sigma_x + \bar{S}_{26}\tau_{xy}) \tag{17}$$

So,

$$\sigma_y = \upsilon_{yx}\sigma_x - \eta_{xy,y}\tau_{xy} \tag{18}$$

By substituting of equation (18) into equation. (16), the generated normal stress, σ_x, becomes

$$\sigma_x = \upsilon_{xy}(\upsilon_{yx}\sigma_x - \eta_{xy,y}\tau_{xy}) - \eta_{xy,x}\tau_{xy} \tag{19}$$

Equation (19) can be written in a form to give a relation between normal stress σ_x and shear stress τ_{xy} as;

$$\sigma_x = \frac{-1}{(1-\upsilon_{xy}\upsilon_{yx})}(\eta_{xy,x} + \upsilon_{xy}\eta_{xy,y})\tau_{xy} \tag{20}$$

Substituting equation (20) into equation (18), the normal stress σ_y can be written as:

$$\sigma_y = [\frac{-\upsilon_{xy}}{(1-\upsilon_{xy}\upsilon_{yx})}(\eta_{xy,x} + \upsilon_{xy}\eta_{xy,y}) - \eta_{xy,y}]\tau_{xy} \tag{21}$$

One can simplify equations (20) and (21) to the following forms:

$$\sigma_x = X1.\tau_{xy} \tag{22}$$

$$\sigma_y = X2.\tau_{xy} \tag{23}$$

where $X1$ and $X2$ are coefficients in terms of the material properties such that:

$$X1 = \frac{-1}{(1-\upsilon_{xy}\upsilon_{yx})}(\eta_{xy,x} + \upsilon_{xy}\eta_{xy,y}) \tag{24}$$

$$X2 = \frac{-\upsilon_{xy}}{(1-\upsilon_{xy}\upsilon_{yx})}(\eta_{xy,x} + \upsilon_{xy}\eta_{xy,y}) - \eta_{xy,y} \tag{25}$$

Similarly, the shear strain γ_{xy} can be obtained, using the constitutive matrix equation, in the following form:

$$\gamma_{xy} = \bar{S}_{16}\sigma_x + \bar{S}_{26}\sigma_y + \bar{S}_{66}\tau_{xy} \tag{26}$$

By substituting of equations. (22) and (23) into equation (26), the shear strain γ_{xy} is obtained as:

$$\gamma_{xy} = (\bar{S}_{16}X1 + \bar{S}_{26}X2 + \bar{S}_{66})\tau_{xy} \tag{21}$$

So,

$$\frac{\tau_{xy}}{\gamma_{xy}} = \frac{1}{(\bar{S}_{16}X1 + \bar{S}_{26}X2 + \bar{S}_{66})} \tag{22}$$

But, the shear modulus of a lamina in the global coordinates (G_{xy}) equals $\frac{1}{\bar{S}_{66}}$, so equation (22) can be written in the form:

$$\frac{\tau_{xy}}{\gamma_{xy}} = \frac{G_{xy}}{(\frac{\bar{S}_{16}}{\bar{S}_{66}}X1 + \frac{\bar{S}_{26}}{\bar{S}_{66}}X2 + \frac{\bar{S}_{66}}{\bar{S}_{66}})}$$

or,

$$G_{eff} = \frac{G_{xy}}{(\eta_{x,xy}X1 + \eta_{y,xy}X2 + 1)} \tag{23}$$

where: *Geff* is the effective shear modulus of the layer with [θ/-θ] configuration. One can observe that the torsional stiffness of a tube with (n) layers of [θ/-θ], can be written as:

$$GJ = \sum_1^n G_{eff,n}(J1 + J2)n \tag{24}$$

where: J_1, J_2 : polar moment of inertia of lamina(+θ) and lamina (-θ), respectively. In order to validate equation (24), the obtained torsional stiffness is compared with the numerical values calculated using the analytical solution presented in [1]. This analytical method is chosen because it is based on the three-dimensional elasticity theory. A MATLAB code is written, and the torsional stiffness is calculated for thick-walled tubes made of Carbon/Epoxy, with $[\theta/-\theta]_{20}$ configuration, and of t/D_o =0.45. Figure (8), shows GJ values from both the derived closed form equation and 3D based analytical solution of [1]. One can observe that the values coincide with each other, which validates the accuracy of derived equation
for all θ.

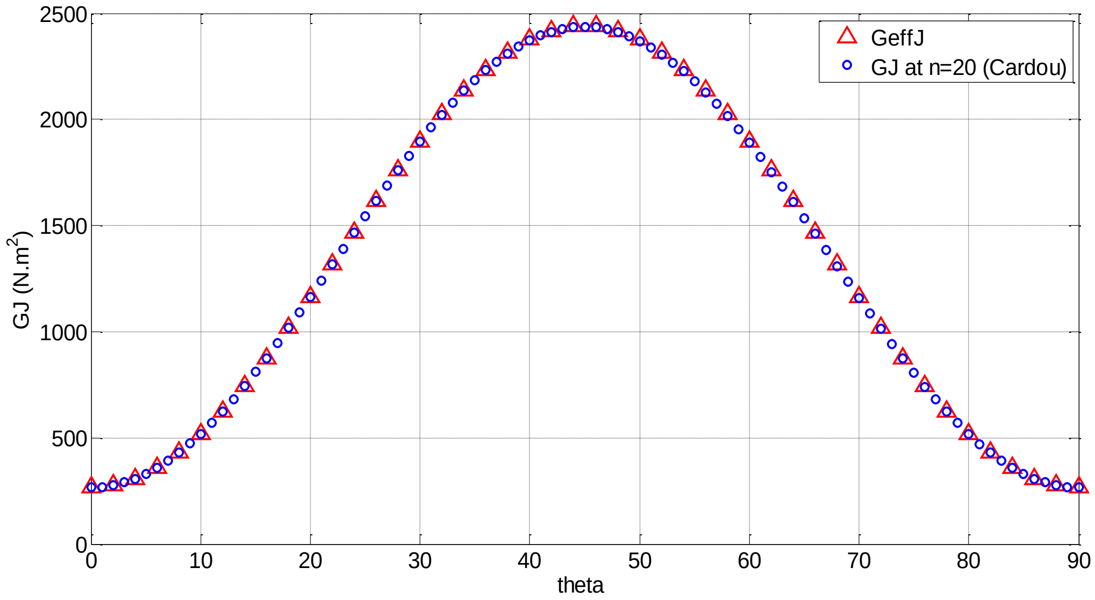

Fig. (8) GJ versus θ for composite tube of $[\theta/-\theta]_n$, t/D_o= 0.45

3. Numerical Example

In this section, the procedure to design a composite tube with a required torsional stiffness value is illustrated through a numerical example. It is required to specify the tube configuration of torsional stiffness equal to 3500 N.m^2. At first, Carbon/Epoxy composite material is chosen for making the composite tube, the mandrel diameter is taken to be 14mm which is considered to be the inner diameter of the composite tube. From figure (8), one can observe that the highest torsional stiffness is obtained at (θ=45^o), which means that the maximum value of G_{eff}.is when the tube has $[45/-45]_n$ configuration. So, this configuration is chosen for the composite tube and it is required to calculate (n) value that give the specified torsional stiffness. The wall thickness of one lamina is assumed to be (t_{lamina}= 0.15mm). So, equations (24) can be arranged to be as follows:

$$GJ = G_{eff} * J_{total} \tag{25}$$

Since
$$J_{total} = \frac{\pi}{32}(2t + D_i)^4 - D_i^4 \tag{26}$$

Making some arrangement to have the following equation:

$$t = \sqrt[4]{\frac{2}{\pi}J_{total} + \frac{D_i^4}{4}} - \frac{D_i}{2} \tag{27}$$

Since
$$t = n * 2t_{lamina} \tag{28}$$

So substitute by equations (25), (27) in equation (28) to have the following equation:

$$n = \frac{\left(\sqrt[4]{\frac{2(GJ)}{\pi G_{eff}}+\frac{D_i^4}{4}}-\frac{D_i}{2}\right)}{2t_{lamina}} \tag{29}$$

So n=29.87, and it must be approximated to the next bigger integer, which means that the tube configuration should be $[45/-45]_{30}$ to have the needed torsional stiffness value. The analytical solution, presented in [1], is used to calculate the torsional stiffness for this configuration and the results are compared in the following table.

Table (2) Torsional stiffness values using the derived equation and the 3D based analytical solution of [1]

Torsional stiffness GJ (N.m^2)		
3D based analytical solution of [1]	Closed Form Formula	% error
3515.5	3500	0.44%

The comparison of results, in table 2, shows that the percentage of error of closed form solution is 0.44%, which means that equation (29) can be used to design the torsional stiffness accurately and simply.

4. Conclusion

A simple and accurate closed form equation for torsional stiffness of multi-layers composite tubes is derived. It can be used for both thin and thick-walled composite tubes. It mainly depends on the understanding the interaction effects at the interface between two laminas of [θ/-θ] configuration. And it is found that main parameters that make this interaction effect are the mutual influence coefficients ($\eta_{xy,x}$), ($\eta_{xy,y}$), ($\eta_{x,xy}$) and ($\eta_{y,xy}$) of these layers.

Lastly, equation (29) provides an accurate and easy tool to design multi-layered composite tubes with the required torsional stiffness value.

5. References

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