Estimation Under a Finite Mixture of Two-Component Topp-Leone Rayleigh Lifetime Model

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Abstract

In this paper, a finite mixture distribution of two-component Topp-Leone Rayleigh lifetime model is introduced. Maximum likelihood and Bayes estimators are derived for the unknown parameters, mixing proportion, reliability and hazard rate functions based on Type II censored samples from the mixture distribution. Confidence and credible intervals for the parameters, mixing proportion, reliability and hazard rate functions are obtained. Bayesian estimation is considered under balanced error loss function. A numerical illustration is carried out to investigate the precision of the maximum likelihood and Bayes estimates. An application using real data is used to insure the simulated results.

Keywords: Topp-Leone Rayleigh distribution; Identifiability; Heterogeneous population; Type II censored sample; Maximum likelihood estimation; Bayesian estimation.

1. Introduction

Topp-Leone (TL) distribution was proposed by Topp and Leone (5); as an alternate model failure data. It is a continuous unimodal distribution with bounded support; therefore it is appropriate for modeling lifetime of distributions with finite support such as limited power supply, maintenance/repair resource, or design life of the system.

The probability density function (pdf) of Topp-Leone distribution is given by

$$f(x;\alpha,b)=\frac{2\alpha}{b}\left(1-\frac{x}{b}\right)\left(2\frac{x}{b}-\left(\frac{x}{b}\right)^2\right)^{\alpha-1},\quad 0<x<b,\alpha>0. \qquad (1)$$

The cumulative distribution function (cdf) of Topp-Leone distribution is as follows:

$$F(x;\alpha,b)=\begin{cases}0, & x\le 0,\\ \left(2\frac{x}{b}-\left(\frac{x}{b}\right)^2\right)^{\alpha}, & 0<x<b,\\ 1, & x\ge b,\end{cases} \qquad (2)$$

where α is a shape parameter and b is a scale parameter, if α is restricted to be in (0,1), then the distribution function in (1) is J-shaped distribution.

Nadarajah and Kotz (2003) showed that TL distribution have bathtub failure rate function with widespread applications in reliability. Some attractive reliability properties were provided by Ghitany et al. (2005), such as the bathtub-shape hazard rate, decreasing reversed hazard rate, upside-down mean residual life, increasing expected inactivity time. Also, Zghoul (2010) studied order statistics from TL distribution and provided expressions for moments of ordered statistics from TL distribution. Feroze and Aslam (2013) derived Bayesian estimation and prediction using a couple of non-informative priors under complete and Type II censored samples. Sindhu et al. (2013) obtained Bayes estimators for the shape parameter and credible intervals based on trimmed samples using different priors.

Maximum likelihood (ML) and Bayesian estimation of the parameters of TL distribution, based on lower record values under symmetric loss function were obtained by Li (2016). Also, he derived the empirical Bayes estimators.

Bayesian estimation of the shape parameter, under simple and mixture priors and different loss functions, is presented by Sultan and Ahmad (2017). [For more details on TL distribution see, Zghoul (2011), Genç (2012), Khan and Khan (2015) and Bayoud (2015)]. Aryal et al. (2016) introduced the TL generated Weibull distribution and derived some structural properties, also they used the ML method to obtain the estimators of the parameters.

Mixtures of distributions is applied in different areas of survival analysis, reliability, economics, medicine, psychology, geology, agriculture, biology and atmospheric sciences. In reliability studies, for instance, failure can occur for more than one reason, and the failure time distribution for each reason can be adequately approximated by a simple density function. Estimation of the parameters or functions of the parameters based on finite mixture models when the components belong to the same family were discussed by several authors for an example Everitt and Hand (1981), Titterington et al. (1985), McLachlan and Basford (1988), Lindsay (1995), McLachlan and Krishnan (1997), and McLachlan and Peel (2000). Reliability and hazard based on finite mixture models was surveyed by AL-Hussaini and Sultan (2001)

Assuming $\mathcal{F}$ is a family of distribution functions. A random variable T is said to have a finite mixture distribution if its distribution function F satisfies

$$F(x) = \sum_{i=1}^{k} p_i F_i(x), \qquad (3)$$

where $F_i \in \mathcal{F}$ are distinct distribution functions of k mixture components or populations and the mixing proportions (weights) satisfy $0 < p_i < 1$, i=1, 2, …, k and $\sum_{i=1}^{k} p_i = 1$.

A heterogeneous population may be described by a finite mixture model with pdf

$$f(x) = \sum_{i=1}^{k} p_i f_i(x), \qquad (4)$$

where $F_i(x)$ and $f_i(x)$ are called the ith components in the finite mixture of distributions (3) and probability functions (4), respectively.

If k=2 in (3) and (4), then a finite mixture distribution of two components can be obtained with the cdf, pdf, rf and hrf as given, respectively, below

$$F(t) = p_1 F_1(t) + p_2 F_2(t) , \quad (5)$$

$$f(t) = p_1 f_1(t) + p_2 f_2(t) , \quad (6)$$

$$R(t) = p_1 R_1(t) + p_2 R_2(t), \quad (7)$$

and

$$h(t) = \frac{f(t)}{R(t)} \quad (8)$$

The Rayleigh distribution is extremely important in communication engineering. For example, the envelope of a narrow band Gaussian random process and the amplitude of atmospheric radio noise caused by the radiation due to lightning discharges in storms have pdfs of the Rayleigh distribution. Also, it is applied in clinical studies dealing with cancer and AIDS patients.

Some important occurrences of the Rayleigh distribution in various practical situations are given by Curelaru and Vod$\tilde{a}$ (1975) as follows:

- **In naval research; to simulate the sea waves.**
- **In telecommunications, to describe the signal fluctuations due to multipath effects in the line of sight links.**
- **In bombing problems; to describe the distribution of distances from target to the actual impact points.**

The pdf of the Rayleigh distribution is given by

$$g(t;\ \sigma) = \frac{t}{\sigma^2} \exp\left(-\frac{1}{2}\left(\frac{t}{\sigma}\right)^2\right), \qquad t > 0, \sigma > 0. \quad (9)$$

The rest of this paper is organized as follows. Section 2 presents Topp-Leon Rayleigh distribution as a composite distribution. The identifiability of the mixture Topp-Leone Rayleigh distribution is proved in Section 3. Maximum likelihood and Bayes estimators under balanced square error function for the parameters, reliability and hazard rate functions are derived in Section 4. In Section 6 a numerical illustration is presented.

2. Topp-Leone Rayleigh Distribution:

Considering b=1 in (2); without any loss of generality, a random variable X is distributed as the TL distribution with parameter α denoted by $X \sim TL(\alpha)$ with a cdf

$$H_{TL}(x) \equiv H_{TL}(x;\alpha) = (2x - x^2)^{\alpha}, \quad 0 < x < 1, \alpha > 0. \qquad (10)$$

The corresponding pdf is

$$h_{TL}(x;\alpha) = 2\,\alpha\,(1-x)(2x-x^2)^{\alpha-1}, \quad 0 < x < 1, \alpha > 0. \qquad (11)$$

The rf and the hrf are, respectively, given by

$$R_{TL}(x;\alpha) = 1 - (2x - x^2)^{\alpha}, \qquad (12)$$

and

$$hr_{TL}(x;\alpha) = \frac{2\alpha\,(1-x)(2x-x^2)^{\alpha-1}}{1-(2x-x^2)^{\alpha}}. \qquad (13)$$

A composition of H, given by (10) and a cdf G, with positive support, yields a new cdf, given below

$$F(t) = H\big(G(t)\big) = (2G(t) - \big(G(t)\big)^2)^{\alpha}, \qquad (14)$$

On composition of distribution functions, see AL-Hussaini (2012).

In particular, if G is Rayleigh distribution; denoted by $R \sim(\sigma)$, with cdf as

$$G(t) \equiv G(t;\sigma) = 1 - \exp(-\tfrac{1}{2}(t/\sigma)^2), \quad t > 0, \; \sigma > 0. \qquad (15)$$

Substituting (15)in (14),the cdf for Topp-Leone Rayleigh distribution ($TLR(\alpha,\sigma)$) is given by

$$F_{TLR}(t) \equiv F_{TLR}(t;\alpha,\sigma) = \left(1 - \exp(-(\tfrac{t}{\sigma})^2)\right)^{\alpha}, \quad t > 0, \; \alpha,\sigma > 0. \qquad (16)$$

The pdf, corresponding to the cdf given in (16), is as follows:

$$f_{TLR}(t;\alpha,\sigma) = 2\,\alpha(t/\sigma^2)\;\exp(-(t/\sigma)^2)\;(1-\exp(-(t/\sigma)^2))^{\alpha-1},$$

$$t > 0, \; \alpha,\sigma > 0, \qquad (17)$$

where α is a shape parameter and σ is a scale parameter.

The rf and hrf for TLR distribution are, respectively, as :

$$R_{TLR}(t;\alpha,\sigma) = 1-\left(1-\exp(-(\tfrac{t}{\sigma})^2)\right)^{\alpha}, \qquad t>0,\ \alpha,\sigma>0, \tag{18}$$

and

$$h_{TLR}(t;\alpha,\sigma) = \frac{2\,\alpha(t/\sigma^2)\,\exp(-(t/\sigma)^2)\left(1-\exp(-(t/\sigma)^2)\right)^{\alpha-1}}{1-\left(1-\exp(-(\frac{t}{\sigma})^2)\right)^{\alpha}}, \qquad t>0,\ \alpha,\sigma>0, \tag{19}$$

From (19) it is noticed that the TLR distribution has an increasing hazard rate and this property means that the TLR distribution is a candidate model for lifetime of components that age rapidly with time.

3. Identifiability of the Mixture Topp-Leone Rayleigh Distribution:

The identifiability of mixtures must be proved before discussing the problem of estimation, testing of hypotheses or classification of random variables, which are based on the observations from a mixture. A mixture is identifiable if there exists a one to one correspondence between the mixing distributions and the resulting mixture.

For details on the identifiability of mixture distributions, see for example Everitt and Hand (1981), AL-Hussaini and Ahmad (1981) and Ahmad and AL-Hussaini (1982), Ahmad (1988), Adham (1996), Baharith (1991), among others. The identifiability of a mixture of two TLR components will be proved as follows:

Considering a linear combination for two different distributions each of them is TLR distribution is given by

$$\sum_{i=1}^{2} p_i\, R_i(t) = 0\,,$$

where $R_i(t) = 1-\left(1-\exp(-(\frac{t}{\sigma})^2)\right)^{\alpha}, \qquad t>0;\ \alpha_i,\sigma_i>0, i=1,2,$

and $R_i(t)$ is the rf of the ith TLR(α,σ) distribution.

The finite mixture of Topp-Leone Rayleigh (MTLR) distributions is identifiable if $R_1(t)$ and $R_2(t)$ are linearly independent. That means $(\alpha_1, \sigma_1) = (\alpha_2, \sigma_2)$ then $p_1 = p_2$.

If $t = 0$ then $R_1(0) = R_2(0) = 1$, $p_1 + p_2 = 0$, $p_1 = -p_2$,

hence

$$p_1 R_1(t) + p_2 R_2(t) = 0,$$

$$p_1 R_1(t) - p_1 R_2(t) = 0,$$

$$p_1 R_1(t) = p_1 R_2(t),$$

$$R_1(t) = R_2(t),$$

$$1 - \left(1 - \exp\left(-\left(\frac{t}{\sigma_1}\right)^2\right)\right)^{\alpha_1} = 1 - \left(1 - \exp\left(-\left(\frac{t}{\sigma_2}\right)^2\right)\right)^{\alpha_2},$$

$$\left(1 - \exp\left(-\left(\frac{t}{\sigma_1}\right)^2\right)\right)^{\alpha_1} = \left(1 - \exp\left(-\left(\frac{t}{\sigma_2}\right)^2\right)\right)^{\alpha_2},$$

$$\sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha_1}{\ell}\exp\left(-\left(\frac{t}{\sigma_1}\right)^2\right)^{\ell} = \sum_{\ell=0}^{\infty}(-1)^{\ell}\binom{\alpha_2}{\ell}\exp\left(-\left(\frac{t}{\sigma_2}\right)^2\right)^{\ell},$$

$$\sum_{\ell=1}^{\infty}(-1)^{\ell}\binom{\alpha_1}{\ell}\exp\left(-\ell\left(\frac{t}{\sigma_1}\right)^2\right) = \sum_{\ell=1}^{\infty}(-1)^{\ell}\binom{\alpha_2}{\ell}\exp\left(-\ell\left(\frac{t}{\sigma_2}\right)^2\right),$$

$$\sum_{\ell=1}^{\infty}\sum_{w=0}^{\infty}(-1)^{\ell}\binom{\alpha_1}{\ell}\left(-\ell\left(\frac{t}{\sigma_1}\right)^2\right)^{w} = \sum_{\ell=1}^{\infty}\sum_{w=0}^{\infty}(-1)^{\ell}\binom{\alpha_2}{\ell}\left(-\ell\left(\frac{t}{\sigma_2}\right)^2\right)^{w}.$$

Comparing the coefficients of $t^2, t^4, t^6, \ldots$ on both sides, one can observe that $\alpha_1 = \alpha_2$ and $\sigma_1 = \sigma_2$ when $p_1 = p_2 = 0$.

Therefore $R_1(t)$ and $R_2(t)$ are linearly independent. Then the finite MTLR (α, σ) distribution components are identifiable.

4. Estimation Using Balanced Square Error Loss Function:

The squared error loss (SEL) function; as a symmetric loss function, has probably been the most popular loss function used in literature. Although the symmetric loss function is relatively easy to handle analytically, in Bayesian

inference the use of symmetric loss functions may be inappropriate for wide range of applications. This was recognized in the literature, (see, for example, Varian (1975), Berger (1980) and Scäbe(1991)), since a given positive error may be serious than a given negative error of the same magnitude or vice versa.

In practice the real loss function is often not symmetric, overestimation of a parameter can lead to more or less severe consequences than underestimation. Examples of such cases are food-processing industries, dam construction and estimating the reliability function.

Varian (1975) introduced the linear exponential (LINEX) loss function as an asymmetric loss function. Ahmadi et al. (2009) suggested the use of the balanced loss (BLS) function; which was originated by Zellner (1994), to be of the form

$$L^{*}(\theta,\delta) = \omega\rho(\delta_0,\delta) + (1-\omega)\rho(\theta,\delta), \qquad \textbf{(20)}$$

where $\rho(\theta,\delta)$ is an arbitrary loss function, δ_0 is a chosen target estimator of δ and the weight $\omega \in (0,1)$.

The BLF in (20) specializes to various choices of loss functions such as the absolute error loss function, entropy, LINEX and generalized SEL functions.

If $\rho(\theta,\delta) = \rho(\delta-\theta)^2$ is substituted in (20), one obtains the Balanced square error loss (BSEL) function, given by

$$L^{**}(\theta,\delta) = \omega\rho(\delta_0-\delta)^2 + (1-\omega)\rho(\delta-\theta)^2, \qquad \textbf{(21)}$$

The estimator $\hat{u}_{BSEL}$ of a function $u(\theta)$, using BSEL function is given by

$$\hat{u}_{BSEL} = \omega\hat{u}_{ML} + (1-\omega)\hat{u}_{SEL}, \qquad \textbf{(22)}$$

where $\hat{u}_{ML}$ is the ML estimator of $u(\theta)$ and $\hat{u}_{SEL}$is its Bayes estimator using SEL function. The estimator of a function using BSEL function is a mixture of the ML estimators of the function and the Bayes estimators using SEL function. Other estimators, such as least squares estimators may replace the ML estimators. Also, LINEX or Quadratic exponential loss functions could be used for $\rho(\theta,\delta)$, [see AL-Hussaini and Hussein (2011)].

The estimators based on BSEL function; having the ML estimators and the Bayes estimators based on SEL function, are given by

$$\hat{p}_{BSEL} = \omega\hat{p}_{ML} + (1-\omega)\hat{p}_{SEL},$$

$$\hat{\alpha}_{1BSEL} = \omega\hat{\alpha}_{1ML} + (1-\omega)\hat{\alpha}_{1SEL},$$

$$\hat{\alpha}_{2BSEL} = \omega\hat{\alpha}_{2ML} + (1-\omega)\hat{\alpha}_{2SEL}, \qquad \textbf{(23)}$$

$$\hat{\sigma}_{1BSEL} = \omega\hat{\sigma}_{1ML} + (1-\omega)\hat{\sigma}_{1SEL},$$

$$\hat{\sigma}_{2BSEL} = \omega\hat{\sigma}_{2ML} + (1-\omega)\hat{\sigma}_{2SEL},$$

$$\hat{R}_{BSEL}(t) = \omega\hat{R}_{ML}(t) + (1-\omega)\hat{R}_{SEL}(t),$$

and

$$\hat{h}_{BSEL}(t) = \omega\hat{h}_{ML}(t) + (1-\omega)\hat{h}_{SEL}(t).$$

4.1 Maximum likelihood estimation:

4.1.1Point estimation:

Considering that $T_{(1)}, T_{(2)}, \dots, T_{(r)}$ are r lifetimes out of a Type II censored sample of size n from MTLR distribution, then the likelihood function is given by

$$L(\underline{\theta}|\underline{t}) \propto \left[\prod_{i=1}^{r} f(t_i)\right]\left[R(t_{(r)})\right]^{n-r}, \qquad (24)$$

where $f(t)$ and $R(t)$ are given, respectively, by (9) and (10), also the components $f_i(t)$ and $R_j(t)$ are defined, respectively, in (6) and (7), the vector $\underline{\theta} = (\alpha_1, \sigma_1, \alpha_2, \sigma_2, p)$ and

$\underline{t} = (t_1, \dots, t_r)$ are the ordered lifetimes.

Hence, one obtains

$$L(\underline{\theta}; \underline{t}) \propto \prod_{i=1}^{r}\{\left[p\,\alpha_1\left(\frac{t_i}{\sigma_1^{\,2}}\right)\exp\left(-\left(\frac{t_i}{\sigma_1}\right)^2\right)\left(1-\exp(-\left(\frac{t_i}{\sigma_1}\right)^2\right)^{\alpha_1-1}\right]$$

$$+\left[(1-p)\,\alpha_2\left(\frac{t_i}{\sigma_2^{\,2}}\right)\exp\left(-\left(\frac{t_i}{\sigma_2}\right)^2\right)\left(1-\exp(-\left(\frac{t_i}{\sigma_2}\right)^2\right)^{\alpha_2-1}\right]\}$$

$$\times\left[p\left(1-\left(1-\exp\left(-\left(\frac{t_{(r)}}{\sigma_1}\right)^2\right)\right)^{\alpha_1}\right)+(1-p)\left(1-\left(1-\exp(-(\frac{t_{(r)}}{\sigma_2})^2)\right)^{\alpha_2}\right)\right]^{n-r}. \quad (25)$$

The natural logarithm of the likelihood function is

$$l \equiv lnL(\underline{t};\underline{\theta}) = \sum_{i=1}^{r} ln\{\left[p\,\alpha_1\left(\frac{t_i}{\sigma_1^2}\right)\exp\left(-\left(\frac{t_i}{\sigma_1}\right)^2\right)\left(1-\exp(-\left(\frac{t_i}{\sigma_1}\right)^2\right)^{\alpha_1-1}\right]$$

$$+\left[(1-p)\,\alpha_2\left(\frac{t_i}{\sigma_2^2}\right)\exp\left(-\left(\frac{t_i}{\sigma_2}\right)^2\right)\left(1-\exp(-\left(\frac{t_i}{\sigma_2}\right)^2\right)^{\alpha_2-1}\right]\}$$

$$+(n-r)ln\left[p\left(1-\left(1-\exp\left(-\left(\frac{t_{(r)}}{\sigma_1}\right)^2\right)\right)^{\alpha_1}\right)+(1-p)\left(1-\left(1-\exp(-(\frac{t_{(r)}}{\sigma_2})^2)\right)^{\alpha_2}\right)\right]. \quad (26)$$

The maximum likelihood (ML) estimators are derived by setting the first partial derivatives of l with respect to $\alpha_1, \sigma_1, \alpha_2, \sigma_2$ and p respectively, to zeros.

Differentiating with respect to $\alpha_1, \sigma_1, \alpha_2, \sigma_2$ and p, a system of non-linear equations are derived and setting to zeros, the ML estimators can be derived using Newton-Raphson method. Also, the ML estimators of the rf and the hrf are obtained using the invariance of ML estimators based on (7) and (8), respectively.

The ML estimators have an asymptotic variance-covariance matrix defined by inverting the information matrix. The asymptotic variance-covariance matrix of the estimators $\hat{\alpha}_1, \hat{\sigma}_1, \hat{\alpha}_2, \hat{\sigma}_2$ and $\hat{p}$ are obtained depending on the inverse asymptotic Fisher information matrix $\tilde{I}$ using the second derivatives of the logarithm of the likelihood function.

The asymptotic Fisher information matrix can be written as follows:

$$\tilde{I} \approx \left[\frac{\partial^2 l}{\partial\theta_i \partial\theta_j}\right], \qquad i,j = 1,2,\ldots,5, \quad (27)$$

where $\theta_1 = \alpha_1$, $\theta_2 = \alpha_2$, $\theta_3 = \sigma_1$, $\theta_4 = \sigma_2$ and $\theta_5 = p$.

4.1.2 The Asymptotic confidence intervals:

For large sample size, the ML estimators under appropriate regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the two sided approximate $100(1-\tau)\%$ **confidence intervals for the ML estimator say,** $\hat{\theta}_i$ **of a population value** θ_i **can be obtained by** $P(-Z \le \frac{\hat{\theta}_i-\theta_i}{\sqrt{V_{\hat{\theta}_i}}} \le Z) = (1-\tau)$ **where Z is the** $100\left(1-\frac{\tau}{2}\right)$**th standard normal percentile. The two sided approximate** $100(1-\tau)\%$ **confidence intervals for** θ_i**, will be given as follows:**

$$L_{\theta_i} = \hat{\theta}_i - Z_{\frac{\tau}{2}}\sqrt{V_{\hat{\theta}_i}}, \qquad and \quad U_{\theta_i} = \hat{\theta}_i + Z_{\frac{\tau}{2}}\sqrt{V_{\hat{\theta}_i}},$$ **i=1, 2, ..,5,**

where $\sqrt{V_{\hat{\theta}_i}}$ **is the standard deviation and** $\hat{\theta}_i$ **is** $\hat{\alpha}_1, \hat{\sigma}_1, \hat{\alpha}_2, \hat{\sigma}_2, \hat{p}$**, rh or hrf respectively.**

4.2 Bayesian estimation under square error loss function:

4.2.1Point estimation:

In this section, Bayes estimators $\hat{u}_{SEL}$ **of a function** $\hat{u}(\theta)$ **for the unknown vector of parameters;** $\underline{\theta} = (\alpha_1, \sigma_1, \alpha_2, \sigma_2, p)$**, of MTLR distribution is considered.**

Assuming that the prior belief of the experimenter is that $p \sim beta(a_1, a_2)$**,** $\alpha_1 \sim Gamma(b_1, b_2)$**,** $\alpha_2 \sim Gamma(b_3, b_4)$**,** $\sigma_1 \sim Gamma(b_5, b_6)$ **a** $\sigma_2 \sim Gamma(b_7, b_8)$**.**

Suppose that the parameters are independent, then the joint prior density function is

$$\pi(\underline{\theta}) \propto p^{a_1-1}(1-p)^{a_2-1}$$

$$* \alpha_1^{b_1-1}\, \alpha_2^{b_3-1}\, \sigma_1^{b_5-1} \sigma_2^{b_7-1}\, e^{-(b_2\alpha_1 + b_4\alpha_2 + b_6\sigma_1 + b_8\sigma_2)}. \qquad \textbf{(28)}$$

The joint posterior density function can be obtained from (25) and (28) as

$$\pi(\underline{\theta}|\underline{t}) \propto L(\underline{\theta}|\underline{t})\pi(\underline{\theta})$$

$$= p^{n+\alpha_1-1}(1-p)^{\alpha_2-1}\,\alpha_1^{n+b_1-1}e^{-\alpha_1 b_2}\,\alpha_2^{b_3-1}e^{-\alpha_2 b_4}\,\sigma_1^{n+b_5-1}e^{-\sigma_1 b_6}\,\sigma_2^{b_7-1}e^{-\sigma_2 b_8}$$

$$\times \prod_{i=1}^{r}\left[\left(\frac{t_i}{\sigma_1{}^2}\right)\exp\left(-\left(\frac{t_i}{\sigma_1}\right)^2\right)\left(1-\exp(-\left(\frac{t_i}{\sigma_1}\right)^2\right)^{\alpha_1-1}\right]$$

$$+p^{\alpha_1-1}(1-p)^{n+\alpha_2-1}\,\alpha_1^{b_1-1}e^{-\alpha_1 b_2}\,\alpha_2^{n+b_3-1}e^{-\alpha_2 b_4}\,\sigma_1^{b_5-1}\,e^{-\sigma_1 b_6}\,\sigma_2^{n+b_7-1}e^{-\sigma_2 b_8}$$

$$\times \prod_{i=1}^{r}\left[\left(\frac{t_i}{\sigma_2{}^2}\right)\exp\left(-\left(\frac{t_i}{\sigma_2}\right)^2\right)\left(1-\exp(-\left(\frac{t_i}{\sigma_2}\right)^2\right)^{\alpha_2-1}\right]$$

$$\times\left[p\left(1-\left(1-\exp\left(-\left(\frac{t_{(r)}}{\sigma_1}\right)^2\right)\right)^{\alpha_1}\right)+(1-p)\left(1-\left(1-\exp(-\left(\frac{t_{(r)}}{\sigma_2}\right)^2)\right)^{\alpha_2}\right)\right]^{n-r}. \qquad \textbf{(29)}$$

The marginal posteriors of $\alpha_1, \sigma_1, \alpha_2, \sigma_2$ and p can be obtained by integrating the joint posterior distribution given by (29) with respect to the other parameters, that is the marginal posterior density is given by

$$\pi_j^*(\underline{\theta} \mid \underline{t}) = \int\int_{\theta_i} \pi(\underline{\theta} \mid \underline{t})\, d\theta_i, \qquad i,j = 1,2,3,\ i \neq j. \qquad \textbf{(30)}$$

Hence, the Bayes estimators under SEL function is given by the mean of the posterior distribution, and can be derived as follows:

$$\theta_{j(SE)}^* = E(\theta_j \mid \underline{t}) = \int_{\underline{\theta}} \theta_j\, \pi_j^*(\underline{\theta} \mid \underline{t})\, d\theta_j, \quad j = 1,2,\dots,5, \qquad \textbf{(31)}$$

where $\int_{\underline{\theta}} = \int_{\theta_1}\int_{\theta_2}\int_{\theta_3}\int_{\theta_4}\int_{\theta_5}$, $\theta_1 = \alpha_1,\ \theta_2 = \sigma_1,\ \theta_3 = \alpha_2, \theta_4 = \sigma_2$ and $\theta_5 = p$.

The Bayes estimators of the rf and the hrf under SEL function; which are the posterior expectations, can be obtained as follows:

$$R_{(SE)}^*(t) = E\left(R(t)\middle|\underline{t}\right) = \int_{\underline{\theta}} R(t)\,\pi(\underline{\theta} \mid \underline{t})\, d\underline{\theta}, \qquad \textbf{(32)}$$

and

$$h_{(SE)}^*(t) = E\left(h(t)\middle|\underline{t}\right) = \int_{\underline{\theta}} h(t)\,\pi(\underline{\theta} \mid \underline{t})\, d\underline{\theta}, \qquad \textbf{(33)}$$

where $\int_{\underline{\theta}} = \int_{\theta_1}\int_{\theta_2}\int_{\theta_3}\int_{\theta_4}\int_{\theta_5}$ and $d\underline{\theta} = d\theta_5\, d\theta_4\, d\theta_3\, d\theta_2\, d\theta_1$.

Equations (31-33) can be solved numerically to obtain the Bayes estimates of the parameters, rf and hrf of the MTLR distribution based on SEL function.

4.2.2Credible intervals

The Bayesian analog to the confidence interval is called a credibility interval. In general, $(L(\underline{t}), U(\underline{t}))$ is $100(1-\tau)\%$ credibility interval of $\underline{\theta}$ if

$$P[L(\underline{t}) < \underline{\theta} < U(\underline{t}) | \underline{t}] = \int_{L(\underline{t})}^{U(\underline{t})} \pi^*(\underline{\theta}|\underline{t})\, d\underline{\theta} = 1 - \tau. \qquad (34)$$

Using the marginal posterior distribution given by (30), then a $100(1-\tau)\%$ credibility interval for θ_j is $(L_j(\underline{t}), U_j(\underline{t}))$, where

$$P[\theta_j > L_j(\underline{t}) | \underline{t}] = \int_{L_j(\underline{t})}^{\infty} \pi_j^*(\underline{\theta} \mid \underline{t})\, d\theta_j = 1 - \frac{\omega}{2}, \quad j = 1,2,\dots,5, \qquad (35)$$

and

$$P[\theta_j > U_j(\underline{t}) | \underline{t}] = \int_{U_j(\underline{t})}^{\infty} \pi_j^*(\underline{\theta} \mid \underline{t})\, d\theta_j = \frac{\tau}{2}, \quad j = 1,2,\dots,5\ . \qquad (36)$$

Remarks

- **When r = n all the results obtained for Type II censoring reduce to the complete sample case**
- **If a mixing proportion is zero (or one) in a finite mixture of two components, then the mixture reduces to a single population case.**
- **Finite mixtures are more appropriate to represent heterogeneous population**.

5. Numerical Illustration:

In this section, Monte Carlo simulation study is conducted to illustrate the performance of the presented ML and Bayes estimates on the basis of generated data from the MTLR distribution. Absolute biases of the parameters, rf and hrf based on Type II censoring are computed. Moreover, confidence and credible intervals of the parameters are calculated, all the results are obtained using R programming language.

a. **For given values of** $\alpha_1, \sigma_1, \alpha_2, \sigma_2$ and p **random samples of size n are generated from a finite mixture of two TLR components whose pdf is given in (6) and (17) observing that if U1 and U2 is uniform distributions (0,1), then**

$$t = \begin{cases} (\log[1 - u_1^{1/\alpha_1}]^{-\sigma_1^2})^{1/2}, & u_1 \le p, \\ (\log[1 - u_2^{1/\alpha_2}]^{-\sigma_2^2})^{1/2}, & u_2 > p. \end{cases}$$

b. **The population parameter values of** $\alpha_1, \sigma_1, \alpha_2, \sigma_2$ and p **used in the simulation are (**$\alpha_1 = 2, \sigma_1 = 3, \alpha_2 = 3, \sigma_2 = 4$ and $p = 0.4$**) and** $t_0 = 1$**.**

c. **For each sample size** $n = 50$, **sort the** $t_i{}'s$**, such that** $t_1 < t_2 < \cdots < t_n$**.**

d. **Choose the number of failures** r **to be less than or equal to the sample size.** n

e. **The number of failures** $r = 45$ (90% of n) **out of the** $n = 50$ **observations, which are assumed to be known. Hence, one obtains** r_1**observations from the first component of the mixture and** r_2**from the second component (**$r = r_1 + r_1 = 45$).

f. **Repeat all the previous steps N=5000 times where N represents a fixed number of simulated samples.**

g. **The ML and Bayes averages of** $\alpha_1, \sigma_1, \alpha_2, \sigma_2, p$**, rf and hrf are computed. The Bayes averages are obtained under SEL function. An estimator of a function using BSEL function is a mixture of the ML and the Bayes estimator using the SEL function and the weight** ω**.**

h. **The estimates of the parameters, rf and hrf under BSEL function are displayed in Table 1 for different weights** ω**, where** $\omega = 0, 0.2, 0.4, 0.6, 0.8, 1$

Table 1. Estimates and absolute biases of the parameters, rf and hrf under BSEL function

$\underline{\theta}$	$\omega = 0$ "B-SEL"	$\omega = 0.2$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$	$\omega = 1$ "MLE"
α_1	1.9998 0.00002	2.11234 0.00004	2.19653 0.00015	2.26987 0.00020	2.32781 0.00023	2.41212 0.00380
σ_1	3.00098 0.00310	3.18859 0.00537	3.27526 0.00590	3.29743 0.00748	3.41272 0.00789	3.52891 0.00946
α_2	2.99997 0.00001	3.12314 0.00002	3.27908 0.00509	3.36582 0.00889	3.37933 0.01018	3.49578 0.01779
σ_2	4.00975 0.00161	4.12130 0.00042	4.18964 0.00074	4.29103 0.00223	4.36941 0.00484	4.38573 0.00845
p	0.40007 0.00029	0.42000 0.00015	0.43855 0.00011	0.44631 0.00143	0.45969 0.00207	0.46411 0.00302
$R(t_0)$	0.21009 0.00121	0.23316 0.00272	0.25324 0.00463	0.26598 0.00580	0.30078 0.00656	0.31211 0.00820
$h(t_0)$	2.1101 0.00019	2.21579 0.00186	2.24387 0.00437	2.27796 0.00682	2.48956 0.00834	2.58632 0.00859

Concluding Remarks

- **In this article, ML and Bayes estimators; based on Type II censored data, are derived for the unknown parameters, reliability and hazard rate functions of a finite mixture of two MTLR components. Also, confidence and credible intervals are obtained.**
- **The estimators are derived under BSEL as an asymmetric loss function and is a weighted average for two loss functions; one reflects goodness of fit and the other reflects the precision of the estimation.**

- **The BSEL function as an asymmetric loss function is considered a compromise between Bayes and non-Bayes estimates.**
- **It is noticed that when $\omega = 1$, one gets the ML estimates, while when $\omega = 0$, one obtains the Bayes estimates under SEL function.**

References

1. Ahmad, K. E. (1988). Identifiability of finite mixtures using a new transform. Annals of Institute of Statistical Mathematics, 40(2), 261-265.
2. Aryal, G., R., Ortega, E. M., Hamedani, G. G. and Yousof, H. M. (2017). The Topp- Leone generated Weibull distribution: regression Model, Characterizations and Applications. International Journal of Statistics and Probability; 6 (1).
3. AL-Hussaini, E. K. and Ahmad, K. E. (1981). On the identifiability of finite mixtures of distributions. IEEE Transactions on Information Theory, IT-27(5), 664-668.
4. AL-Hussaini, E. K. and Ahmad, K. E. (1984). Information matrix for a mixture of two inverse Gaussian distributions. Communications in Statistics-Simulation and Computation 13, 785-800.
5. AL-Hussaini, E. K., AL-Dayian, G. R. and Adham, S. A. (2000). On finite mixture of two-component Gompertz lifetime model. Journal of Statistical Computation and Simulation, 67, 1-20.
6. AL-Hussaini, E. K., and Sultan, K. S. (2001). Reliability and hazard based on finite mixture models in Advances in Reliability, N. Balakrishnan and C. R. Rao, Eds., of Handbook of Statistics, 20, 139–183, Amsterdam, The Netherlands, North-Holland.
7. AL-Hussaini, E. K. and Hussein, M. (2011). Estimation using censored data from exponentiated Burr Type XII Population. American Open Journal of Statistics, 1, 33-45.
8. AL-Hussaini, E. K. (2012). Composition of cumulative distribution functions. Journal of Statistical Theory and Applications, 11(4), 323-336.
9. Baharith, L. A. (1997). On the four-parameter Burr type XII lifetime model. Master's Thesis, King Abdulaziz University, Jeddah-Kingdom of Saudi Arabia.
10. Bayoud, H. A. (2015). Admissible minimax estimators for the shape parameter of Topp-Leone distribution. Communications in Statistics - Theory and Methods, 45 (1):71–82.

11. Berger, J. O. (1980). Statistical decision theory: Foundations, concepts and methods, New York: Springer-Verlag.

12. Curelaru and Vod$\tilde{\tilde{a}}$ (1975). Some notes on Rayleigh distribution. Revista Colombian a de Matem$\acute{a}$ticas IX, 9-22

13. Everitt, B. S. and Hand, D. J. (1981). Finite mixture distributions. Springer, London, New York.

14. Feroze, N. and Aslam, M. (2013). On Bayesian analysis of failure rate under Topp-Leone distribution using complete and censored samples. International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 7(3).

15. Genç, A. I. (2012). Moments of order statistics of Topp-Leone distribution. Statistical Papers, 53 (1):117–31.

16. Ghitany, M. E., Kotz, S. and Xie, M. (2005). On some reliability measures and their stochastic orderings for the Topp-Leone distribution. Journal of Applied Statistics, 32 (7):715–22.

17. Lindsay, B. G. (1995). Mixture Models: Theory, Geometry and Applications. The Institute of Mathematical Statistics, Hayward, Calif, USA.

18. Li, L. (2016). Bayes estimation of Topp-Leone distribution under symmetric entropy loss function based on lower record values. Science Journal of Applied Mathematics and Statistics, 4 (6):284–88.

19. McLachlan, G. J. and Basford, K. E. (1988). Mixture models: applications to clustering, Marcel Dekker, New York, New York, USA.

20. McLachlan, G. J. and Krishnan, T. (1997). The EM algorithm and extensions, John Wiley and Sons, New York, New York, USA.

21. McLachlan, G. J. and Peel, D. (2000). Finite mixture models, John Wiley and Sons, New York, New York, USA.

22. Nadarajah, S. and Kotz, S. (2003). Moments of some J-shaped distributions. Journal of Applied Statistics, 30:311–17.

23. Scäbe, H. (1991). Bayes estimates under asymmetric loss. IEEE Transactions on Reliability, 19, 13-16.

24. Sindhu, T. N., Saleemb, M. and Aslam, M. (2013). Bayesian estimation for Topp-Leone distribution under trimmed samples. Journal of Basic and Applied Scientific Research 3(10), 347-360,

25. Sultan, H. and Ahmad, S.P. (2016). Bayesian analysis of Topp-Leone distribution under different loss functions and different priors. Journal of Statistics Applications and Probability Letters, 3(3), 109-118.

26. Thompson, R. D. and Basu, A. P. (1996). Asymmetric loss Function for Estimating System Reliability, In: D. A. Berry, K. M. Chaloner and J. K. Geweke, Eds., Bayesian Analysis in Statistics and Econometrics, Wiley Series in Probability and Statistics, New York.

27. Titterington, D. M., Smith A. F. M. and Makov,U. E. (1985). Statistical Analysis of Finite Mixture Distributions. John Wiley and Sons.

28. Topp, C. W., and Leone, F. C. (5). A family of J-shaped frequency functions. Journal of the American Statistical Association, 50: 209–19.

29. Varian, R. (1975). A Bayesian approach to real estate assessment. In studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage, eds. Stephen E. Fienberg and Arnold Zellner, Amsterdam : North-Holland, -208.

30. Zellner, A. (1994). Bayesian and non-Bayesian estimation using balanced loss functions. In: Berger, J. O., Gupta, S. S. Eds., Statistical Decision Theory and Related Topics V. Springer-Verlag, New York, 337–390.

31. Zghoul, A. A. (2010). Order statistics from a family of J-shaped distributions. Metron, 68 (2):127–36.

32. Zghoul, A. A. (2011). Record values from a family of J-shaped distributions. Statistica, 71 (3):355–65.