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Derivation of Optimal Guidance and Control Problem

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Abstract: New tactical missile requirements are so stringent that weapon subsystem technology must be utilized at the highest possible level consistent with cost, reliability, and performance. This is particularly true with the guidance and control subsystems, which are the nerve center or backbone of a weapon. As a result of this, there is continuing requirement for more and better tools for analyzing performance, predicting requirements, determining error sources, and selecting suitable concepts. Among these concepts is the optimal guidance and control which is indispensable for advanced guidance processes where ever-increasing performance requirements are to be achieved with minimum control or actuation and minimum cost. This cost is the performance index that mathematically weights different flight variables including the time constraints, the guidance commands or actuating signals or commanded acceleration and miss distance especially near the target interception. That is, tackling such problems necessitates formulation, solution and design with synthesis which is mathematically cumbersome and boring for researchers. Therefore, this paper is devoted to formulate the problem in a systematic and concise approach with detailed and complete derivation of riccatti equations and its impact on the controller\autopilot design. Then, the theory is also derived for the regulator and servomechanism\ tracking problems. Each theory is augmented with analytical case study to clarify the impact of optimality and riccatti equation solution upon the system's performance.

Keywords: Guidance and Control, Optimal Control, Riccatti Equations

1. Introduction

Among the huge control design techniques is the optimal control in which a performance index (P.I.) is to be specified for penalizing the states, the tracking error the control effort or both in accordance to the performance requirements. That is, the key to a successful problem formulation is the translation of performance requirements into the mathematical P.I. No matter what theoretical techniques are used to develop the optimal control strategy, they will always be based on the minimization (or maximization) of that performance index which necessitates the translation of its ingredients into concise mathematical terms. In addition to the performance index, two other formulations will impact the optimal control strategy, namely, the mathematical model of the system and the additional equality and inequality constraints to be placed on the system.

In general, a more detailed system model results in a more accurate control strategy, but this is achieved at the expense of additional complexity in both the derivation and resulting algorithms. The selection of appropriate equality/inequality constraints can be based upon either actual system parameters or trajectory properties (operational characteristics) that the

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optimal solution should possess. Some of the modern control techniques that have been investigated and/or applied are (1) reachable set theory, (2) singular perturbation theory, (3) differential game theory, (4) robust control theory and (5) adaptive control theory.

The performance index (cost functional) study is a fundamental but extremely complex problem as the choice of the parameters/states constituting the performance index is influenced by the performance objectives and the interrelationships of the steps involved in the modern control problem formulation. Specifically, for every different performance index or cost functional there is a different optimal guidance law, the measure of its performance will be the ability of the missile to intercept the target with, say, minimum terminal miss distance in various engagement scenarios. Additional measures of performance involve considerations about (1) launch envelope, (2) fuel considerations, (3) flight time, (4) power supply, (5) maneuver capability and (6) external disturbances.

Unfortunately, the cited pertinent literature (more than the listed) presents the theory in a discrete and very complex form such that applicants and researchers got bored from tailoring this theory to their real applications. In addition, it will be very difficult to own the know how in this context and consequently lagging from time to time in an era of ever-increasing requirements with accelerating technologies. Therefore, this paper is devoted to present a novel derivation for the state-space optimal control theory starting from the P.I. until the riccatti equation and the optimal controller with implementation and evaluation. The novelty of this approach stems from formulating the problem in a systematic and concise detailed approach towards the riccatti equations and its impact on the controller design. The theory is tailored for the two control problems, regulator and servomechanism\ tracking, augmented with analytical case study for each of them.

2. Optimal Control Theory

This section is devoted to the derivation of the optimal control problem via following the Hamiltonian and Lagrangian approach.

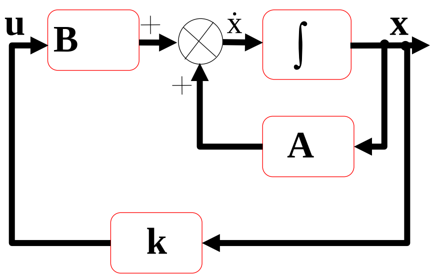

Fig. 1: State Feedback Closed Loop

2.1 Proof

The optimal control problem is to find an admissible control vector $u^* \in U$ that makes the system $\dot{X} = f(x,u,t)$ follow an admissible state trajectory $x^* \in X$ such that the following performance index (P.I.) is minimized

$$I = F\{X(t_f), t_f\} + \int_{t_o}^{t_f} L[x(t), u(t), t]\, dt \qquad (1a)$$

With the following assumptions:

- System state space
- The initial time t_o and the initial state $x(t_o) = x_o$ are specified
- The state and the control regions are not bounded

$\dot{X} = f(x,u,t)$ $\dot{X}(t) = A(t)\,X(t) + B(t)\,u(t)$ $Y(t) = C(t)\,X(t)$	(1b)

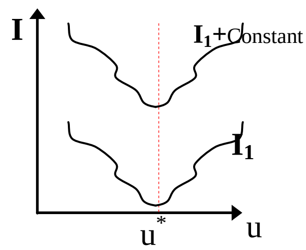

Fig. 2: Global minima

- The integral $\int_{t_o}^{t_f} \frac{d}{dt} F\{x(t), t\}\ dt = F\{x(t_f), t_f\} - F\{x(t_o), t_o\}$ (1c)
- Initial time t_o and final time is t_f

Thus, the performance index (P.I.) can be put in the following form:

$$I = \int_{t_o}^{t_f} \left\{ L[x(t), u(t), t] + \frac{d}{dt} F\{x(t), t\} \right\} dt + F\{x(t_o), t_o\} \tag{2a}$$

Where $F\{x(t_o), t_o\}$ is a constant term and it does not affect the calculation of the P.I. and its value because it is independent of the control vector u. Thus, this term can be discarded from subsequent discussions and consequently the P.I. is shortly rewritten as;

$$I = \int_{t_o}^{t_f} \left\{ L + \frac{dF}{dt} \right\} dt \tag{2b}$$

That is

$$I = \int_{t_o}^{t_f} \left\{ L[x(t), u(t), t] + \frac{\partial}{\partial x}[F\{x(t), t\}]^T \dot{X} + \frac{\partial}{\partial t}[F\{x(t), t\}] \right\} dt \tag{3}$$

The solution of the state equation (1b), $\dot{X} = f(x, u, t)$, is obtained as follows

$$0 = f[X(t), u(t), t] - \dot{X} \tag{4a}$$

This relation can be adjoined to the cost function by means of a Lagrangian multiplier λ as follows:

$$\lambda^T \{f(x, u, t) - \dot{X}\} = 0 \tag{4b}$$

Thus the augmented P.I. has the following form:

$$I = \int_{t_o}^{t_f} \left\{ L[x(t), u(t), t] + \frac{\partial}{\partial x}[F\{x(t), t\}]^T \dot{X} + \frac{\partial}{\partial t}[F\{x(t), t\}] + \lambda^T [f\{x(t), t\} - \dot{X}] \right\} dt \tag{5a}$$

$$L_a\{x, \dot{x}, u, \lambda, t\} = \left\{ L[x(t), u(t), t] + \frac{\partial}{\partial x}[F\{x(t), t\}]^T \dot{X} + \frac{\partial}{\partial t}[F\{x(t), t\}] + \lambda^T [f\{x(t), t\} - \dot{X}] \right\} \tag{5b}$$

Where, $X = [x_1 \quad x_2 \quad .... \quad x_n]^T$, $\frac{\partial F}{\partial x} = \begin{bmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & .... & \frac{\partial F}{\partial x_n} \end{bmatrix}^T$, $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} f_1 - \dot{x}_1 \\ f_2 - \dot{x}_2 \\ \vdots \\ f_n - \dot{x}_n \end{bmatrix}$

That is augmented P.I. becomes as

$$I_a = \int_{t_o}^{t_f} L_a\{x, \dot{x}, u, \lambda, t\}\ dt \tag{6}$$

Considering the perturbations ($\delta_x, \delta_{\dot{x}}, \delta_u, \delta_\lambda, \delta_{t_f}$) the perturbation in the P.I. should be zero and it is given as follows:

$$\delta_I(u^*) = \int_{t_o}^{t_f} \left\{ \left[\frac{\partial L_a}{\partial x}\right]^T \delta_x + \left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_{\dot{x}} + \left[\frac{\partial L_a}{\partial u}\right]^T \delta_u + \left[\frac{\partial L_a}{\partial \lambda}\right]^T \delta_\lambda \right\} dt + L_a\{x^*(t_f), \dot{x}^*(t_f), u^*(t_f), \lambda^*(t_f), t_f\}\, \delta_{t_f} \tag{7a}$$

$$I_a = I_a + \delta_I \qquad \& \qquad L_a^f = L_a + \delta_{L_a} = L_a + \dot{L}_a\ \delta_{t_f} \tag{7b}$$

The integration of the derivative of two multiplied functions;

$$\int_{t_o}^{t_f} \frac{d}{dt}\left\{\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x\right\} dt \;=\; \int_{t_o}^{t_f}\left\{\frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x + \left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \frac{d}{dt}\delta_x\right\} dt$$
$$= \int_{t_o}^{t_f}\left\{\frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x + \left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_{\dot{x}}\right\} dt \tag{8}$$

Also this integration can be carried out normally as follows:

$$\int_{t_o}^{t_f} \frac{d}{dt}\left\{\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x\right\} dt = \left\{\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x\right\}_{t_o}^{t_f} = \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \delta_x(t_f) - \left[\frac{\partial L_a(t_o)}{\partial \dot{x}}\right]^T \delta_x(t_o) \tag{9}$$

Considering $\delta_x(t_o) = 0$ yields that Eq^n (9) becomes

$$\int_{t_o}^{t_f} \frac{d}{dt}\left\{\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x\right\} dt = \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \delta_x(t_f) \tag{10}$$

From Eq^n (8) and Eq^n (10) it is clear that

$$\int_{t_o}^{t_f}\left\{\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_{\dot{x}}\right\} dt = \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \delta_x(t_f) - \int_{t_o}^{t_f}\left\{\frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x\right\} dt \tag{11}$$

Substituting Eq^n (11) into Eq^n (7) yields

$$\delta_I(u^*) = 0 = \int_{t_o}^{t_f}\left\{\left[\frac{\partial L_a}{\partial x}\right]^T \delta_x - \frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x + \left[\frac{\partial L_a}{\partial u}\right]^T \delta_u + \left[\frac{\partial L_a}{\partial \lambda}\right]^T \delta_\lambda\right\} dt$$
$$+ \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \delta_x(t_f) + L_a\{x^*(t_f), \dot{x}^*(t_f), u^*(t_f), \lambda^*(t_f), t_f\}\,\delta_{t_f} \tag{12}$$

Since, $\delta_{X_f} = \delta_X(t_f) + \dot{X}(t_f)\,\delta_{t_f}$; then $\delta_X(t_f) = \delta_{X_f} - \dot{X}(t_f)\,\delta_{t_f}$ and consequently, Eq^n (12) can be written as follows:

$$\delta_I(u^*) = 0 = \int_{t_o}^{t_f}\left\{\left[\frac{\partial L_a}{\partial x}\right]^T \delta_x - \frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x + \left[\frac{\partial L_a}{\partial u}\right]^T \delta_u + \left[\frac{\partial L_a}{\partial \lambda}\right]^T \delta_\lambda\right\} dt \tag{13}$$
$$+ \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \{\delta_{X_f} - \dot{X}(t_f)\,\delta_{t_f}\} + L_a\{x^*(t_f), \dot{x}^*(t_f), u^*(t_f), \lambda^*(t_f), t_f\}\,\delta_{t_f}$$

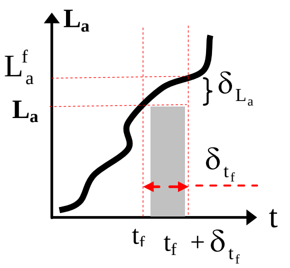

Fig. 3: Function perturbation

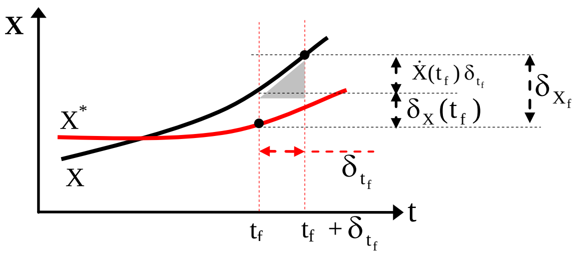

Fig. 4: Function increments

Rearranging this equation yields

$$\delta_I(u^*) = 0 = \int_{t_o}^{t_f} \left\{ \left[\frac{\partial L_a}{\partial x}\right]^T \delta_x - \frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T \delta_x + \left[\frac{\partial L_a}{\partial u}\right]^T \delta_u + \left[\frac{\partial L_a}{\partial \lambda}\right]^T \delta_\lambda \right\} dt$$
$$+ \left\{ L_a\{x^*(t_f), \dot{x}^*(t_f), u^*(t_f), \lambda^*(t_f), t_f\} - \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \dot{X}(t_f) \right\} \delta_{t_f}$$
$$+ \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \delta_{X_f} \qquad (14)$$

From this equation, the following equalities can be considered:

$$\left[\frac{\partial L_a}{\partial x}\right]^T - \frac{d}{dt}\left[\frac{\partial L_a}{\partial \dot{x}}\right]^T = 0 \qquad (15\text{-a})$$

$$\frac{\partial L_a}{\partial u} = 0 \qquad (15\text{-b})$$

$$\frac{\partial L_a}{\partial \lambda} = 0 \qquad (15\text{-c})$$

$$\left\{ L_a\{x^*(t_f), \dot{x}^*(t_f), u^*(t_f), \lambda^*(t_f), t_f\} - \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \dot{X}(t_f) \right\} \delta_{t_f} + \left[\frac{\partial L_a(t_f)}{\partial \dot{x}}\right]^T \delta_{X_f} = 0 \qquad (15\text{-d})$$

Since λ is a function of time;

$$L_a = L + \frac{\partial F}{\partial x}\dot{X} + \frac{\partial F}{\partial t} + \lambda^T[f - \dot{X}] \qquad (16\text{-a})$$

$$\frac{\partial L_a}{\partial x} = \frac{\partial L}{\partial x} + \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{X} + \frac{\partial}{\partial x}\frac{\partial F}{\partial t} + \lambda^T \frac{\partial f}{\partial x} \qquad \text{(partial derivative of (16-a) w.r.t. x)} \qquad (16\text{-b})$$

$$\frac{\partial L_a}{\partial \dot{x}} = \frac{\partial F}{\partial x} - \lambda^T \qquad \text{(partial derivative of (16-a) w.r.t. } \dot{x}\text{)} \qquad (16\text{-c})$$

Then, Eq^n (15-a) with the definition of L_a (16) yield

$$\frac{\partial L_a}{\partial x} - \frac{d}{dt}\left\{\frac{\partial L_a}{\partial \dot{x}}\right\} = 0$$

$$\frac{\partial L}{\partial x} + \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{X} + \frac{\partial}{\partial x}\frac{\partial F}{\partial t} + \lambda^T \frac{\partial f}{\partial x} - \frac{d}{dt}\left\{\frac{\partial F}{\partial x} - \lambda^T\right\} = 0$$

Then

$$\frac{\partial L}{\partial x} + \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{X} + \frac{\partial}{\partial x}\frac{\partial F}{\partial t} + \lambda^T \frac{\partial f}{\partial x} - \left\{\frac{\partial}{\partial x}\frac{\partial F}{\partial x}\dot{X} + \frac{\partial}{\partial t}\frac{\partial F}{\partial x} - \dot{\lambda}^T\right\} = 0$$

$$\rightarrow \frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda} = 0 \qquad (17\text{-a})$$

$$\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0 \qquad \text{\{using Eq}^n \text{ (15-b) with the definition of } L_a \text{ (16)\}} \qquad (17\text{-b})$$

$$f(x, u, t) - \dot{X} = 0 \qquad \text{\{using Eq}^n \text{ (15-c) with the definition of } L_a \text{ (16)\}} \qquad (17\text{-c})$$

$$\left\{L(t_f)+\cancel{\frac{\partial F(t_f)}{\partial x}}\dot{X}(t_f)+\frac{\partial F(t_f)}{\partial t}+\lambda^T(t_f)\,f(t_f)-\lambda^T\xcancel{(t_f)}\,\cancel{\dot{X}(t_f)}-\cancel{\frac{\partial F(t_f)}{\partial x}}\dot{X}(t_f)+\lambda^T\xcancel{(t_f)}\cancel{\dot{X}(t_f)}\right\}\delta_{t_f}$$
$$+\left[\frac{\partial F(t_f)}{\partial x}-\lambda(t_f)\right]^T\delta_{X_f}=0$$

i.e.

$$\left\{L(t_f)+\lambda^T(t_f)\,f(t_f)+\frac{\partial F(t_f)}{\partial t}\right\}\delta_{t_f}+\left[\frac{\partial F(t_f)}{\partial x}-\lambda(t_f)\right]^T\delta_{X_f}=0 \qquad (18)$$

Let us define the Hamiltonian matrix as follows:

$$H=L(x,u,t)+\lambda^T f(x,u,t) \qquad (19)$$

Then the following equations can be obtained

State equation $\dot{X}=\frac{\partial H}{\partial \lambda}$ $\{Eq^n$ (19) yields $\frac{\partial H}{\partial \lambda}=f(x,u,t)=\dot{X}\,\}$

Costate equation $\dot{\lambda}=-\frac{\partial H}{\partial x}$ $\{Eq^n$ (17-a) yields $\dot{\lambda}=-\left[\frac{\partial L}{\partial x}+\lambda^T\frac{\partial f}{\partial x}\right]\overset{(19)}{=}-\frac{\partial H}{\partial x}\,\}$

Control equation $0=\frac{\partial H}{\partial u}$ $\{Eq^{ns}$ (17-b) and (19) $\}$ (20)

Boundary conditions obtained by substituting (19) into (18) to yield

$$\left\{H\{x(t_f),.......,t_f\}+\frac{\partial F(t_f)}{\partial t}\right\}\delta_{t_f}+\left[\frac{\partial F(t_f)}{\partial x}-\lambda(t_f)\right]^T\delta_{X_f}=0$$

According to whether t_f and/or X_f are specified, there are different cases that can be tackled following different approaches.

2.2 Case Study-1

The fin drive of a guided missile is described by the following state equations;

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -x_2(t)+u(t)$$

and it is required to be controlled such that the control effort is conserved assuming that the admissible states and controls are not bounded.

(a) Find the necessary conditions that must be satisfied for optimal control
(b) Calculate and plot the optimal states and control functions for x(0)=0 and x(2)=[5 2]^T
(c) Solve part (b) for the following P.I.

$$I=\frac{1}{2}[x_1(2)-5]^2+\frac{1}{2}[x_2(2)-2]^2+\frac{1}{2}\int_0^2 u^2\,dt\text{, where } x(0)=0$$

Solution:

- It is clear that $t_o=0$ and $t_f=2$
- The cost function has the following form

$$I=F\{X(t_f),t_f\}+\int_{t_o}^{t_f}L[x(t),u(t),t]\,dt$$

with number of states n=2 and control m=1.

- For minimum control effort

$$I = \int_{t_o}^{t_f} [u^T \ R \ u] \ dt$$

Let R=1/2

Then $I = \frac{1}{2}\int_{t_o}^{t_f} u^2 \ dt$ with, F=0 and $L = u^2/2$

- The system is given by

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{X} = f(x, u, t)$$

- The Hamiltonian has the following form

$$\begin{aligned} H &= L(x,u,t) + \lambda^T f(x,u,t) \\ &= \frac{u^2}{2} + [\lambda_1 \quad \lambda_2] \begin{bmatrix} x_2 \\ -x_2 + u \end{bmatrix} \\ &= \frac{u^2}{2} + \lambda_1 x_2 + \lambda_2 [-x_2 + u] \end{aligned}$$

- The Costate equations $\dot{\lambda} = -\frac{\partial H}{\partial x}$ are

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 + \lambda_2$$

- The State equations $\dot{X} = \frac{\partial H}{\partial \lambda}$ are

$$\dot{x}_1 = \frac{\partial H}{\partial \lambda_1} = x_2$$

$$\dot{x}_2 = \frac{\partial H}{\partial \lambda_2} = -x_2 + u$$

- The Control equation $0 = \frac{\partial H}{\partial u}$ is

$$u + \lambda_2 = 0$$

i.e. $u = -\lambda_2$

- Substituting the control equation into the state equations and considering the other equations yield

$$\dot{\lambda}_1 = 0$$

$$\dot{\lambda}_2 = \lambda_2 - \lambda_1$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - \lambda_2$$

with boundary conditions $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x(2) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

- Since both t_f and x_f are specified, there is no need for the boundary condition; where $t_f = 2$ and $x(t_f) = x(2) = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$
- The integration of the above equations can be carried out as follows:

$$\dot{\lambda}_1 = 0 \quad \Rightarrow \quad \lambda_1 = c_1$$

$$\begin{aligned}
\dot{\lambda}_2 = \lambda_2 - c_1 \quad &\Rightarrow \quad \frac{1}{\lambda_2 - c_1} d\lambda_2 = dt \\
&\Rightarrow \quad \ell n(\lambda_2 - c_1) = t + \ell n(c_2) \\
&\Rightarrow \quad \frac{\lambda_2 - c_1}{c_2} = e^t \\
&\Rightarrow \quad \lambda_2 = c_2\, e^t + c_1
\end{aligned}$$

$$\dot{x}_2 = -x_2 - \lambda_2 \qquad \Rightarrow \qquad \dot{x}_2 + x_2 = -\lambda_2 = -c_2\, e^t - c_1$$

Since

$$\begin{aligned}
\dot{x} + p\,x = q \quad \Rightarrow \quad & e^{\int pdt}\, \dot{x} + e^{\int pdt}\, p\,x && = e^{\int pdt}\, q \\
\Rightarrow \quad & \int \frac{d}{dt}\left\{x\, e^{\int pdt}\right\} dt && = \int \left\{e^{\int pdt}\, q\right\} dt \\
\Rightarrow \quad & x\, e^{\int pdt} && = \int \left\{e^{\int pdt}\, q\right\} dt \\
\Rightarrow \quad & x && = e^{-\int pdt} \int \left\{e^{\int pdt}\, q\right\} dt
\end{aligned}$$

The above equation can be solved as follow;

$$\begin{aligned}
x_2 \quad &= \quad e^{-t} \int e^t\, [-c_1 - c_2\, e^t]\, dt \\
&= \quad -e^{-t} \int [c_1\, e^t + c_2\, e^{2t}]\, dt \\
&= \quad -e^{-t} \{[c_1\, e^t + \frac{1}{2} c_2\, e^{2t}] + c_3\} \\
&= \quad -c_1 - \frac{1}{2} c_2\, e^t - c_3\, e^{-t}
\end{aligned}$$

$$\dot{x}_1 = x_2$$

$$\begin{aligned}
\Rightarrow \; x_1 \quad &= \quad \int x_2\, dt \\
&= \quad -c_1\, t - \frac{1}{2} c_2\, e^t + c_3\, e^{-t} + c_4
\end{aligned}$$

- Considering the initial and final conditions, the integration constants can be found as follows:

$$x_1(0) = 0 = -\frac{1}{2} c_2 + c_3 + c_4 \tag{a1}$$

$$x_2(0) = 0 = -c_1 - \frac{1}{2} c_2 - c_3 \tag{a2}$$

$$x_1(2) = 5 = -2\, c_1 - \frac{1}{2} c_2\, e^2 + c_3\, e^{-2} + c_4 \tag{a3}$$

$$x_2(2) = 2 = -c_1 - \frac{1}{2} c_2\, e^2 - c_3\, e^{-2} \tag{a4}$$

(a1)+(a2) yields $c_1 + c_2 - c_4 = 0$ (a5) (a3)-(a4) yields $-c_1 + 2c_3 e^{-2} + c_4 = 3$ (a6)

(a5)+(a6) yields $c_2 = 3 - 2c_3 e^{-2}$ (a7)

(a2)-(a4) yields

$$\frac{1}{2}c_2(e^2 - 1) + c_3(e^{-2} - 1) = -2$$

$$\Rightarrow 3.19\ c_2 - 0.864\ c_3 = -2$$

$$\Rightarrow 3.19\ (3 - 2c_3 e^{-2}) - 0.864\ c_3 = -2$$

$$\Rightarrow\ c_3 = 6.697 \tag{a8}$$

$$\Rightarrow\ c_2 = 1.187$$

$$\Rightarrow\ c_1 = -0.5c_2 - c_3 = -7.291$$

$$\Rightarrow\ c_4 = 0.5c_2 - c_3 = -6.1$$

- The optimal control is given by:

$$u^* = -\lambda_2 = 7.289 - 13.392\,e^t$$

- The optimal states are given by:

$$x_1^* = 7.289\,t - 6.103 + 6.696\,e^{-t} - 0.593\,e^t$$
$$x_2^* = 7.289 - 6.696\,e^{-t} - 0.593\,e^t$$

Now, the optimal variables (control and states) are programmed within MATLAB environments from which the system responses are plotted versus time as shown in Fig. 5.

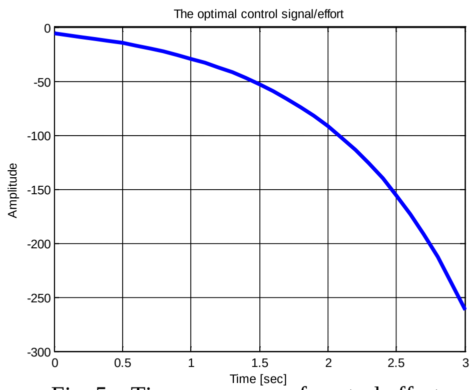

Fig. 5a: Time responses of control effort

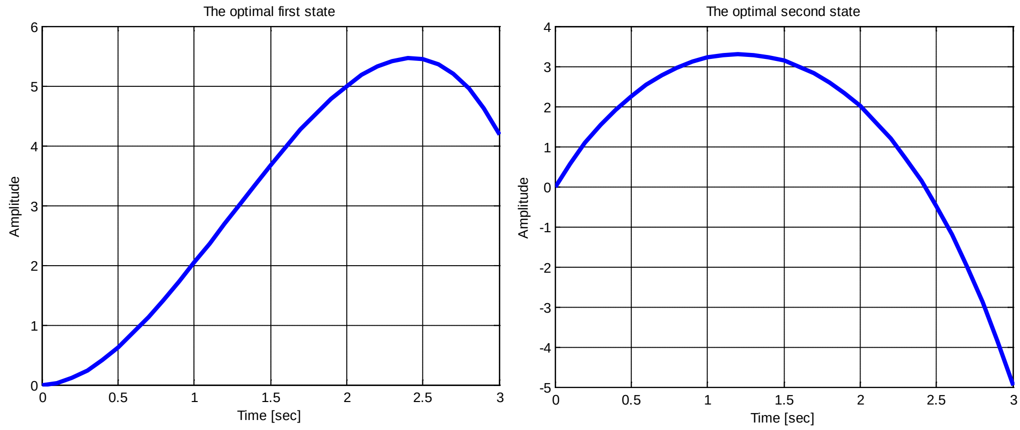

Fig. 5b: Time responses of state x_1 and state x_2

c) Since

$$I = \frac{1}{2}[x_1(2)-5]^2 + \frac{1}{2}[x_2(2)-2]^2 + \frac{1}{2}\int_0^2 u^2\,dt = F\{X(t_f), t_f\} + \int_{t_o}^{t_f} L[x(t), u(t), t]\,dt$$

Then

$$F = \frac{1}{2}\{x_1(2)-5\}^2 + \frac{1}{2}\{x_2(2)-2\}^2$$

$$L = \frac{1}{2}u^2$$

Where t_f is specified while x_f is free and $x(0) = x_0$.

$\lambda(t_f) = \dfrac{\partial F\{x(t_f), t_f\}}{\partial x}$ which represent the boundary equation.

$$\frac{\partial F\{t_f\}}{\partial x_1} = \lambda_1(t_f) = x_1(2) - 5$$

$$\frac{\partial F\{t_f\}}{\partial x_2} = \lambda_2(t_f) = x_2(2) - 2$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda}$$

$$0 = \frac{\partial H}{\partial u}$$

The Hamiltonian has the following form

$$\begin{aligned} H = L(x,u,t) + \lambda^T f(x,u,t) &= \frac{u^2}{2} + \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} \\ &= \frac{u^2}{2} + \lambda_1\,\dot{x}_1 + \lambda_2\,\dot{x}_2 \\ &= \frac{u^2}{2} + \lambda_1\,x_2(t) + \lambda_2\,\{-x_2(t) + u(t)\} \end{aligned}$$

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 \qquad \Rightarrow \lambda_1 = C_1$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = \lambda_2 - \lambda_1 = \lambda_2 - C_1$$

i.e.

$$\begin{aligned} \dot{\lambda}_2 = \lambda_2 - c_1 \quad &\Rightarrow \quad \frac{1}{\lambda_2 - c_1}\,d\lambda_2 = dt \\ &\Rightarrow \quad \ell n(\lambda_2 - c_1) = t + \ell n(c_2) \\ &\Rightarrow \quad \frac{\lambda_2 - c_1}{c_2} = e^t \\ &\Rightarrow \quad \lambda_2 = c_2\,e^t + c_1 \end{aligned}$$

The State equations $\dot{X} = \dfrac{\partial H}{\partial \lambda}$ are

$$\dot{x}_1 = \frac{\partial H}{\partial \lambda_1} \quad = \quad x_2$$
$$\dot{x}_2 = \frac{\partial H}{\partial \lambda_2} \quad = \quad -x_2 + u$$

The Control equation $0 = \dfrac{\partial H}{\partial u}$ is

$$u + \lambda_2 = 0 \qquad \Rightarrow u = -\lambda_2$$

Thus $\dot{x}_2 = -x_2 - \lambda_2 \qquad \Rightarrow \qquad \dot{x}_2 + x_2 = -\lambda_2 = -c_2\, e^{t} - c_1$

$$\Rightarrow \begin{array}{lll} x_2(t) & = & -c_1 - \frac{1}{2} c_2\, e^{t} - c_3\, e^{-t} \\ x_1(t) & = & -c_1\, t - \frac{1}{2} c_2\, e^{t} + c_3\, e^{-t} + c_4 \end{array}$$

Considering the initial and final conditions, the integration constants can be found and the optimal variables (control and states) can be plotted versus time as before.

3. Linear Regulator Problem

3.1 Theory

It is required to find the admissible optimal control vector $u^* \in U$ for the linear system described by the state equation (1) to follow an admissible state trajectory $x^* \in X$ such that the following performance index (P.I.) is minimized

$$I = \frac{1}{2} X^T(t_f)\, S\, X(t_f) + \frac{1}{2}\int_0^{t_f} [X^T(t)\, Q\, X(t) + u^T(t)\, R\, u(t)]\, dt \tag{21}$$

Where; S, Q, and R are square symmetrical weighting matrices to be selected by the designer. If the initial time t_o and the final time t_f are specified, the final state $x(t_f) = x_f$ is free and the admissible state and control regions are not bounded. Comparing Eq^n (21) with the general form of cost function (1a) yields:

$$\begin{aligned} F &= \frac{1}{2} X^T(t_f)\, S\, X(t_f) \\ L &= \frac{1}{2}\left\{X^T(t)\, Q\, X(t) + u^T(t)\, R\, u(t)\right\} \end{aligned} \tag{22}$$

Thus, the Hamiltonian matrix, Eq^n (19), can be obtained as follows:

$$\begin{aligned} H \quad &= \quad L(x,u,t) + \lambda^T f(x,u,t) \\ &= \quad \frac{1}{2} X^T(t)\, Q\, X(t) + \frac{1}{2} u^T(t)\, R\, u(t) + \lambda^T [A\, X + B\, u] \end{aligned} \tag{23}$$

That is, the following equations can be obtained

- State equation $\dot{X} = \dfrac{\partial H}{\partial \lambda} = A\, X + B\, u$
- Costate equation $\dot{\lambda} = -\dfrac{\partial H}{\partial x} = -Q\, X(t) - A^T \lambda$ (24)
- Control equation $\dfrac{\partial H}{\partial u} = 0 = R\, u(t) + B^T \lambda$ or

$$u(t) = -R^{-1} B^T \lambda = -R^{-1} B^T P\, x = -K\, x \qquad \Rightarrow \quad K = R^{-1} B^T P$$

Manipulating these equations yields the following differential equations

$$\dot{X} = A\,X - B\,R^{-1}\,B^{T}\,\lambda$$
$$\dot{\lambda} = -Q\,X(t) - A^{T}\,\lambda \tag{25}$$

These equations can be put in the following matrix form

$$\begin{bmatrix} \dot{X} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -B\,R^{-1}\,B^{T} \\ -Q & -A^{T} \end{bmatrix} \begin{bmatrix} X \\ \lambda \end{bmatrix} \tag{26}$$

Therefore, the block diagram representing the operation of this system can be drawn as shown in Fig. 1 and the system equations (26) can be solved either following the state-transition matrix or the riccati equation approach. The later approach is the objective of this paper.

Since x_f is free the following equation holds;

$$\lambda(t_f) = \frac{\partial F}{\partial x} = S\,X(t_f) \tag{27}$$

Looking at Eq^n (27), the Lagrangian multiplier $\lambda(t)$ can be calculated through the following formula:

$$\lambda(t) = P(t)\,X(t) \tag{28}$$

Where P is a square matrix obtained from the solution of Riccati equation with $P(t_f) = S$. Thus, differentiating Eq^n (28) and using both (24, 26) and (28) yield the following:

$$\dot{P}\,X + P\,\dot{X} = -Q\,X - A^{T}\,\lambda = -Q\,X - A^{T}\,P\,X \tag{29}$$

The left side can be substituted for using Eq^{ns}(1) and (24) to yield

$$\dot{P}\,X + P\,A\,X - P\,B\,R^{-1}\,B^{T}\,P\,X = -Q\,X - A^{T}\,P\,X \tag{30}$$

Rearranging yields the Riccati equation as follows:

$$\dot{P} = -P\,A - A^{T}\,P - Q + P\,B\,R^{-1}\,B^{T}\,P \tag{31}$$

where $P(t_f) = S$. This equation can be solved in P that is used to obtain the Lagrangian multiplier $\lambda(t)$ and consequently the control signal u(t). The process of design, implementation and evaluation can be described in block diagram as shown in Fig. 6.

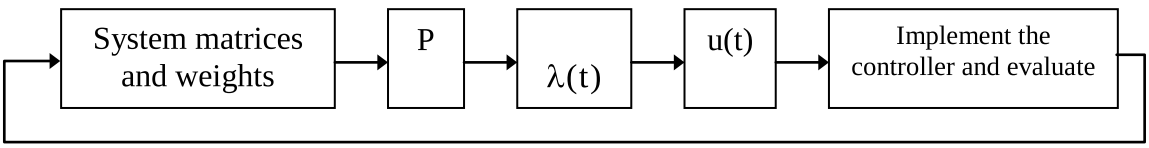

Fig. 6: Sequence of optimal control regulator design

For time invariant systems $\dot{P} = 0$ and consequently Eq^n (4-16) reduces to the following form:

$$P\,A + A^{T}\,P + Q - P\,B\,R^{-1}\,B^{T}\,P = 0 \tag{32}$$

3.2 Case Study-2

Find out the solution of a linear regulator problem concerning the system described by

$$\dot{x} = a\,x + u$$

such that the following performance index (P.I.) is minimized

$$I = \frac{1}{2}\,S\,X^{2}(t_f) + \frac{1}{4}\int_{0}^{t_f} u^{2}\;dt$$

Where a=-2, x(0)=5, t_f=15 [sec] and S=5, or 0.05

Solution:
Substituting the given data (A=a=-2, B=1, R=1/2, Q=0) into the Riccati equation (31) yields

$$\begin{aligned}\dot{P} &= -P\,a - a\,P - 0 + 2\,P^2 \\ &= -2\,a\,P + 2\,P^2 \\ &= 2\;P\,(P-a)\end{aligned} \tag{b1}$$

That is

$$\frac{dP}{P\,(P-a)} = 2\,dt \tag{b2}$$

Since

$$\frac{1}{P\,(P-a)} = \frac{c_1}{P} + \frac{c_2}{(P-a)} \quad \Rightarrow \quad c_1 = -1/a \quad \& \quad c_2 = 1/a$$

Then Eq^n (b2) becomes as follows:

$$\begin{aligned}\frac{dP}{P} - \frac{dP}{(P-a)} = -2\,a\,dt \quad &\Rightarrow \quad \ell n(P) - \ell n(P-a) = -2\,a\,t + \ell n(c) \\ &\Rightarrow \quad \ell n\left\{\frac{P}{c\,(P-a)}\right\} = -2\,a\,t \\ &\Rightarrow \quad \frac{P}{c\,(P-a)} = e^{-2\,a\,t} \\ &\Rightarrow \quad P = c\,P\,e^{-2\,a\,t} - c\,a\,e^{-2\,a\,t} \\ &\Rightarrow \quad P\left\{1 - c\;e^{-2\,a\,t}\right\} = -c\,a\,e^{-2\,a\,t} \\ &\Rightarrow \quad P = \frac{-c\,a\,e^{-2\,a\,t}}{1 - c\;e^{-2\,a\,t}}\end{aligned} \tag{b3}$$

The value of c can be obtained from the boundary conditions as follows:
At $t=t_f$ the unknown P is given by $P(t_f) = S$ i.e.

$$\begin{aligned}S = \frac{-c\,a\,e^{-2\,a\,t_f}}{1 - c\;e^{-2\,a\,t_f}} \quad &\Rightarrow \quad S - S\,c\;e^{-2\,a\,t_f} = -c\,a\,e^{-2\,a\,t_f} \\ &\Rightarrow \quad S = S\,c\;e^{-2\,a\,t_f} - c\,a\,e^{-2\,a\,t_f} \\ &\Rightarrow \quad S = c\;e^{-2\,a\,t_f}\,(S-a) \\ &\Rightarrow \quad c = \frac{S\,e^{2\,a\,t_f}}{S-a}\end{aligned} \tag{b4}$$

Substituting Eq^n (b4) into Eq^n (b3) yields

$$P = \frac{-c\,a\,e^{-2\,a\,t}}{1 - c\;e^{-2\,a\,t}} = \frac{-\dfrac{S\,a\,e^{2\,a\,t_f}}{S-a}\;e^{-2\,a\,t}}{1 - \dfrac{S\,e^{2\,a\,t_f}}{S-a}\;e^{-2\,a\,t}} = \frac{-S\,a\,e^{2\,a\,(t_f - t)}}{S - a - S\,e^{2\,a\,(t_f - t)}} \tag{b5}$$

Considering the given data

$$P = \frac{2\,S\,e^{-4\,(15-t)}}{2 + S\,\{1 - e^{-4\,(15-t)}\}} = \begin{cases} \dfrac{10\,e^{-4\,(15-t)}}{2 + 5\,\{1 - e^{-4\,(15-t)}\}} & \text{for} \quad S = 5 \\ \dfrac{0.1\,e^{-4\,(15-t)}}{2 + 0.05\,\{1 - e^{-4\,(15-t)}\}} & \text{for} \quad S = 0.05 \end{cases} \tag{b6}$$

The system states as function of time can be obtained as follows:

$$\begin{aligned}\dot{x} &= a\,x + u \\ &= a\,x - R^{-1} B^T \lambda \\ &= a\,x - R^{-1} B^T P\,x \\ &= \{a - R^{-1} B^T P\}x \\ &= \{-2 - 2\,P)\,x \qquad \text{(b7)}\end{aligned}$$

Since the initial condition upon x(t) is given, Eq^n (b7) can be solved numerically as follows:

$$x_i(n) = x_i(n-1) + \dot{x}_i(n-1)\,\Delta_t \qquad \text{(b8)}$$

Where i is a counter for the states, n for the step of integration and analogous to time, and Δ_t is the sampling period or integration time-step i.e. $t = n\,\Delta_t$.

Now, the solution of the problem in a step-wise form can be summarized as follows:

1. the intermediate variable $P = \begin{cases} \dfrac{10\,e^{-4(15-t)}}{2+5\{1-e^{-4(15-t)}\}} & \text{for} \quad S = 5 \\ \dfrac{0.1\,e^{-4(15-t)}}{2+0.05\{1-e^{-4(15-t)}\}} & \text{for} \quad S = 0.05 \end{cases}$
2. the instantaneous state $x_i(n) = x_i(n-1) + \dot{x}_i(n-1)\,\Delta_t$
3. the instantaneous state-rate $\dot{x} = \{-2 - 2\,P)\,x$
4. the instantaneous lagrangian multiplier $\lambda = P\,x$
5. the instantaneous control vector $u(t) = -R^{-1} B^T \lambda = -2\,\lambda$

These equations are solved against time, say, t=0 to 15 [sec].

4. Servo Mechanism (Tracking) Problem

4.1 Servomechanism Theory

It is required to find the admissible optimal control vector $u^* \in U$ for the linear system described by the state equation (1b) to follow an admissible state trajectory $x^* \in X$ such that the following performance index (P.I.) is minimized;

$$I = \frac{1}{2}\left[x(t_f) - r(t_f)\right]^T S \left[x(t_f) - r(t_f)\right] + \frac{1}{2}\int_0^{t_f} \left\{\left[x(t) - r(t)\right]^T Q \left[x(t) - r(t)\right] + u^T(t)\,R\,u(t)\right\} dt \qquad \text{(33a)}$$

Or

$$I = \frac{1}{2}\left[y_r(t_f) - y(t_f)\right]^T S \left[y_r(t_f) - y(t_f)\right] + \frac{1}{2}\int_0^{t_f} \left\{\left[y_r(t) - y(t)\right]^T Q \left[y_r(t) - y(t)\right] + \left[u_r(t) - u(t)\right]^T R \left[u_r(t) - u(t)\right]\right\} dt \qquad \text{(33b)}$$

Where; S, Q, and R are square symmetrical weighting matrices to be selected by the designer. If the initial time t_o and the final time t_f are specified, then the final state $x(t_f) = x_f$ is free and the admissible state and control regions are not bounded. The variables y_r and u_r represent respectively the reference output and control signals. According to the nature of these variables there are different control design problems;

- If $y_r = \text{constant} = 0$, $u_r = \text{constant} = 0$ and C=I, then the problem is known as state regulator problem.
- If $y_r = \text{constant} = 0$, $u_r = \text{constant} = 0$ and $C \neq I$, then the problem is known as output regulator problem.

Comparing Eq^n (33) with the general form of cost function (1) yields:

$$F = \frac{1}{2}\left[x(t_f) - r(t_f)\right]^T S\left[x(t_f) - r(t_f)\right] \tag{34a}$$

$$L = \frac{1}{2}\left[x(t) - r(t)\right]^T Q\left[x(t) - r(t)\right] + u^T(t)\, R\, u(t)$$

or

$$F = \frac{1}{2}\left[y_r(t_f) - y(t_f)\right]^T S\left[y_r(t_f) - y(t_f)\right] \tag{34b}$$

$$L = \frac{1}{2}\left[y_r(t) - y(t)\right]^T Q\left[y_r(t) - y(t)\right] + \frac{1}{2}\left[u_r(t) - u(t)\right]^T R\left[u_r(t) - u(t)\right]$$

Thus, the Hamiltonian matrix (19) can be obtained as follows:

$$\begin{aligned} H &= L(x,u,t) + \lambda^T f(x,u,t) \\ &= \frac{1}{2}\left[x(t) - r(t)\right]^T Q\left[x(t) - r(t)\right] + u^T(t)\, R\, u(t) + \lambda^T [A X + B u] \end{aligned} \tag{35a}$$

or

$$\begin{aligned} H &= L(x,u,t) + \lambda^T f(x,u,t) \\ &= \frac{1}{2}\left[y_r(t) - y(t)\right]^T Q\left[y_r(t) - y(t)\right] + \frac{1}{2}\left[u_r(t) - u(t)\right]^T R\left[u_r(t) - u(t)\right] + \lambda^T [A X + B u] \end{aligned} \tag{35b}$$

That is, the following equations can be obtained

- State equation $\dot{X} = \dfrac{\partial H}{\partial \lambda} = A X + B u$
- Costate equation $\dot{\lambda} = -\dfrac{\partial H}{\partial x} = -\left\{-C^T Q\left[y_r - C X\right] + A^T \lambda\right\}$ (36)
- Control equation $\dfrac{\partial H}{\partial u} = 0 = -R[u_r - u] + B^T \lambda$ or $u(t) = u_r - R^{-1} B^T \lambda$

Manipulating these equations yields the following differential equations

$$\begin{aligned} \dot{X} &= A X - B R^{-1} B^T \lambda + B u_r \\ \dot{\lambda} &= -C^T Q C X(t) - A^T \lambda + C^T Q y_r \end{aligned} \tag{37}$$

Since x_f is free, the final value of the Lagrangian multiplier (boundary condition) is given by

$$\lambda(t_f) = \frac{\partial F}{\partial x} = -C^T S\left[y_r(t_f) - C X(t_f)\right] = C^T S C X(t_f) - C^T S y_r(t_f) \tag{38}$$

Let us consider the Lagrangian multiplier as follows:

$$\lambda(t) = P(t) X(t) - \zeta(t), \text{ where } P(t_f) = C^T S C \quad \& \quad \zeta(t_f) = C^T S y_r(t_f) \tag{39}$$

Differentiating Eq^n (39) and substituting into Eq^n (37) yields

$$\dot{P} X + P \dot{X} - \dot{\zeta} = -C^T Q C X - A^T P X + A^T \zeta + C^T Q y_r \tag{40}$$

$$\dot{P} X + P[A X - B R^{-1} B^T (P X - \zeta) + B u_r] - \dot{\zeta} = -C^T Q C X - A^T P X + A^T \zeta + C^T Q y_r$$

$$\left\{\dot{P} + P A + A^T P + C^T Q C - P B R^{-1} B^T P\right\} X + \left\{-\dot{\zeta} + [P B R^{-1} B^T - A^T]\zeta + P B u_r - C^T Q y_r\right\} = 0$$

Thus, assuming that $X \neq 0$ yields the following two Riccati equations for the servo mechanism problem:

$$\dot{P} = -P\,A - A^T\,P - C^T\,Q\,C + P\,B\,R^{-1}\,B^T\,P \tag{41a}$$

$$\dot{\zeta} = [P\,B\,R^{-1}\,B^T - A^T]\,\zeta + P\,B\,u_r - C^T\,Q\,y_r \tag{41b}$$

With the optimal controller as

$$u(t) = u_r - R^{-1}\,B^T\,P\,X + R^{-1}\,B^T\,\zeta \tag{42}$$

This equation can be solved in P and ζ which are used to obtain the Lagrangian multiplier $\lambda(t)$ and consequently the optimal control signal u(t). The process of servo design, implementation and evaluation can be described in block diagram as shown in Fig. 7.

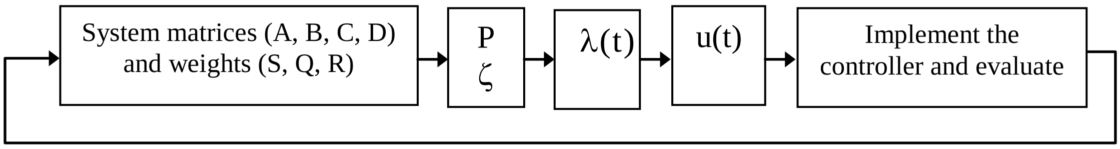

Fig. 7: Sequence of optimal control servomechanism design

For time invariant systems $\dot{P} = 0$, $\dot{\zeta} = 0$ and consequently Eq^n (41) reduces to the following form:

$$P\,A + A^T\,P + C^T\,Q\,C - P\,B\,R^{-1}\,B^T\,P = 0 \tag{43a}$$

$$[P\,B\,R^{-1}\,B^T - A^T]\,\zeta + P\,B\,u_r - C^T\,Q\,y_r = 0 \tag{43b}$$

4.2 Case Study-3

Design an optimal controller to form a tracking problem with the following system,

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = u$$
$$y = x_1$$

such that the following performance index is minimized

$$I = \frac{1}{2}\int_0^\infty [q_{11}\,(x_{1r} - x_1)^2 + u^2]\; dt$$

Assuming that the weight $q_{11} = 16$ and the reference state is $x_{1r} = 1 - e^{-t}$ (i.e. $y_r = 1 - e^{-t}$), **find** the closed loop transfer function, analyze the system performance using MATLAB and comment your result.

Solution:

It is clear that the state equation representing the plant is obtained as follows:

$$\dot{X} \equiv \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{c1}$$

i.e.

$$\dot{X} = AX + BU$$
$$Y = CX$$

Consequently, the different system matrices are as follows:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 0 \end{bmatrix},\ D = [0] \tag{c2}$$

$$S = 0,\ R = 1,\ Q = q_{11}$$

Thus

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix},\quad \zeta = \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} \text{ and } \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \tag{c3}$$

Substituting the given data into the Riccati equations (43) yield:

$$-\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_{11} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1]\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = 0 \tag{c4}$$

$$\left\{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1]\begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} u_r - \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_{11}\, y_r = 0$$

The algebraic manipulation of this equation (c4) yields five algebraic equations as follows:

$$\begin{bmatrix} 0 & -p_{11} \\ 0 & -p_{12} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -p_{11} & -p_{12} \end{bmatrix} + \begin{bmatrix} -q_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p_{12}\, p_{21} & p_{12}\, p_{22} \\ p_{22}\, p_{21} & p_{22}^2 \end{bmatrix} = 0 \tag{c5}$$

$$\left\{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} u_r - \begin{bmatrix} q_{11}\, y_r \\ 0 \end{bmatrix} = 0$$

or

$$\begin{bmatrix} -q_{11} & -p_{11} \\ -p_{11} & -2\,p_{12} \end{bmatrix} + \begin{bmatrix} p_{12}\, p_{21} & p_{12}\, p_{22} \\ p_{22}\, p_{21} & p_{22}^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} p_{12}\, \zeta_2 \\ p_{22}\, \zeta_2 \end{bmatrix} - \begin{bmatrix} 0 \\ \zeta_1 \end{bmatrix} + \begin{bmatrix} p_{12}\, u_r \\ p_{22}\, u_r \end{bmatrix} - \begin{bmatrix} q_{11}\, y_r \\ 0 \end{bmatrix} = 0$$

i.e.

$$\begin{bmatrix} -q_{11} + p_{12}\, p_{21} & -p_{11} + p_{12}\, p_{22} \\ -p_{11} + p_{22}\, p_{21} & -2\,p_{21} + p_{22}^2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -q_{11}\, y_r + p_{12}\, \zeta_2 + p_{12}\, u_r \\ -\zeta_1 + p_{22}\, \zeta_2 + p_{22}\, u_r \end{bmatrix} = 0$$

which yields the following equations

$$\begin{aligned} -q_{11} + p_{12}\, p_{21} &= 0 \\ -p_{11} + p_{12}\, p_{22} &= 0 \\ -2\,p_{12} + p_{22}^2 &= 0 \\ -q_{11}\, y_r + p_{12}\, \zeta_2 + p_{12}\, u_r &= 0 \\ -\zeta_1 + p_{22}\, \zeta_2 + p_{22}\, u_r &= 0 \end{aligned} \tag{c6}$$

The solution of these equations yields the elements of the Recatti matrix P as follows:

$$\begin{aligned} p_{11} &= \sqrt{2}\; q_{11}^{3/4} \\ p_{12} &= \sqrt{q_{11}} \\ p_{22} &= \sqrt{2}\; q_{11}^{1/4} \\ \zeta_1 &= p_{22}\,\zeta_2 = \sqrt{2}\; q_{11}^{3/4}\;\; y_r \\ \zeta_2 &= \frac{q_{11}\, y_r}{p_{12}} = \sqrt{q_{11}}\;\; y_r \end{aligned} \tag{c7}$$

The optimal controller is obtained as follows:

$$\begin{aligned} u &= -R^{-1} B^T\, P\, X + R^{-1} B^T\, \zeta \\ &= -\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\zeta_1 \\ \zeta_2\end{bmatrix} \\ &= -\, p_{12}\, x_1 - p_{22}\, x_2 + \zeta_2 \\ &= -\sqrt{q_{11}}\;\; x_1 - \sqrt{2}\;\; q_{11}^{1/4}\; x_2 + \sqrt{q_{11}}\;\; y_r \end{aligned} \tag{c8}$$

Or

$$\begin{aligned} u &= \sqrt{q_{11}}\;\; y_r - \sqrt{q_{11}}\;\; y - \sqrt{2}\;\; q_{11}^{1/4}\; \dot{y} \\ &= \sqrt{q_{11}}\;\; \{y_r - y\} - \sqrt{2}\;\; q_{11}^{1/4}\; \dot{y} \end{aligned} \tag{c9}$$

Using the given value of $q_{11} = 16$ and the reference roll angle yield

$$\begin{aligned} p_{11} &= 8\sqrt{2} \\ p_{12} &= 4 \\ p_{22} &= 2\sqrt{2} \\ \zeta_1 &= 8\sqrt{2}\;\; (1 - e^{-t}) \\ \zeta_2 &= 4\;\; (1 - e^{-t}) \\ u &= 4\;\; (1 - e^{-t}) - 4\, y - 2\sqrt{2}\;\dot{y} \end{aligned} \tag{c10}$$

Thus, the optimal roll-autopilot is

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 4 & 2\sqrt{2} \end{bmatrix} \tag{c11}$$

Now, the designed closed loop has the form shown in Fig. 8, from which the system output is given by:

$$y = \frac{4}{s^2 + 2\sqrt{2}\;\; s + 4}\, y_r = \frac{4\;\; (1 - e^{-t})}{s^2 + 2\sqrt{2}\;\; s + 4} = \frac{4\;\; (1 - e^{-t})}{s^2 + 2.8284\;\; s + 4}$$

The response of this system can be evaluated using the program built by the author within the MATLAB environments as shown in Fig. 9.

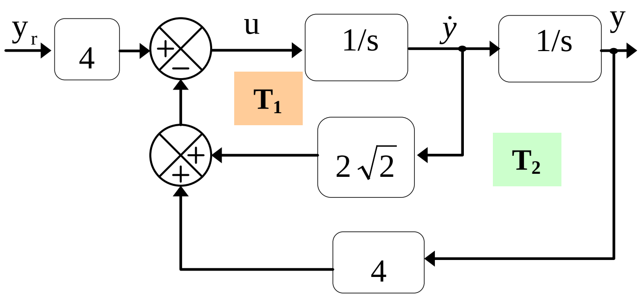

Fig. 8: Designed stabilization system

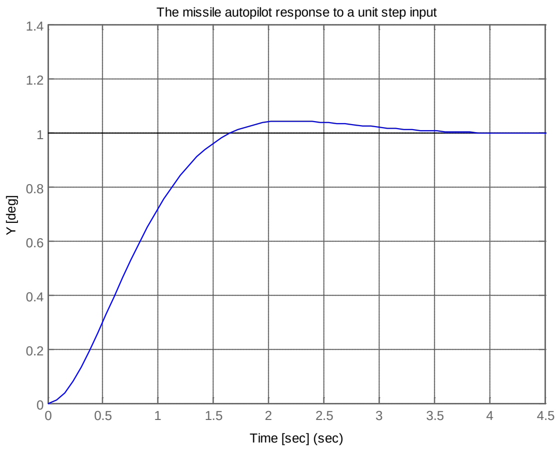

Fig. 9a: Step Response

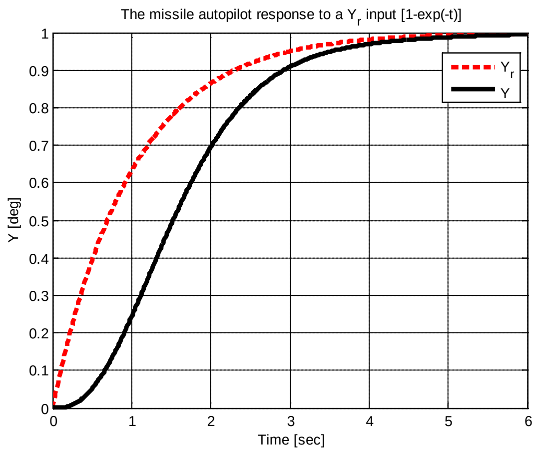

Fig. 9b: Command Following Response

5. Conclusions

This paper presented the derivation of the optimal control theory, state space approach, in a novel form following a systematic approach which is more concise, clear and general. The derivation was based on a general system structure which contains colored input disturbance and measurement noise. The theory is presented in a more concise, clear and general form to help those looking to use it without any details as well as those looking for detailed understanding and tailoring the theory to their real problems. The cost function weights may be dynamical (frequency-dependent) to allow various performance characteristics (including integral action) to be easily introduced and robustness characteristics to be modified. The derivation is complemented with the two control problems; regulator and servo-mechanism in addition to case study for each to clarify the application of these theories. The future work is concerned with the numerical solution procedure of the riccatti equation(s) and carrying the derivation procedure presented here with quantification of different sources of uncertainty and its impact upon the controller design.

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