Sohag Journal of Science
An International Journal

http://dx.doi.org/10.18576/sjs/030201

A solution of system of linear matrix equations by matrix of matrices

Z. Kishka[1], *M. Abdalla*[2] *and A. Elrawy*[2,*]

[1] Dept. of Math., Faculty of Science, Sohag University, Sohag 82524, Egypt.
[2] Dept. of Math., Faculty of Science, South Valley University, Qena 83523, Egypt.

Received: 2 Feb. 2018, Revised: 27 Mar. 2018, Accepted: 2 Apr. 2018
Published online: 1 May 2018

Abstract: It is well-known that the matrix equations play an important role in engineering and applied sciences. In this paper, we define and study a system of linear matrix equations. Also, we define the notions of vector of matrices (for short, VMs), linearly dependent, independent VMs and rank of the matrix of matrices (for short, MMs) which introduced recently by Kishka and et al. [1,2]. We propose three methods for computing the solution of this system.

Keywords: MMs, system of linear matrix equations, rank of MMs

1 Introduction

In recent years, several articles are written about the system of matrix equations. Mitra [3], first study the system

$$A_1XB_1 = C_1,$$
$$A_2XB_2 = C_2,$$

over the complex number $\mathbb{C}$. Many authors contributed to this system (see [4,5,6,7,8,9,10]). To instance, in 1987, Van der Woude investigated this system over a field. In 1996, Wang studied the system over an arbitrary division ring.
Kiska et al. (cf. [1,2]) introduced the concept of MMs which provide the possibility of extending the many topics of matrices in addition to saving time and speed in the work of many mathematical calculations. Our fundamental outcomes, define a linear system of matrix equations, and we introduce several methods to solve this system. The remainder of this article is organized as follows:
In section 2, we present the notions of VMs, linearly dependent, linearly independent VMs and rank of MMs. Section 3, is devoted to show some methods for computing the solution of linear system of matrix equations. The conclusion is reported in section 4.
Throughout this paper, the following notations are used:
Let $\mathbb{K}$ be an arbitrary field and $M_l(\mathbb{K})$ be the set of all $l\times l$ matrices over $\mathbb{K}$, $M_l^{'}(\mathbb{K})\subset M_l(\mathbb{K})$ be commutative ring I and O stand for the identity matrix and the zero matrix in $M_l(\mathbb{K})$, respectively. We denote by $\mathscr{M}_{m\times n}(M_l(\mathbb{K}))$ the set of all $m\times n$ MMs over $M_l(\mathbb{K})$, the elements of $\mathscr{M}_{m\times n}(M_l(\mathbb{K}))$ are denoted by $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$, $\mathscr{G}$, ...etc.
Now, it is useful to define MMs and the system of linear matrix equations.

Definition 1.[2] *Let $\mathscr{A}\in\mathscr{M}_{m\times n}(M_l(\mathbb{K}))$, then it can be written as a rectangular table of elements A_{ij}; $i=1,2,\ldots,m$ and $j=1,2,\ldots,n$, as follows:*

$$\mathscr{A}=\begin{pmatrix} A_{11} & \ldots & A_{1n}\\ \vdots & \ddots & \vdots\\ A_{m1} & \ldots & A_{mn}\end{pmatrix}.$$

* Corresponding author e-mail: amrmath1985@yahoo.com

The system of m matrix equations in n unknowns defined as follows:

$$\begin{array}{c} A_{11}X_1+......+A_{1n}X_n=B_1,\\ A_{21}X_1+......+A_{2n}X_n=B_2,\\ \vdots \\ A_{m1}X_1+.....+A_{mn}X_n=B_m, \end{array} \tag{1.1}$$

where A_{ij}, X_i and $B_j \in M_l(\mathbb{K})$, $i=1,2,...,n,$ $j=1,2,...,m$, and A_{ij} are called the coefficients, X_i are variables, and B_j are scalars matrices called the constants.

We can write the above system (1.1) in the form

$$\mathscr{A}\mathscr{X}=\mathscr{B}, \tag{1.2}$$

by using the concept MMs.

*Remark.*It is said that system of matrix equations that homogeneous if $B_i=O$, where O is zero matrix. Otherwise called non-homogeneous.

Definition 2.*Two systems of matrices equations involving the same matrices of variables are said to be equivalent if they have the same solution set.*

Theorem 1.*Any system of matrix equations has one of the following exclusive conclusions:*
(1) No solution.
(2) Unique solution.
(3) Infinitely many solutions.

Before presenting solution of the system of the linear matrix equations, it is useful to introduce a few notations.

2 Basic points

In this section, we introduce a definition for VMs, linearly dependent, linearly independent VMs and rank of MMs. Also, we show the type of solution.

Definition 3.*Let $M_l(\mathbb{K})$ be a ring with unity, then the set*

$$\mathscr{V}^n=\{(A_1,A_2,...,A_n):A_1,A_2,...,A_n\in M_l(\mathbb{K})\},$$

is called VMs.

Definition 4.*The set of n VMs is linearly independent if*

$$\sum_{k=1}^{n}E_k\mathscr{V}_k=\mathscr{V}_0,$$

and $E_k=O$, where $\mathscr{V}_k=\left(A_{kj}\right)$, $\mathscr{V}_0=\left(O_{1,k}\right)\in\mathscr{M}_{1\times n}(M_l(\mathbb{K}))$, O, $E_k\in M_l(\mathbb{K})$ and $k=1,2,...,n$. If one of $E_k\neq O$, then n VMs is linearly dependent.

Definition 5.*Let $\mathscr{A}=(A_{ij})\in\mathscr{M}_{m\times n}(M_l(\mathbb{K}))$, then*

$$\begin{aligned} rank(\mathscr{A}) &= \max_{1\le j\le n}\{rank\left(A_{1j}\right)\}+...+\max_{1\le j\le n}\{rank(A_{pj})\}\\ &=\sum_{i=1}^{p}\max_{1\le j\le n}\{rank(A_{ij})\}, \end{aligned}$$

where p denoted the number of linear independent rows.

If the augmented MMs $(\mathscr{A}|\mathscr{B})$ is obtained from (1.2), then we have the following properties

(1)If $rank(\mathscr{A})=rank(\mathscr{A}|\mathscr{B})=\rho$, then the system has a unique solution.
(2)If $rank(\mathscr{A})=rank(\mathscr{A}|\mathscr{B})<\rho$, then the system has $\infty-$many solutions.
(3)If $rank(\mathscr{A})<rank(\mathscr{A}|\mathscr{B})$, then the system has no solution.

Where ρ is the number of rows in $\mathscr{X}$ product size of elements.

3 Solve the system of the linear matrix equations

In this section, we introduce some methods to solve the system of the linear matrix equations.

3.1 The inverse MMs method

In this subsection, we use the inverse of MMs to solve the system of the linear matrix equations.
We consider $M_l'(\mathbb{K})$ be commutative matrices, and $\mathscr{A} \in \mathscr{M}_n\left(M_l'(\mathbb{K})\right)$, $\mathscr{B} \in \mathscr{M}_{n\times 1}\left(M_l'(\mathbb{K})\right)$ and $\mathscr{X} \in \mathscr{M}_{n\times 1}\left(M_l'(\mathbb{K}[x])\right)$.

Furthermore, we consider the system,

$$\mathscr{A}\mathscr{X} = \mathscr{B},$$

then the solution of this system is

$$\mathscr{X} = \mathscr{A}^{-1}\mathscr{B}.$$

*Example 1.*Let

$$\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} X_1 + \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} X_2 = I_2,$$

$$\begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} X_1 + \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} X_2 = O_2,$$

where $X_1 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $X_2 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, I_2 and $O_2 \in M_2'(\mathbb{K})$, then

$$rank(\mathscr{A}) = rank(\mathscr{A}|\mathscr{B}) = 4,$$

so the system has a unique solution. Since the inverse $\mathscr{A}$

$$\mathscr{A}^{-1} = \begin{pmatrix} \begin{pmatrix} 1 & -\frac{9}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{pmatrix} & \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{pmatrix} \\ \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{pmatrix} & \begin{pmatrix} \frac{1}{7} & -\frac{3}{7} \\ -\frac{1}{7} & 0 \end{pmatrix} \end{pmatrix},$$

then the solution is

$$\mathscr{X} = \begin{pmatrix} \begin{pmatrix} 1 & -\frac{9}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{pmatrix} \\ \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{pmatrix} \end{pmatrix}.$$

*Example 2.*Let

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 2 & 3 & 1 \end{pmatrix} X_1 + \begin{pmatrix} 10 & -1 & 1 \\ 0 & 0 & 10 \\ 0 & 8 & 2 \end{pmatrix} X_2 = O_3,$$

$$\begin{pmatrix} 12 & -1 & 1 \\ 0 & 3 & 10 \\ 0 & 8 & 4 \end{pmatrix} X_1 + I_3 X_2 = I_3,$$

where $X_1 = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$, $X_2 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{pmatrix}$, I_3 and $O_3 \in M_3'(\mathbb{K})$, then

$$rank(\mathscr{A}) = rank(\mathscr{A}|\mathscr{B}) = 6,$$

so the system has a unique solution. Since the inverse $\mathscr{A}$

$$\mathscr{A}^{-1}=\begin{pmatrix} \begin{pmatrix} -\frac{1407}{167276} & -\frac{17}{17608} & \frac{83}{167276} \\ -\frac{51}{334552} & -\frac{545}{35216} & \frac{2143}{334552} \\ -\frac{43}{334552} & \frac{231}{35216} & -\frac{4753}{334552} \end{pmatrix} & \begin{pmatrix} \frac{7035}{83638} & -\frac{109}{8804} & \frac{714}{41819} \\ \frac{255}{167276} & -\frac{905}{17608} & \frac{11885}{83638} \\ \frac{215}{167276} & \frac{1999}{17608} & -\frac{3099}{83638} \end{pmatrix} \\ \begin{pmatrix} \frac{4221}{41819} & -\frac{167}{17608} & \frac{1345}{83638} \\ \frac{153}{83638} & -\frac{175}{35216} & \frac{21627}{167276} \\ \frac{129}{83638} & \frac{3305}{35216} & -\frac{1445}{167276} \end{pmatrix} & \begin{pmatrix} -\frac{391}{41819} & \frac{-135}{8804} & -\frac{1197}{41819} \\ -\frac{765}{41819} & \frac{333}{17608} & -\frac{2340}{41819} \\ -\frac{645}{41819} & -\frac{755}{17608} & \frac{487}{41819} \end{pmatrix} \end{pmatrix}$$

then the solution is

$$\mathscr{X}=\begin{pmatrix} \begin{pmatrix} \frac{7035}{83638} & -\frac{109}{8804} & \frac{714}{41819} \\ \frac{255}{167276} & -\frac{905}{17608} & \frac{11885}{83638} \\ \frac{215}{167276} & \frac{1999}{17608} & -\frac{3099}{83638} \end{pmatrix} \\ \begin{pmatrix} -\frac{391}{41819} & \frac{-135}{8804} & -\frac{1197}{41819} \\ -\frac{765}{41819} & \frac{333}{17608} & -\frac{2340}{41819} \\ -\frac{645}{41819} & -\frac{755}{17608} & \frac{487}{41819} \end{pmatrix} \end{pmatrix}.$$

3.2 Gaussian elimination on MMs

Gaussian elimination method is the standard method for solving linear equations (see [11]). In this subsection, we offer Gaussian elimination method in the case of MMs to solve the system of the linear matrix equations.
Now, we introduce elementary row operations and row echelon forms.

Definition 6.*There are three kinds of elementary row operations on matrices:*

(a)Adding a multiple of one row to another row;
(b)Multiplying all entries of one row by a nonzero matrix constant;
(c)Interchanging two rows.

Definition 7.*A MMs is in row echelon form when it satisfies the following conditions:*

(i)The first non-zero matrix element in each row, called the leading entry, is identity matrix.
(ii)Each leading entry is in a column to the right of the leading entry in the previous row.
(iii)Rows with all zero matrix elements, if any, are below rows having a non-zero matrix element.

Definition 8.*A MMs is in reduced row echelon form when it satisfies the following conditions:*

(i)The MMs is in row echelon form (i.e., it satisfies the three conditions listed above).
(ii)The leading entry in each row is the only non-zero matrix entry in its column.

*Example 3.*Applying the Gaussian elimination method on the same Example 1, it follows that

$$\left(\begin{array}{cc|c} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} & \vdots\ I_2 \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} & \vdots\ O_2 \end{array}\right) \xrightarrow{\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}^{-1} r_1}$$

$$\left(\begin{array}{ccc} I_2 & \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} & \vdots \begin{pmatrix} 1 & -1 \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} & \vdots \quad O_2 \end{array}\right) \xrightarrow{-\begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} r_1 + r_2}$$

$$\left(\begin{array}{ccc} I_2 & \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} & \vdots \begin{pmatrix} 1 & -1 \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ O_2 & \begin{pmatrix} 0 & -7 \\ \frac{-7}{3} & \frac{-7}{3} \end{pmatrix} & \vdots \begin{pmatrix} 0 & -2 \\ -\frac{2}{3} & -\frac{2}{3} \end{pmatrix} \end{array}\right) \xrightarrow{\begin{pmatrix} 0 & -7 \\ \frac{-7}{3} & \frac{-7}{3} \end{pmatrix}^{-1} r_2}$$

$$\begin{pmatrix} I_2 & \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} & \vdots & \begin{pmatrix} 1 & -1 \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \\ O_2 & I_2 & \vdots & \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{pmatrix} \end{pmatrix} \xrightarrow{-\begin{pmatrix} 0 & 1 \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}^{-1} r_2 + r_1}$$

$$\begin{pmatrix} I_2 & O_2 & \vdots & \begin{pmatrix} 1 & -\frac{9}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{pmatrix} \\ O_2 & I_2 & \vdots & \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{pmatrix} \end{pmatrix},$$

so $A = \begin{pmatrix} 1 & -\frac{9}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{pmatrix}$.

*Example 4.*Let

$$\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} X_1 + 3I_2X_2 + \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} X_3 = I_2,$$

$$\begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} X_1 + \begin{pmatrix} 3 & 3 \\ 1 & 6 \end{pmatrix} X_2 + \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} X_3 = 2I_2,$$

$$\begin{pmatrix} 3 & 9 \\ 3 & 6 \end{pmatrix} X_1 + \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix} X_2 + 2I_2X_3 = O_2,$$

where $X_1 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $X_2 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, $X_3 = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$, I_2 *and* $O_2 \in M_2(\mathbb{K})$, then

$$\begin{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} & 3I_2 & \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} & \vdots & I_2 \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 3 & 3 \\ 1 & 6 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} & \vdots & 2I_2 \\ \begin{pmatrix} 3 & 9 \\ 3 & 6 \end{pmatrix} & \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix} & 2I_2 & \vdots & O_2 \end{pmatrix} \xrightarrow{\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}^{-1} r_1}$$

$$\begin{pmatrix} I_2 & \begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} & \vdots & \begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 3 & 3 \\ 1 & 6 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} & \vdots & 2I_2 \\ \begin{pmatrix} 3 & 9 \\ 3 & 6 \end{pmatrix} & \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix} & 2I_2 & \vdots & O_2 \end{pmatrix} \xrightarrow[-\begin{pmatrix} 3 & 9 \\ 3 & 6 \end{pmatrix} r_1 + r_3]{-\begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} r_1 + r_2 \&}$$

$$\begin{pmatrix} I_2 & \begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} & \vdots & \begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \\ O_2 & \begin{pmatrix} 3 & -3 \\ -1 & 4 \end{pmatrix} & \begin{pmatrix} 0 & -7 \\ -\frac{7}{3} & -\frac{7}{3} \end{pmatrix} & \vdots & \begin{pmatrix} 4 & -6 \\ \frac{4}{3} & -\frac{8}{3} \end{pmatrix} \\ O_2 & \begin{pmatrix} 1 & -3 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & -6 \\ -2 & -3 \end{pmatrix} & \vdots & \begin{pmatrix} 3 & -9 \\ 2 & -7 \end{pmatrix} \end{pmatrix} \xrightarrow{\begin{pmatrix} 3 & -3 \\ -1 & 4 \end{pmatrix}^{-1} r_2}$$

$$\begin{pmatrix} I_2 & \begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix} & \vdots & \begin{pmatrix} 1 & -1 \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \\ O_2 & I_2 & \begin{pmatrix} -\frac{7}{9} & -\frac{35}{9} \\ -\frac{7}{9} & -\frac{14}{9} \end{pmatrix} & \vdots & \begin{pmatrix} \frac{20}{9} & -\frac{32}{9} \\ \frac{8}{9} & \frac{14}{9} \end{pmatrix} \\ O_2 & \begin{pmatrix} 1 & -3 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} -1 & -6 \\ -2 & -3 \end{pmatrix} & \vdots & \begin{pmatrix} 3 & -9 \\ 2 & -7 \end{pmatrix} \end{pmatrix} \xrightarrow[-\begin{pmatrix} 1 & -3 \\ -1 & 0 \end{pmatrix} r_2 + r_2]{-\begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix} r_2 + r_1 \&}$$

$$\begin{pmatrix} I_2 & O_2 & \begin{pmatrix} 0 & 8 \\ \frac{10}{9} & \frac{-4}{9} \end{pmatrix} & \vdots & \begin{pmatrix} -4 & 7 \\ \frac{1}{9} & \frac{2}{9} \end{pmatrix} \\ O_2 & I_2 & \begin{pmatrix} -\frac{7}{9} & -\frac{35}{9} \\ -\frac{7}{9} & -\frac{14}{9} \end{pmatrix} & \vdots & \begin{pmatrix} \frac{20}{9} & -\frac{32}{9} \\ \frac{8}{9} & \frac{14}{9} \end{pmatrix} \\ O_2 & O_2 & \begin{pmatrix} -\frac{5}{9} & -\frac{16}{9} \\ -\frac{11}{9} & -\frac{34}{9} \end{pmatrix} & \vdots & \begin{pmatrix} \frac{31}{9} & -\frac{91}{9} \\ -\frac{38}{9} & -\frac{95}{9} \end{pmatrix} \end{pmatrix} \xrightarrow{\begin{pmatrix} -\frac{5}{9} & -\frac{16}{9} \\ -\frac{11}{9} & -\frac{34}{9} \end{pmatrix}^{-1} r_3}$$

$$\left(\begin{array}{ccccc} I_2 & O_2 & \begin{pmatrix} 0 & 8 \\ \frac{10}{9} & \frac{-4}{9} \end{pmatrix} & \vdots & \begin{pmatrix} -4 & 7 \\ \frac{1}{9} & \frac{2}{9} \end{pmatrix} \\ O_2 & I_2 & \begin{pmatrix} -\frac{7}{9} & -\frac{35}{9} \\ -\frac{7}{9} & -\frac{14}{9} \end{pmatrix} & \vdots & \begin{pmatrix} \frac{20}{9} & -\frac{32}{9} \\ \frac{8}{9} & \frac{14}{9} \end{pmatrix} \\ O_2 & O_2 & I_2 & \vdots & \begin{pmatrix} \frac{223}{3} & -\frac{787}{3} \\ -\frac{151}{9} & \frac{263}{3} \end{pmatrix} \end{array}\right) \xrightarrow[-\begin{pmatrix} -\frac{7}{9} & -\frac{35}{9} \\ -\frac{7}{9} & -\frac{14}{9} \end{pmatrix} r_3 + r_2]{-\begin{pmatrix} 0 & 8 \\ \frac{10}{9} & \frac{-4}{9} \end{pmatrix} r_3 + r_1 \&}$$

$$\left(\begin{array}{ccccc} I_2 & O_2 & O_2 & \vdots & \begin{pmatrix} \frac{592}{3} & -\frac{2083}{3} \\ -\frac{281}{3} & \frac{992}{3} \end{pmatrix} \\ O_2 & I_2 & O_2 & \vdots & \begin{pmatrix} -\frac{227}{6} & \frac{400}{3} \\ \frac{176}{9} & -\frac{623}{9} \end{pmatrix} \\ O_2 & O_2 & I_2 & \vdots & \begin{pmatrix} \frac{223}{3} & -\frac{787}{3} \\ -\frac{151}{9} & \frac{263}{3} \end{pmatrix} \end{array}\right),$$

so $X_1 = \begin{pmatrix} \frac{592}{3} & -\frac{2083}{3} \\ -\frac{281}{3} & \frac{992}{3} \end{pmatrix}$, $X_2 = \begin{pmatrix} -\frac{227}{6} & \frac{400}{3} \\ \frac{176}{9} & -\frac{623}{9} \end{pmatrix}$ and $X_3 = \begin{pmatrix} \frac{223}{3} & -\frac{787}{3} \\ -\frac{151}{9} & \frac{263}{3} \end{pmatrix}$.

*Example 5.*Let

$$I_2X_1 + 2\, I_2X_2 = O_2,$$
$$3\, I_2X_1 + 4\, I_2X_2 = O_2,$$

where $X_1 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $X_2 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, I_2 and $O_2 \in M_2(\mathbb{K})$, then the system has $\infty -$many solutions.

*Example 6.*Let

$$I_2X_1 + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_2 + I_2X_3 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X_1 + O_2X_2 + 2I_2X_3 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},$$

where $X_1 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $X_2 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, $X_3 = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$, I_2 and $O_2 \in M_2(\mathbb{K})$, then the system has no solution.

3.3 Crammer's rule on MMs

In this subsection, we give Crammer's rule on MMs and use them in solving system of linear matrix equations.

Throughout this subsection, we consider $M_l'(\mathbb{K}) \subset M_l(\mathbb{K})$ be commutative matrices, and $\mathscr{A} \in \mathscr{M}_n\left(M_l'(\mathbb{K})\right)$, $\mathscr{B} \in \mathscr{M}_{n\times 1}\left(M_l'(\mathbb{K})\right)$ and $\mathscr{X} \in \mathscr{M}_{n\times 1}\left(M_l'(\mathbb{K}(x))\right)$ and let $\mathscr{A}_i$ be the MMs obtained from $\mathscr{A}$ by replacing the ith column of $\mathscr{A}$ by the column vector $\mathscr{B}$. Furthermore, let

$$D = \det \mathscr{A},\ N_1 = \det \mathscr{A}_1, \ldots,\ N_n = \det \mathscr{A}_n.$$

The fundamental relationship between determinants and the solution of the system $\mathscr{A}\mathscr{X} = \mathscr{B}$ will be given in the following theorem.

Theorem 2.*The square system $\mathscr{A}\mathscr{X} = \mathscr{B}$ has a solution if and only if D is non-singular. In this case, the unique solution is given by*

$$X_i = D^{-1}N_i,\ i \in \{1, 2, \ldots n\}.$$

*Proof.*The (square) system $\mathscr{A}\mathscr{X} = \mathscr{B}$ has a unique solution if and only if $\mathscr{A}$ is invertible, and $\mathscr{A}$ is invertible if and only if $D = \det \mathscr{A}$ is non-singular. Now suppose D non-singular, then $\mathscr{A}^{-1} = D^{-1} adj\mathscr{A}$. Multiplying $\mathscr{A}\mathscr{X} = \mathscr{B}$ by $\mathscr{A}^{-1}$, we obtain

$$\mathscr{X} = \mathscr{A}^{-1}\mathscr{A}\mathscr{X} = D^{-1}(adj\mathscr{A})\mathscr{B}. \tag{3.1}$$

Note that the ith row of $\frac{1}{D}(adj\mathscr{A})$ is $\frac{1}{D}(\mathscr{A}_{1i}, \mathscr{A}_{2i}, \ldots, \mathscr{A}_{ni})$. If $\mathscr{B} = (B_1, B_2, \ldots, B_n)^t$, then by (3.1),

$$X_i = D^{-1}(B_1\mathscr{A}_{1i} + B_2\mathscr{A}_{2i} + \ldots + B_n\mathscr{A}_{ni}).$$

However, $B_1\mathscr{A}_{1i}+B_2\mathscr{A}_{2i}+\ldots+B_n\mathscr{A}_{ni}=N_i$, the determinant of the matrix obtained by replacing the ith column of $\mathscr{A}$ by the column vector $\mathscr{B}$. Thus,

$$X_i = D^{-1}N_i.$$

*Example 7.*Applied Theorem 2 in Example 1

$$\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} X_1 + \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} X_2 = I_2,$$

$$\begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} X_1 + \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} X_2 = O_2,$$

where $X_1 = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, $X_2 = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$, then

$$D = \det\mathscr{A} = \det\begin{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} & \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -7 & -21 \\ -7 & -14 \end{pmatrix},$$

and

$$N_{X_1} = \det\begin{pmatrix} I_2 & \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \\ O_2 & \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix},$$

also

$$N_{X_2} = \det\begin{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} & I_2 \\ \begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix} & O_2 \end{pmatrix} = -\begin{pmatrix} 2 & 6 \\ 2 & 4 \end{pmatrix},$$

the unique solution is given by

$$X_1 = D^{-1}N_{X_1} = \begin{pmatrix} 1 & -\frac{9}{7} \\ -\frac{3}{7} & \frac{4}{7} \end{pmatrix},\ X_2 = D^{-1}N_{X_2} = \begin{pmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{pmatrix}.$$

4 Conclusion

In this work, we have introduced a definition for VMs, linearly dependent, independent VMs and rank of the MMs. Also, the system of the linear matrix equations are solved by using some methods. It is worthy to ensure that when we write the above matrix linear equations in the case of systems of linear equations whose coefficients are the scalar number instead of matrices and solve any system by using the classical methods we shall obtain the same results but we shall need a lot of papers and time.

Acknowledgment

The authors are very grateful to the anonymous referees for many valuable comments and suggestions which helped to improve the paper.

References

[1] Z. Kishka, M. Saleem, S. Sharqawy and A. Elrawy, On matrix of matrices over semirings, Math. Sci. Lett., 7(2), (2018), 107-110.

[2] Z. Kishka, M. Saleem, M. Abdalla and A. Elrawy, On Hadamard and Kronecker products over matrix of matrices, General Letters in Mat., 4(1), (2018), 13-22.

[3] S.K. Mitra, A pair of simultaneous linear matrix equations $A_1XB_1 = C_1$and $A_2XB_2 = C_2$, Proc. Cambridge Philos. Soc. 74, (1973), 213–216.

[4] A.B. Ozgüler and N. Akar, A common solution to a pair of linear matrix equations over a principle domain, Linear Algebra Appl. 144, (1991), 85-99.

[5] A. Navarra, P.L. Odell and D.M. Young, A representation of the general common solution to the matrix equations $A_1XB_1 = C_1$and $A_2XB_2 = C_2$ with applications, Comput. Math. Appl. 41, (2001), 929–935.

[6] J. Van der Woulde, Almost noninteracting control by measurement feedback, Systems Control Lett.9, (1987), 7–16.

[7] J. Van der Woulde, Feedback Decoupling and Stabilization for Linear System with Multiple Exogenous Variables, Ph.D. Thesis, Technical University of Eindhoven, Netherlands, (1987).

[8] N. Shinozaki and M. Sibuya, Consistency of a pair of matrix equations with an application, Keio Eng. Rep. 27, (1974), 141–146.

[9] Q.W. Wang, The decomposition of pairwise matrices and matrix equations over an arbitrary skew field, Acta Math. Sinica 39, (1996), 396–403.

[10]S.K. Mitra, A pair of simultaneous linear matrix equations and a matrix programming problem, Linear Algebra Appl. 131, (1990), 97–123.

[11]S. Gharib, S. R. Ali, R. Khan, N. Munir and M. Khanam, System of linear equations, Gaussian elimination, Global J. Inc., (2015).

Zanhom M. Kishka has got Ph.D. degree from Assiut University 1984 in the field of complex analysis. He is professor in mathematics department, faculty of science, Sohag University. His research interests are in the areas complex analysis and theory of matrices. He has published research articles in reputed international journals of mathematical.

Mohamed. A. Abul-Dahab earned his M.Sc. degree. In 2010 in pure mathematics from South Valley University, faculty of science, Qena, Egypt. In 2013, he received his Ph.D. degree from South Valley University, faculty of science, Qena, Egypt. His research interests include complex analysis, special function, and matrix analysis. He has published research articles in reputed international journals of mathematical sciences. He is referee and editor of mathematical journals.

Amr M. Elrawy has M.Sc. degree in mathematics science from South Valley University 2013. He assistant lecturer in mathematics department, faculty of science, South Vallely University. His research interests are in the areas of algebraic geometry and algebra. He has published research articles in reputed international journals of mathematical.