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ON THE REPRESENTATION OF ANALYTIC FUNCTIONS BY SERIES OF DERIVED BASES OF POLYNOMIALS IN HYPERELLIPTICAL REGIONS

Mohamed S. Al-Sheikh [a,*], Gamal F. Hassan [b], Abd Almonem M. Ibrahim [a], Ahmed M. Zahran [a]

[a] Mathematics department, Faculty of Science, Al-Azhar University, Assiut branch, Egypt.
[b] Mathematics department, Faculty of Science, Assiut University, Assiut, Egypt.
*Corresponding Author: medoalshekh77@gmail.com

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ABSTRACT

One of the important themes in complex analysis is the expansion of analytic functions by infinite series in a given sequence of bases of polynomials. In the present paper, we investigated the representation of analytic functions in different domains of derived bases of polynomials. The behaviour of the associated representation of whole functions is directly related to determining the convergence properties (effectiveness) of such bases. The representation domains are closed hyperellipses, open hyperellipses, and closed regions surrounding a closed hyperellipse. Also, some results concerning the order of derived bases in hyperellipse are obtained. The results obtained are natural generalisations of the results obtained in hyperspherical regions.

Keywords: Bases of polynomials; Effectiveness; Hyperelliptical regions; Derived bases.

1. INTRODUCTION

The base of polynomials is considered a powerful theory with many applications in analysis, mathematical physics, approximation theory, Geometry, partial differential equations and mathematical physics. The basic sets (bases) of polynomials of one complex variable was first introduced by Whittaker in [1] who laid down the definition of bases, basic series, effectiveness, and order of a base. Many well-known polynomials, including Lagendre, Laguerre, Bernoulli, Euler, Hermite, Bessel, and Chebyshev polynomials [2-6], have simple bases.

The authors of [3,5] proved that Bernoulli and Euler's polynomials were not found to be effective anywhere. Furthermore, they determined that each of these polynomials is of order 1. In [2,6] Bessel polynomials were shown to be everywhere effective. Besides, the authors of [7] studied the effectiveness of the Chebyshev polynomials in the unit disk. In [8,9] Cannon provided the necessary and sufficient conditions for the effectiveness of bases in classes of holomorphic functions with finite regularity radius and entire functions. Mursi and Makar [10] introduced the theory of bases of polynomials in several complex variables in polycylindrical regions (complete Reinhart domains). Also, the bases of polynomials in several complex variables in hyperspherical and hyperelliptical regions are discussed by Nassif [11], kishka and others [12-19].

In [20], Abul-Ez and Constales applied the theory of polynomial bases in one variable to the context of Clifford analysis. Many authors studied the bases of polynomials in Clifford analysis [5, 20-30]. Also, there are studies on bases of polynomials in Faber regions [31, 32]. The topic of derivative base of polynomials in one complex variable has been studied by the authors [33-35], they considered the disks in the complex plane. For several complex variables (see [16,18], the representation domains are hyperspherical and hyperelliptical regions. Recently, in [26,36] the authors investigated this problem in Clifford setting which is called hypercomplex derivative bases of special monogenic polynomials, where the representation in closed balls.

In this paper we study the convergence properties (The effectiveness of the derived base) in several domains (closed hyperellipse, open hyperellipse, closed regions surrounding closed hyperellipse). Moreover, we shall study the order of the derived base in closed hyperellipse.These results indicate the generalisation of previous studies on effectiveness in the hyperspherical regions.

2 NOTATION AND BASIC RELATIONS

The following notations are used throughout this work to prevent long scripts (see [11, 16, 17]).

$$\mathrm{m} = m_1, m_2, \dots, m_k; \quad < \mathrm{m} \geq m_1 + m_2 + \dots + m_k;$$
$$\mathrm{h} = h_1, h_2, \dots, h_k; \quad < \mathrm{h} \geq h_1 + h_2 + \dots + h_k;$$
$$\mathrm{z} = z_1, z_2, \dots, z_k; \quad z^m = z_1^{m_1}.z_2^{m_2}\dots.z_k^{m_k}; \mathbf{0} = (^0, ^0, \dots, ^0;$$
$$|z|^2 = |z_1|^2 + |z_2|^2 + \dots + |z_k|^2; \quad R = R_1, \boldsymbol{R}_2, \dots, R_k;$$
$$R_i = R_i^{(1)}, R_i^{(2)}, \dots\dots, R_i^{(k)}; \quad \alpha R = \alpha_1 \boldsymbol{R}, \alpha_2 R_2, \dots, \alpha_k \boldsymbol{R};$$

$$\alpha([r],[R]) = \max\left\{ r_1 \prod_{s=2}^{n} R_s, r_\nu \prod_{s=1, s\neq \nu}^{n} R_s, r_n \prod_{s=1}^{n-1} R_s \right\};$$

where $m_1, m_2, \dots, m_k$ and $h_1, h_2, \dots, h_k$ are non-negative integers, $\nu = \{2,3,\dots,k-1\}$.

In the space $\mathbb{C}^k$, an open hyperelliptical region $\sum_{s=0}^{k} \frac{|z_s|^2}{R_s^2} < \mathbf{1}$ is here denoted by $E_{[R]}$ and its closure $\sum_{s=1}^{k} \frac{|z_s|^2}{R_s^2} \leq \mathbf{1}$, by $\bar{E}_{[R]}$, where $R_s, \boldsymbol{s} \in \boldsymbol{I}$ are positive numbers. In terms of the introduced notations, these regions satisfy the following inequalities:

$$\bar{E}_{[R]} = \{ \mathcal{W} : |\mathcal{W}| \leq 1 \},$$
$$E_{[R]} = \{ \mathcal{W} : |\mathcal{W}| < 1 \},$$
$$D(\bar{E}_{[R]}) = \{ \mathcal{W}^* : |\mathcal{W}^*| \leq 1 \}, \tag{2.1}$$

where $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2, \dots, \mathcal{W}_k), \mathcal{W}_s = \frac{z_s}{R_s}$ and $\mathcal{W}^* = (\mathcal{W}_1^*, \mathcal{W}_2^*, \dots, \mathcal{W}_k^*), \mathcal{W}_s^* = \frac{z_s}{R_s^+}, s \in \boldsymbol{I}$.

Definition 2.1 [11, 15] A base of polynomials

$$\{P_m[z]\} = \{P_0[z], P_1[z], P_2[z], \dots, P_n[z], \dots\},$$

is said to be base when every polynomial in the complex variables $z_s, \boldsymbol{s} \in \boldsymbol{I}$, may only be described as a finite linear combination of the elements of the base $\{P_m[z]\}$. Thus, according to [10] the set $\{P_m[z]\}$ will be base if and and only if there exists a unique row-finite matrix $\bar{\boldsymbol{P}}$ such that

$$P\,\bar{P} = \bar{P}P = I, \tag{2.2}$$

where $P = [P_{m;h}]$ and $\bar{P} = [P_{m;h}]$ are the coefficient and operator matrices of the bases $\{P_m[z]\}$ respectively, and I is the unit matrix**.** Suppose that $f(z)$, is given by

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \tag{2.3}$$

is regular in $\overline{E}_{[R]}$ and

$$A[f; \overline{E}_{[R]}] = \sup_{\overline{E}_{[R]}} |f(z)|. \tag{2.4}$$

For the base $\{P_m[z]\}$, we have

$$P_m[z] = \sum_h P_{m,h} z^h, \tag{2.5}$$

$$z^m = \sum_h \overline{P}_{m,h} P_h[z], \tag{2.6}$$

For the function $f(z)$ in (2.3), substituting for z^m from (2.6) we get

$$f(z) = \sum_m \Pi_m P_m[z], \tag{2.7}$$

where

$$\Pi_m = \sum_h \overline{P}_{h,m} a_h, \tag{2.8}$$

The series $\sum_m \Pi_m P_m[z]$ is the basic series associated with $f(z)$.

Definition 2.2 A base $\{P_m[z]\}$ is effective in $\overline{E}_{[R]}$ if (2.7) converges uniformly to every analytic function in $\overline{E}_{[R]}$. Similar inclusion can be applied for $E_{[R]}$ and $D(\overline{E}_{[R]})$.

We use the following notations for Cannon sums to investigate the convergence properties of such polynomial bases in hyperelliptical regions (cf. [15, 16]).

$$A(\overline{P}_m, \overline{E}_{[R]}) = \sup_{\overline{E}_{[R]}} |\overline{P}_m[z]| \tag{2.9}$$

$$H(P_m, \overline{E}_{[R]}) = \sum_h |\overline{P}_{m,h}| A(P_h, \overline{E}_{[R]}) \tag{2.10}$$

$$\Psi(P_m, \overline{E}_{[R]}) = \sigma_m \prod_{s=1}^{k} R_s^{<m>-m_s} H(P_m, \overline{E}_{[R]}) \tag{2.11}$$

where

$$\sigma_m = \inf_{|t|=1} \frac{1}{t^m} = \frac{\{<m>\}^{\frac{<m>}{2}}}{\Pi_{s=1}^{k} m_s^{\frac{m_s}{2}}}, \tag{2.12}$$

and $1 \le \sigma_m \le (\sqrt{k})^{<m>}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0;\ s \in I$.

In hyperelliptical regions, a Cannon function was defined for the base of polynomials as follows:

$$\Psi[P,\overline{E}_{[R]}] = \limsup_{<m>\to\infty}\{\Psi[P_m,\overline{E}_{[R]}]\}^{\frac{1}{<m>}}. \tag{2.13}$$

Let $N_m = N_{m_1,m_2,\ldots,m_k}$ be the number of coefficients $\overline{P}_{m,h}$ that are non-zero in (2.6). A base $\{P_m[z]\}$, satisfying the condition

$$\lim_{<m>\to\infty}\{N_m\}^{\frac{1}{<m>}} = a, a > 1 \tag{2.14}$$

is called general base and if a = 1, then the base is called Cannon base [10].

Theorem concerning the effectivenss of bases of polynomials in hyperelliptical regions are due to [15, 16].

Theorem 2.1 The necessary and sufficient condition for a base $\{P_m[z]\}$ of polynomials to be effective in $\overline{E}_{[R]}$, $E_{[R]}$ or $D(\overline{E}_{[R]})$ is that

$\Psi(P,\overline{E}_{[R]}) = \prod_{s=1}^{k} R_s$, $\Psi(P,E_{[R]}) < \alpha([r],[R])$ or $\Psi(P,D(\overline{E}_{[R]})) = \prod_{s=1}^{k} R_s$ respectively.

The order of a base $\{P_m[z]\}$ in the hyperellipse $\overline{E}_{[\alpha R]}$ is defined in [15] by

$$\omega = \lim_{R\to\infty}\limsup_{<m>\to\infty}\frac{\log\Psi[P_m;\overline{E}_{[\alpha R]}]}{<m>\log<m>}. \tag{2.15}$$

The fact of order ω lies in that if the base of polynomials $\{P_m[z]\}$ is of finite order ω, it will represent in any finite ellipse, every entire function of order less than $\frac{1}{\omega}$. We refer to the work of [3-5, 24, 37-39] in relation to this order of the bases. For more information on the study of polynomials of bases [2, 18, 2, 39-40].

3 DERIVED BASES OF POLYNOMIALS

The $\mathfrak{D}^{(N)}$-operator which is defined and studied in [40] in the case of three complex variables and is defined as follows in the case of several complex variables acting also on monomial z^m:

$$\mathfrak{D}^{(N)}z^m = \begin{cases}(\mathfrak{D}_1+\mathfrak{D}_2+\cdots+\mathfrak{D}_k)^N z^m; & m\neq 0\\ 1; & m=0\end{cases} \tag{3.1}$$

where $\mathfrak{D}_s = z_s\frac{\partial}{\partial z_s}, s\in I$.

N_s-times the derivatives are applied; $s\in I$. Thus,

$$\mathfrak{D}^{(N)}z^m = \begin{cases}\prod_{i=0}^{N-1}(<m>-i)z^m; & m\neq 0\\ 1; & m=0\end{cases} \tag{3.2}$$

Applying $\mathfrak{D}^{(N)}$ into (2.6) we have

$$\begin{cases} \prod_{i=0}^{N-1} (<m> - i) z^m = \sum_h \overline{P}_{m,h} P_h^{(N)}[z]; & m \neq 0 \\ 1 = \sum_h \overline{P}_{0,h} P_h^{(N)}[z]; & m = 0 \end{cases}$$

where

$$\begin{aligned} P_m^{(N)}[z] &= \mathfrak{D}^{(N)} P_m[z] \\ &= P_{m,0} + \sum_{h \geq 1} P_{m,h} \prod_{i=0}^{N-1} (<h> - i) z^h \\ &= \sum_h \delta_{N,h} \mathbf{P}_{m,h} z^h \end{aligned}$$

and

$$\delta_{N,h} = \begin{cases} \prod_{i=0}^{N-1} (<h> - i); & h \neq 0 \\ 1; & h = 0 \end{cases}$$

The set $P_m^{(N)}[z]$ is said to be derived base of polynomials. The basic property of the set $P_m^{(N)}[z]$ is constructed as follows:

$$\mathfrak{D}^{(N)} P_m[z] = \sum_h \delta_{N,h} P_{m,h} z^h = \sum_h P_{m,h}^{(N)} z^h$$

where $P^{(N)} = (\delta_{N,h} P_{m,h})$ is the matrix of coefficients of the base $P_m^{(N)}[z]$. Also, the matrix of operators $\overline{P}^{(N)}$ follows from the representation

$$z^m = \frac{1}{\delta_{N,m}} \sum_h \overline{P}_{m,h} P_h^{(N)}[z] = \sum_h \overline{P}_{m,h}^{(N)} P_h^{(N)}[z]$$

that is to say $\overline{P}^{(N)} = \left(\frac{1}{\delta_{N,m}} \overline{P}_{m,h} \right)$. Therefore

$$\begin{aligned} P^{(N)} \overline{P}^{(N)} &= \left(\sum_h P_{m,h}^{(N)} \overline{P}_{h,k}^{(N)} \right) \\ &= \left(\sum_h \delta_{N,h} \mathbf{P}_{m,h} \frac{1}{\delta_{N,h}} \overline{P}_{h,k} \right) \\ &= P\overline{P} = \boldsymbol{I}. \end{aligned}$$

Similarly, we find that

$$\overline{P}^{(N)} P^{(N)} = \left(\frac{\delta_{N,k}}{\delta_{N,m}} \delta_k^m \right) = \boldsymbol{I},$$

where δ_k^m is the symbol for Kronneker. Therefore the bases property of $\mathfrak{D}^{(N)}$ operator bases $\{P_m^{(N)}[z]\}$ follows directly from (2.2).

4 EFFECTIVENESS OF DERIVED BASE OF POLYNOMIALS IN CLOSED HYPERELLIPSE

We consider the following question: In a closed hyperellipse $\overline{E}_{[R]}$, If the set $\{P_m[z]\}$ is effective. Does the base $\{P_m^{(N)}[z]\}$ still effective in the same region? The answer to this question will be given in this section. Suppose that $\{P_m[z]\}$ be a base of polynomials and $\{P_m^{(N)}[z]\}$ be base associated to $\{P_m[z]\}$. Let $\Psi(P_m^{(N)}, \bar{E}_{[R]})$ be the Cannon sum of the base $\{P_m^{(N)}[z]\}$ for $\bar{E}_{[R]}$, then

$$\Psi(P_m^{(N)}, \bar{E}_{[R]}) = \sigma_m \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \sum_h \left|\bar{P}_{m,h}^{(N)}\right| A\left(P_h^{(N)}, \bar{E}_{[R]}\right)$$

$$= \frac{\sigma_m}{\delta_{N,h}} \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \sum_h |\bar{P}_{m,h}| A\left(P_h^{(N)}, \bar{E}_{[R]}\right) \quad (4.1)$$

where

$$A(P_m^{(N)}, \bar{E}_{[R]}) = \sup_{\bar{E}_{[R]}} |P_m^{(N)}[z]|$$

Let, D_m be the degree of the polynomial with the highest degree in the representation (2.6). Hence by Cauchy's inequality we get

$$A(P_m^{(N)}, \bar{E}_{[R]}) = \sup_{\bar{E}_{[R]}} |P_m^{(N)}[z]| \le \sum_h \frac{|P_{m,h}^{(N)}| \left(\prod_{s=1}^{k} \{R_s\}^{h_s}\right)}{\sigma_m}$$

$$= \sum_h \frac{\delta_{N,h} |P_{m,h}| \left(\prod_{s=1}^{k} \{R_s\}^{h_s}\right)}{\sigma_m}$$

$$\le A\left(P_m, \bar{E}_{[R]}\right) \sum_h \delta_{N,h}$$

$$\le A\left(P_m, \bar{E}_{[R]}\right) \left[1 + \sum_{h \ge 1} \left(\prod_{i=0}^{N-1} (<h> - i)\right)\right] \quad (4.2)$$

$$\le L D_m^{N+1} A\left(P_m, \bar{E}_{[R]}\right)$$

where L is a constant.

From the relations (4.1) and (4.2) may be used to derive the relation between the Cannon sums of the two bases $\{P_m[z]\}$ and $\{P_m^{(N)}[z]\}$

$$\Psi(P_m^{(N)}, \bar{E}_{[R]}) \le \frac{L D_m^{N+1}}{\delta_{N,h}} \Psi(P_m, \bar{E}_{[R]}) \quad (4.3)$$

Consider the condition

$$\lim_{<m> \to \infty} \{D_m\}^{\frac{1}{<m>}} = 1 \quad (4.4)$$

we have

$$\Psi(P^{(N)}, \bar{E}_{[R]}) = \limsup_{<m> \to \infty} \left[\Psi(P_m^{(N)}, \bar{E}_{[R]})\right]^{\frac{1}{<m>}}$$

$$\le \limsup_{<m> \to \infty} \left[\frac{L D_m^{N+1}}{\delta_{N,h}} \Psi\left(P_m, \bar{E}_{[R]}\right)\right]^{\frac{1}{<m>}}$$

$$\le \Psi\left(P, \bar{E}_{[R]}\right) = \prod_{s=1}^{k} R_s \quad (4.5)$$

But

$$\Psi(P^{(N)},\overline{E}_{[R]}) \geq \prod_{s=1}^{k} R_s \quad (4.6)$$

Then,

$$\Psi(P^{(N)},\overline{E}_{[R]}) = \prod_{s=1}^{k} R_s \quad (4.7)$$

According to (4.7) and **Theorem 2.1**, we may conclude that the effectiveness of the original set $\{P_m[z]\}$ in $\overline{E}_{[R]}$ implies the effectiveness of derived base $\{P_m^{(N)}[z]\}$ in $\overline{E}_{[R]}$. Hence, we get the following theorem:

Theorem 4.1. If the base $\{P_m[z]\}$ of polynomials for which the condition (4.4) is satisfied, is effective in $\overline{E}_{[R]}$, then the derived base $\{P_m^{(N)}[z]\}$ of polynomials associated with the base $\{P_m[z]\}$ will be effective in $\overline{E}_{[R]}$. If, condition (4.4) is not satisfied then the base $\{P_m^{(N)}[z]\}$ can not be effective in $\overline{E}_{[R]}$, where the base $\{P_m[z]\}$ is effective in $\overline{E}_{[R]}$. To ensure this, we give the following example.

Example 4.1 Consider the base $\{P_m[z]\}$ of polynomials given by

$$P_m[z] = \begin{cases} \sigma_m z^m + \sigma_{cm} z^{cm} & ; m \neq 0 \\ \sigma_m z^m & ; \text{otherwise}, \end{cases}$$

where $c = d^{<m>}, \mathbf{d} > \mathbf{1}$, then

$$z^m = \frac{1}{\sigma_m [P_m[z] - P_{cm}[z]]}. \quad (4.8)$$

The Cannon sum $\Psi(P_m, \overline{E}_{[R]})$ will given by

$$\Psi(P_m, \overline{E}_{[R]}) = \prod_{s=1}^{k} \left[R_s^{<m>} + 2R_s^{<m>+(c-1)m_s}\right] \quad (4.9)$$

It turns out that

$$\Psi(P, \overline{E}_{[1]}) \leq \limsup_{<m>\to\infty} [\Psi(P_m, \overline{E}_{[1]})]^{\frac{1}{<m>}} = \mathbf{1}. \quad (4.10)$$

That is mean that the base $\{P_m[z]\}$ is effective in $\overline{E}_{[1]}$ for $R_s = 1; \boldsymbol{s} \in \boldsymbol{I}$.

Now, construct derived base $\{P_m^{(N)}[z]\}$ as follows;

$$P_m^{(N)}[z] = \begin{cases} \sigma_m \delta_{N,m} z^m + \sigma_{cm} \delta_{N,cm} z^{cm} & ; m \neq 0 \\ \sigma_m \delta_{N,m} z^m & ; \text{otherwise}, \end{cases}$$

Hence,we find

$$z^m = \frac{1}{\sigma_m \delta_{N,m} \left[P_m^{(N)}[z] - P_{cm}^{(N)}[z]\right]} \quad (4.11)$$

and $\Psi(P_m^{(N)}, \overline{E}_{[R]})$ will produce the Cannon sum

$$\Psi(P_m^{(N)}, \overline{\boldsymbol{E}}_{[R]}) = \sigma_m \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \sum_h \left|\overline{P}_{m,h}^{(N)}\right| \boldsymbol{A}\left(P_h^{(N)}, \overline{E}_{[R]}\right)$$

$$= \prod_{s=1}^{k} \square\ R_s^{<m>} + \Im(c) \prod_{s=1}^{k} \square\ R_s^{<m>+(c-1)m_s}$$

where $\Im(c) = \frac{2\delta_{N,cm}}{\delta_{N,m}} > 1$ is a constant that only depends on c and

$$\Psi(P, \overline{E}_{[1]}) = \limsup_{<m>\to\infty} [1 + \Im(c)]^{\frac{1}{<m>}} > 1.$$

That is, the derived base $\{P_m^{(N)}[z]\}$ is not effective in $\overline{E}_{[R]}$ for $R_s = 1, s \in I$, but the original set $\{P_m[z]\}$ is effective in $\overline{E}_{[R]}$. The reason for this is that the set $\{P_m[z]\}$ does not satisfy condition (4.4) as necessary.

5 Effectiveness of derived base of polynomials in open hyperellipse and the region $D(\overline{E}_{[R]})$.

The effectiveness property for the derived base $\{P_m^{(N)}[z]\}$ in open hyperllipse and the Region $D(\overline{E}_{[R]})$ is established in this section. Assume that the Cannon sum $\{P_m[z]\}$ is effective in $E_{[R]}$. Then, based on the properties of Cannon functions, [15], it follows that

$$\Psi(P, \overline{E}_{[R]}) < \alpha([r],[R]), \qquad \forall \quad 0 < R_s < r_s, s \in I. \tag{5.1}$$

Constructing the sets of numbers $\{r_i^s, s \in I\}$, (cf.[15]), in such a way that $0 < R_s < r_0^s, s \in I$ and

$$\frac{r_0^{(s)}}{r_0^{(j)}} = \frac{r_s}{r_j}, \qquad j, s \in I, \tag{5.2}$$

$$r_{i+1}^{(s)} = \frac{1}{2\left(r_s + r_i^{(s)}\right)}, s \in I; i \geq 0. \tag{5.3}$$

It follows, easily, from (5.2) and (5.3) that

$$\frac{r_i^{(s)}}{r_i^{(j)}} = \frac{r_s}{r_j}, \qquad j, s \in I, ; i \geq 0 \tag{5.4}$$

Therefore it follows that

$$R_s < r_i^{(s)} < r_s, s \in I; i \geq 0. \tag{5.5}$$

Now, since the base $\{P_m[z]\}$ accord to (5.1), (2.11) and (2.13), then corresponding to the numbers $r_i^{(s)}, s \in I$, there exists a constant $K \geq 1$ such that

$$\sigma_m \prod_{s=1}^{k} \square \left\{r_i^{(s)}\right\}^{<m>-m_s} H\left(P_m, \overline{E}_{[r_i]}\right) < K\left\{r_{i+1}^{(1)} \prod_{s=2}^{k} \square\ r_i^{(s)}\right\}^{<m>} \tag{5.6}$$

In view of (5.4), we obtain the following inequality

$$H\left(P_m, \overline{E}_{[r_i]}\right) < \frac{K}{\sigma_m \left\{\frac{r_{i+1}^{(1)}}{r_i^{(1)}}\right\}^{<m>} \prod_{s=1}^{k} \square} \left\{r_i^{(s)}\right\}^{m_s}$$

$$= \frac{K}{\sigma_m}\prod_{s=1}^{k} \left\{\frac{r_{i+1}^{(1)}}{r_i^{(1)}} r_i^s\right\}^{m_s}$$

$$= \frac{K}{\sigma_m}\prod_{s=1}^{k} \left\{\frac{r_{i+1}^{(s)}}{r_i^{(s)}} r_i^s\right\}^{m_s} \quad \textbf{(5.7)}$$

$$= \frac{K}{\sigma_m}\prod_{s=1}^{k} \left\{r_{i+1}^{(s)}\right\}^{m_s}, (m_s \geq \mathbf{0}; s \in I)$$

Therefore, we have at least one of the following cases for the integers $R_s; r_s; s \in \boldsymbol{I}$:

(a) $\frac{R_1}{R_s} \leq \frac{r_1}{r_s}; s \in I$ or

(b) $\frac{R_v}{R_s} \leq \frac{r_v}{r_s}; s \in I, v = 2 \text{ or } 3 \text{ or} \dots \text{or } k-1$ or

(c) $\frac{R_k}{R_s} \leq \frac{r_k}{r_s}; s \in I$.

Assuming the relation (a) is satisfied, we can deduce from the construction of the sets $\{r_i^{(s)}, s \in I\}$, that

$$\frac{R_1}{R_s} \leq \frac{r_1}{r_s} = \frac{r_{i+1}^{(1)}}{r_{i+1}^{(s)}}; s \in I \quad (5.8)$$

Using eq.(5.7) and (5.8), the cannon sum of the base $\{P_m^{(N)}[z]\}$ for $E_{[R]}$, we obtain

$$\Psi(P_m^{(N)}, \overline{E}_{[R]}) = \sigma_m \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \sum_h \left|\overline{P}_{m,h}^{(N)}\right| A\left(P_h^{(N)}, \overline{E}_{[R]}\right)$$

$$= \frac{\sigma_m}{\delta_{N,m}} \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \sum_h \left|\overline{P}_{m,h}\right| A\left(P_h^{(N)}, \overline{E}_{[R]}\right)$$

$$< L\frac{\sigma_m}{\delta_{N,m}} \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \sum_h \left|\overline{P}_{m,h}\right| A\left(P_h, \overline{E}_{[r]}\right)$$

$$= L\frac{\sigma_m}{\delta_{N,m}} \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} H\left(P_m, \overline{E}_{[r]}\right)$$

$$< \frac{KL}{\delta_{N,m}} \prod_{s=1}^{k} \{R_s\}^{<m>-m_s} \left\{r_{i+1}^{(s)}\right\}^{m_s}$$

$$= \frac{KL}{\delta_{N,m}} \prod_{s=1}^{k} \left\{r_{i+1}^{(s)}\right\}^{m_s} \left\{\frac{R_1}{R_s}\right\}^{m_s} \prod_{s=2}^{k} \{R_s\}^{<m>}$$

$$\leq \frac{KL}{\delta_{N,m}} \prod_{s=1}^{k} \left\{r_{i+1}^{(s)}\right\}^{m_s} \left\{\frac{r_1}{r_s}\right\}^{m_s} \prod_{s=2}^{k} \{R_s\}^{<m>}$$

$$= \frac{KL}{\delta_{N,m}} \prod_{s=1}^{k} \left\{r_{i+1}^{(s)}\right\}^{m_s} \left\{\frac{r_{i+1}^{(1)}}{r_{i+1}^{(s)}}\right\}^{m_s} \prod_{s=2}^{k} \{\boldsymbol{R}_s\}^{<m>}$$

$$= \frac{KL}{\delta_{N,m}\left\{r_{i+1}^{(1)} \prod_{s=2}^{k} R_s\right\}^{<m>}}$$

which implies that

$$\Psi(P^{(N)},\bar{E}_{[R]}) = \limsup_{<m>\to\infty}\left[\Psi\left(P_m^{(N)},\bar{E}_{[R]}\right)\right]^{\frac{1}{<m>}}$$

$$\leq r_{i+1}^{(1)}\prod_{s=2}^{k} R_s$$

$$< r_1\prod_{s=2}^{k} \boldsymbol{R}_s \tag{5.9}$$

where

$$L = 1 + \sum_{h\geq 1}\left(\prod_{i=0}^{N-1}(<h>-i)\right)\prod_{s=1}^{k}\left\{\frac{R_i^{(s)}}{r_i^{(s)}}\right\}^{h_s} \quad \forall\ \mathbf{0} < R_s < r_s;\ \boldsymbol{s}\in I.$$

In addition, if relation (b) holds for $\nu = 2 \text{ or } \mathbf{3}\ \boldsymbol{or} \ldots \boldsymbol{or}\ k-1$, we will get

$$\frac{R_\nu}{R_s} \leq \frac{r_\nu}{r_s} = \frac{r_{i+1}^{(\nu)}}{r_{i+1}^{(s)}}; s\in I. \tag{5.10}$$

Thus (5.7) and (5.10) lead to

$$\Psi(P_m^{(N)},\bar{\boldsymbol{E}}_{[R]}) < \frac{KL}{\delta_{N,m}}\prod_{s=1}^{k}\{R_s\}^{<m>-m_s}\left\{r_{i+1}^{(s)}\right\}^{m_s}$$

$$= \frac{KL}{\delta_{N,m}}\prod_{s=1}^{k}\left\{r_{i+1}^{(s)}\right\}^{m_s}\left\{\frac{R_\nu}{R_s}\right\}^{m_s}\prod_{s=1,s\neq\nu}^{k}\{R_s\}^{<m>}$$

$$\leq \frac{KL}{\delta_{N,m}}\prod_{s=1}^{k}\left\{r_{i+1}^{(s)}\right\}^{m_s}\left\{\frac{r_\nu}{r_s}\right\}^{m_s}\prod_{s=1,s\neq\nu}^{k}\{R_s\}^{<m>}$$

$$= \frac{KL}{\delta_{N,m}}\prod_{s=1}^{k}\left\{r_{i+1}^{(s)}\right\}^{m_s}\left\{\frac{r_{i+1}^{(\nu)}}{r_{i+1}^{(s)}}\right\}^{m_s}\prod_{s=1,s\neq\nu}^{k}\{R_s\}^{<m>}$$

$$= \frac{KL}{\delta_{N,m}\left\{r_{i+1}^{(\nu)}\prod_{s=1,s\neq\nu}^{k} R_s\right\}^{<m>}}$$

Therefore,

$$\Psi(P^{(N)},\bar{E}_{[R]}) = \limsup_{<m>\to\infty}\left[\Psi\left(P_m^{(N)},\bar{E}_{[R]}\right)\right]^{\frac{1}{<m>}}$$

$$\leq r_{i+1}^{(\nu)}\prod_{s=1,s\neq\nu}^{k} R_s < r_\nu\prod_{s=1,s\neq\nu}^{k} R_s \tag{5.11}$$

Similarly, if relation (c) is satisfied, we proceed as before to demonstrate.

$$\Psi(P^{(N)},\bar{E}_{[R]}) < r_k\prod_{s=1}^{k-1} R_s. \tag{5.12}$$

Thus, it follows in view of (5.9), (5.11) and (5.12) that

$$\Psi(P^{(N)},\bar{E}_{[R]}) < \boldsymbol{\alpha}([r],[R]). \tag{5.13}$$

As a result of (5.13) and **Theorem 2.2**, the derived base $\{P_m^{(N)}[z]\}$ is effective for $E_{[r]}$ when the original base $\{P_m[z]\}$ is effective for $E_{[r]}$. Hence, we get the following theorem:

Theorem 5.1. If the base $\{P_m[z]\}$ of polynomials is effective for $E_{[R]}$, then the derived base $\{P_m^{(N)}[z]\}$ of polynomials associated with the base $\{P_m[z]\}$ will be effective for $E_{[R]}$. Now, using a reasoning similar to that used to prove **Theorem 5.1**, the following relationship emerges.

$$\Psi\left(P^{(N)}, D(\bar{E}_{[R]})\right) = \prod_{s=1}^{k} R_s \quad when \quad \Psi\left(P, D(\bar{E}_{[R]})\right) = \prod_{s=1}^{k} R_s \tag{5.14}$$

As a result of (5.14), and **Theorem 2.3**, hence the following theorem

Theorem 5.2. If the Cannon base $\{P_m[z]\}$ of polynomials is effective for $D(\bar{E}_{[R]})$, then the derived base $\{P_m^{(N)}[z]\}$ of polynomials associated with the base $\{P_m[z]\}$ will be effective for $D(\bar{E}_{[R]})$.

6 THE ORDER OF DERIVED BASE

Let $\{P_m[z]\}$ be a base of order ρ and the derived base $\{P_m^{(N)}[z]\}$ is of order $\rho^{(N)}$. The following theorem gives the relation between the orders of the two bases $\{P_m[z]\}$ and $\{P_m^{(N)}[z]\}$.

Theorem 6.1. If the base $\{P_m[z]\}$ is of order ρ and satisfying the condition

$$D_m = O[<m>^a] \quad , a \geq 1. \tag{6.1}$$

Then the base $\{P_m^{(N)}[z]\}$ will be of order $\rho^{(N)} \leq \rho$ .The upper bound is attainable .

Proof. From (4.3),we have

$$\lim_{R\to\infty}\limsup_{<m>\to\infty} \frac{\log\Psi\left(P_m^{(N)}, \bar{E}_{[\alpha R]}\right)}{<m>\log<m>} \leq \lim_{R\to\infty}\limsup_{<m>\to\infty} \frac{(N+1)\log D_m - \log\delta_{N,h} + \log\Psi\left(P_m, \bar{E}_{[\alpha R]}\right)}{<m>\log<m>}.$$

Through the definition of order, we have the order $\rho^{(N)}$ of the base $\{P_m^{(N)}[z]\}$ is at most ρ. To show that the two bases $\{P_m[z]\}$ and $\{P_m^{(N)}[z]\}$ is of the same order, we give the following example:

Example 6.1 Let $\{P_m[z]\}$ *be base given by*

$$P_m[z] = <m>^{<m>} + \sigma_m z^m \quad , P_0[z] = 1$$

Hence

$$z^m = \frac{1}{\sigma_m [P_m[z] - <m>^{<m>}]} \tag{6.2}$$

and

$$\Psi(P_m, \bar{E}_{[\alpha R]}) = 2<m>^{<m>} + R_s^{<m>} \prod_{s=1}^{k} \alpha_s$$

$$= <m>^{<m>}\left[2 + \prod_{s=1}^{k} \alpha_s \left(\frac{R_s}{<m>}\right)^{<m>}\right]$$

Hence this base is of order $\rho = 1$ .Construct the base $\{P_m^{(N)}[z]\}$ as follows:

$$P_m^{(N)}[z] = <m>^{<m>} + \sigma_m \delta_{N,m} z^m \quad , P_0^{(N)}[z] = 1$$

then

$$\Psi\left(P_m^{(N)}, \overline{E}_{[\alpha R]}\right) = \frac{< m >^{<m>}}{\delta_{N,m}\left[2 + \delta_{N,m}\left(\frac{R_s}{< m >}\right)^{<m>}\right]} \tag{6.3}$$

Therefore the order of $\{P_m^{(N)}[z]\}$ is of order 1 .That is to say that each of the bases $\{P_m[z]\}$ and $\{P_m^{(N)}[z]\}$ are of the same order. Now ,we will give an example to show that the condition (6.1) is necessary for the validity of

Theorem 6.1.

Example 6.2 *Consider the base* $\{P_m[z]\}$ *given by*

$$P_m[z] = \begin{cases} \sigma_m z^m + \frac{\mu}{b^{2\mu}}\sigma_\mu z^\mu & ; m \neq 0 \\ \sigma_m z^m & ; \text{otherwise}, \end{cases}$$

Applying the definition of the order, To obtain the result $\rho^{(N)} > \rho$, we can follow the same steps as in example. i.e., Theorem is not verified .

7 CONCLUSION

The derived set of polynomials forms a base, as demonstrated in this study. Also, a study concerning the convergence properties of derived base of polynomials, such as effectiveness and the order in hyperelliptical will be carried out. The current work suggests exploring other possible generalizations using other derivative in different regions (e.g., polycylindrical regions, Faber regions). Also, in the future, it is likely to study the convergent properties of new sets of polynomials of several complex variables in different regions (e.g., Laguerre, Legendre, Hermit, and Gontcharoff polynomials) where the derived of these sets can be studied in the same regions. To derive the results for effectiveness and order in hyperspherical regions as special cases from the results for hyperelliptical regions, put $r_s = r, s \in I = \{1,2,\ldots,k\}$ in **Theorem 4.1**, **5.1**, **5.2** and **6.1.** When the original base, $\{P_m[z]\}$ is general base, similar results for the derived base $\{P_m^{(N)}[z]\}$ can be found in hyperelliptical regions.

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تمثيل الدوال التحليلية بواسطة متسلسلة في مشتقة اساسات عديدات الحدود في مناطق ناقصية

محمد صبري الشيخ[1] , جمال حسن فرغلي[2] , عبدالمنعم محمد ابراهيم [1], احمد محمد زهران [1]

1. قسم الرياضيات –كلية العلوم – جامعة الازهر فرع اسيوط
2. قسم الرياضيات –كلية العلوم – جامعة اسيوط

الملخص

احد الموضوعات الهامة في التحليل المركب هو مفكوك الدوال التحليلية بواسطة متسلسلة لا نهائية لمتتابعة من اساسات عديدات الحدود . في هذا البحث تم فحص تمثيل الدوال التحليلية في مناطق ناقصية مغلقة ومفتوحة ومناطق تحتوي مناطق ناقصية مغلقة . أيضا تم الحصول علي نتائج تختص بحساب الرتبة لمشتقة اساسات عديدات الحدود في مناطق ناقصية . نتائجنا تعتبر تعميم لنتائج سابقة في حالة المناطق الكروية.