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Interaction of two long waves in shallow water using Hirota-Satsuma model and similarity transformations

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ABSTRACT
Nonlinear evolution equations in one, two and three dimensions are the corner stone to understand and analyze numerous physical phenomena. The engineering applications comprise ocean waves, shallow water waves, plasma waves, fluid dynamics and many other applications. One of the most known models describing evolution equations is Hirota-Satsuma coupled system of equation in (1+1)-dimensions which describes the interaction of two long waves in shallow water. As the system is coupled and nonlinear, the group transformation technique was employed to transform the system of partial differential equations (PDEs) into a simpler system of ordinary differential equations (ODEs). The resultant system is numerically solved using Runge-Kutta and shooting methods. Two cases of similarity variable were attained leading to the interacted two waves. The results were illustrated graphically. The first solution led to two-solitons solution with large span as the wave conserves its energy and dissipates slowly. The second solution led to multiple solitons solution with higher lumps which enforces the wave to dissipate quicker than the first case.

Keywords: Evolution equations; Hirota-Satsuma equation; Group transformation method

1. Introduction

The investigation of exact traveling wave solutions to nonlinear partial differential equations plays an important role in the study of nonlinear physical phenomena. Nonlinear wave phenomena appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics and geochemistry. In 1981, Hirota and Satsuma first proposed the well-known Hirota – Satsuma KdV system of equations which describes the interaction of two long waves with different dispersion relations. Different methods were used to obtain several solutions of these equations such as Backlund transform (Yadong,2007), auxiliary function method (Feng et al,2008), extended tanh method (Bekir,2008), Lie group method (Zedan,2008, Singh et al,2006), soliton solutions (Wazwaz,2008), Darboux transformation (Yong et al,2005), the projective Riccati equations method (Hu et al,2003, Lu et al,2007), sine-cosine method (El-Wakil et al,2008) and the variation iteration method (Soliman et al,2007). One of the most powerful techniques is the similarity transformation methods including Lie infinitesimals, group transformation methods and hidden symmetries which could employed to construct either exact solutions or reduce the partial differential equation (PDE) model into ordinary differential equation (ODE) model (Rashed et al,2008, Kassem et al,2009, Rashed et al,2014, Mabrouk et al,2017, Kassem et al,2019, Mabrouk et al,2019, Rashed,2019, Rashed et al,2020, Rashed et al,2020, Rashed et al,2020, Rashed et al,2020, Saleh et al,2020, Rashed et al,2021, Rashed et al,2021, Rashed et al,2021, Saleh et al,2021)

2. Mathematical formulation

Consider Hirota-Satsuma system of equations describing the interaction of two long waves (Feng,1994);

$$u_t^* - u_{xxx}^* - 6u^* u_x^* - 2v^* v_x^* = 0 \tag{2.1}$$

$$v_t^* + v_{xxx}^* + 3u^* v_x^* = 0 \tag{2.2}$$

where u* and v* are real valued functions of x and t representing the evolution of two long waves in shallow water . The system, (2.1) and (2.2), is subjected to the initial conditions;

$$\begin{aligned} u^*(x,0) &= \phi(x) \\ \mathrm{v}^*(x,0) &= \psi(x) \end{aligned} \tag{2.3}$$

The following transformations are used to normalize the conditions in (2.3);

$$u(x,t) = \frac{u^*(x,t)}{\phi(x)}, \qquad \mathrm{v(x,\,t)} = \frac{\mathrm{v}^*(x,t)}{\psi(x)} \tag{2.4}$$

Hence, the equations, (2.1) and (2.2), are rewritten as;

$$\phi u_t - \phi u_{xxx} - 3\phi_x u_{xx} - 3\phi_{xx} u_x - \phi_{xxx} u - 6\phi^2 u u_x - 6\phi\phi_x u^2 - 2\psi^2 \mathrm{v}\mathrm{v}_\mathrm{x} - 2\psi\psi_x \mathrm{v}^2 = 0 \tag{2.5}$$

$$\psi \mathrm{v}_\mathrm{t} + \psi \mathrm{v}_\mathrm{xxx} + 3\psi_x \mathrm{v}_\mathrm{xx} + 3\psi_{xx} \mathrm{v}_\mathrm{x} + \psi_{xxx} \mathrm{v} + 3\phi\psi u \mathrm{v}_x + 3\phi\psi_x u \mathrm{v} = 0 \tag{2.6}$$

subjected to the initial conditions;

$$u(x,0) = 1, \quad \mathrm{v}(x,0) = 1 \tag{2.7}$$

3. Group Formulation of the problem

The partial differential system of equations (PDEs) (2.5) and (2.6) is reduced to a system of ordinary differential equations (ODEs), through a one parameter group G defined as follows;

$$G : \bar{S} = Q^s(a) S + T^s(a) \tag{3.1}$$

where S and $\bar{S}$ stand for the system variables before and after transformation, Q^s and T^s are real valued coefficients at least differentiable in the group parameter (a). First and second partial derivatives of the dependent variables with respect to dependent variables are defined as;

$$\left.\begin{aligned} \bar{S}_{\bar{i}} &= \left(\frac{Q^s}{Q^i}\right) S_i \\ \bar{S}_{\bar{i}\,\bar{j}} &= \left(\frac{Q^s}{Q^i Q^j}\right) S_{i\,j} \end{aligned}\right\} \quad i, j = x, t \tag{3.2}$$

where S stands for dependent variables $(u, \mathrm{v}, \phi \; and \; \psi)$. The invariant transformation of the differential equations (2.5) and (2.6) leads after a procedure well described in (Rashed et al,2008, Kassem et al,2009, Rashed et al,2020, Rashed et al,2020, Rashed et al,2020, Rashed et al,2020) to the following group G;

$$G \begin{cases} G_1 \begin{cases} \bar{x} = Q^x x + T^x \\ \bar{t} = \left(Q^x\right)^3 t \end{cases} \\ G_2 \begin{cases} \bar{u} = u \\ \bar{\mathrm{v}} = \mathrm{v} \\ \bar{\phi} = \dfrac{1}{\left(Q^x\right)^2} \phi \\ \bar{\psi} = \dfrac{1}{\left(Q^x\right)^2} \psi \end{cases} \end{cases} \tag{3.3}$$

where G_1, G_2 are the subgroups describing the independent and dependent system variables.

4. Invariant transformation of the system variables

The independent, dependent variables and initial conditions ($x,t; u,v, \Phi, \psi$) are transformed throughout applying Morgan theorem;

$$\sum_{i=1}^{6}(\alpha_i \Xi_i + \beta_i)\frac{\partial \overline{\Xi}_i}{\partial \Xi_i} = 0 \qquad (4.1)$$

where Ξ and $\overline{\Xi}$ stand for the system variables and initial conditions before and after transformation, while α's and β's are defined as;

$$\alpha_i = \frac{\partial Q^{\Xi_i}(a)}{\partial a} \qquad (4.2)$$

$$\beta_i = \frac{\partial T^{\Xi_i}(a)}{\partial a} \qquad (4.3)$$

such that Q and T are the group components described in (3.1) and a is the group parameter identity element.

4.1. Transformation of the independent variables

The similarity variable $\eta(x,t)$ is obtained by applying Morgan theorem (4.1)

$$(\alpha_1 x + \beta_1)\frac{\partial \eta}{\partial x} + \alpha_2 t\frac{\partial \eta}{\partial t} = 0 \qquad (4.4)$$

The solution of this equation yields a similarity variable of the form;

$$\eta(t,x) = t\,\pi(x) \qquad (4.5)$$

where $\pi(x)$ is to be determined later throughout the transformation procedures.

4.2. Transformation of the dependent variables

The transformation of dependent variables of u and v is directly deduced from the group structure (3.3);

$$\overline{u}(\overline{x},\overline{t}) = u(\eta) \qquad (4.6)$$

$$\overline{\mathrm{v}}(\overline{x},\overline{t}) = \mathrm{v}(\eta) \qquad (4.7)$$

while the initial conditions $\phi(x)$ and $\psi(x)$ transformation are obtained invoking Morgan theorem (4.1);

$$\overline{\phi}(\overline{x}) = \theta(x) \qquad (4.8)$$

$$\overline{\psi}(\overline{x}) = \Gamma(x) \qquad (4.9)$$

where $\theta(x)$ *and* $\Gamma(x)$ will be determined later.

5. Reduction of the problem to a system of ordinary differential equations

Replacing for the transformed variables η, u, v, ϕ and ψ in the system of equations (2.5) and (2.6) leads to;

$$\eta^3\frac{d^3u}{d\eta^3} + [A_1 + A_2]\eta^2\frac{d^2u}{d\eta^2} + [A_3 + A_4 + A_5]\eta\frac{du}{d\eta} - A_6\frac{du}{d\eta} + 6A_7\eta u\frac{du}{d\eta} + A_8 u + A_9 u^2 + 2A_{10}\eta\,\mathrm{v}\frac{d\mathrm{v}}{d\eta} + A_{11}\mathrm{v}^2 = 0 \qquad (5.1)$$

$$\eta^3\frac{d^3\mathrm{v}}{d\eta^3} + [B_1 + B_2]\eta^2\frac{d^2\mathrm{v}}{d\eta^2} + [B_3 + B_4 + B_5]\eta\frac{d\mathrm{v}}{d\eta} + A_6\frac{d\mathrm{v}}{d\eta} + A_8\mathrm{v} + 3A_7\eta u\frac{d\mathrm{v}}{d\eta} + \frac{A_9}{2}u\mathrm{v} = 0 \qquad (5.2)$$

where;

$$A_1 = B_1 = 3\pi \frac{\pi_{xx}}{(\pi_x)^2}, \quad A_2 = 3\frac{\pi}{\pi_x}\frac{\theta_x}{\theta}, \quad B_2 = 3\frac{\pi}{\pi_x}\frac{\Gamma_x}{\Gamma}, \quad A_3 = B_3 = \pi^2 \frac{\pi_{xxx}}{\pi_x^{\ 3}}, \quad A_4 = 3\pi^2 \frac{\theta_x}{\theta}\frac{\pi_{xx}}{\pi_x^{\ 3}}, \quad B_4 = 3\pi^2 \frac{\Gamma_x}{\Gamma}\frac{\pi_{xx}}{\pi_x^{\ 3}}, \quad A_5 = 3\frac{\theta_{xx}}{\theta}\left(\frac{\pi}{\pi_x}\right)^2, \quad B_5 = 3\frac{\Gamma_{xx}}{\Gamma}\left(\frac{\pi}{\pi_x}\right)^2 \tag{5.3}$$

$$A_6 = \frac{\pi^4}{(\pi_x)^3}, \quad A_7 = \theta\left(\frac{\pi}{\pi_x}\right)^2, \quad A_8 = \frac{\theta_{xxx}}{\theta}\left(\frac{\pi}{\pi_x}\right)^3, \quad A_9 = 6\left(\frac{\pi}{\pi_x}\right)^3 \theta_x, \quad A_{10} = \frac{\Gamma^2}{\theta}\left(\frac{\pi}{\pi_x}\right)^2, \quad A_{11} = 2\frac{\Gamma\Gamma_x}{\theta}\left(\frac{\pi}{\pi_x}\right)^3 \tag{5.4}$$

The coefficients in (5.3) and (5.4) must be constants or function of η in order to reduce (5.1) and (5.2) to a system of ordinary differential equations.

5.1. Case 1 $\eta(x,t) = t(\alpha_1 x + \beta_1)^m \ , \ m = -\frac{\alpha_2}{\alpha_1}$

Assuming $\frac{A_{10}}{A_7} = 1$ leads to;

$$\Gamma(x) = \theta(x) \tag{5.5}$$

This result is exploited to get the values of A's and B's described in (5.3) as;

$$B_2 = A_2, \quad B_4 = A_4, \quad B_5 = A_5 \tag{5.6}$$

Then setting $A_7 = 1 = A_{10}$ yields;

$$\theta(x) = \Gamma(x) = \frac{m^2 \alpha_1^{\ 2}}{(\alpha_1 x + \beta_1)^2} \tag{5.7}$$

This result corresponds to the transformation of $\phi(x)$ and $\psi(x)$ described in (3.3)

$$\bar{\phi}(\bar{x}) = \frac{1}{(Q^x)^2}\phi = \theta(x), \quad \bar{\psi}(\bar{x}) = \frac{1}{(Q^x)^2}\psi = \Gamma(x) \tag{5.8}$$

The remaining constants are obtained following the same procedures as;

$$A_6 = \frac{(\alpha_1 x + \beta_1)^{3+m}}{m^3 \alpha_1^{\ 3}} \tag{5.9}$$

For A_6 to be constant, the following condition must apply;

$$m = -3 \tag{5.10}$$

Thus, A_6 is reduced to;

$$A_6 = -\frac{1}{27\alpha_1^{\ 3}} \tag{5.11}$$

Setting $\alpha_1 = \frac{1}{3}$ results in A_6=-1 and the remaining coefficients are simplified to;

$$A_1 = 4, \quad A_2 = B_2 = 2, \quad A_3 = \frac{20}{9}, \quad A_4 = B_4 = \frac{8}{3}, \quad A_5 = B_5 = 2, \quad A_6 = -1, \quad A_7 = 1, \quad A_8 = \frac{8}{9}, \quad A_9 = 4, \quad A_{10} = 1, \quad A_{11} = \frac{4}{3} \tag{5.12}$$

Replacing for them in (5.1), (5.2) results in a system of ordinary differential equations;

$$\eta^3 \frac{d^3u}{d\eta^3} + 6\eta^2 \frac{d^2u}{d\eta^2} + \frac{62}{9}\eta \frac{du}{d\eta} + \frac{du}{d\eta} + 6\eta u \frac{du}{d\eta} + \frac{8}{9}u + 4u^2 + 2\eta \mathrm{v} \frac{d\mathrm{v}}{d\eta} + \frac{4}{3}\mathrm{v}^2 = 0 \tag{5.13}$$

$$\eta^3 \frac{d^3\mathrm{v}}{d\eta^3} + 6\eta^2 \frac{d^2\mathrm{v}}{d\eta^2} + \frac{62}{9}\eta \frac{d\mathrm{v}}{d\eta} - \frac{d\mathrm{v}}{d\eta} + \frac{8}{9}\mathrm{v} + 3\eta u \frac{d\mathrm{v}}{d\eta} + 2u\mathrm{v} = 0 \tag{5.14}$$

subjected to the initial conditions;

$$u(0) = 1,\ \mathrm{v}(0) = 1 \tag{5.15}$$

5.2. Case 2

Following similar procedures as in case 1, the coefficients A's and B's described in (5.3), (5.4) are evaluated. Starting with the same assumption $A_7 = A_{10} = 1$ leads to;

$$\Gamma(x) = \theta(x) = 1 \tag{5.16}$$

where $\theta = \frac{1}{\pi^2}\left(\frac{d\pi}{dx}\right)^2$ and $\pi(x) = 1.$ So,

$$\eta = t \tag{5.17}$$

So, the system, (5.1) and (5.2), is simplified to a system of ODEs;

$$\eta^3 \frac{d^3u}{d\eta^3} + 3\eta^2 \frac{d^2u}{d\eta^2} + \eta \frac{du}{d\eta} - \frac{du}{d\eta} + 6\eta u \frac{du}{d\eta} + 2\eta \mathrm{v} \frac{d\mathrm{v}}{d\eta} = 0 \tag{5.18}$$

$$\eta^3 \frac{d^3\mathrm{v}}{d\eta^3} + 3\eta^2 \frac{d^2\mathrm{v}}{d\eta^2} + \eta \frac{d\mathrm{v}}{d\eta} + \frac{d\mathrm{v}}{d\eta} + 3\eta u \frac{d\mathrm{v}}{d\eta} = 0 \tag{5.19}$$

subjected to the initial conditions (5.15) .

6. Results and discussion

The resultant systems for the two cases, (5.13) - (5.14) and (5.18) - (5.19), are numerically solved using Runge-Kutta and shooting methods. The results show the interaction and dissipation of two long waves in shallow water. The two interacting and colliding waves, *u* and *v*, are depicted in Figs. 1 and 2 for the first case and Figs. 3 and 4 for the second. The figures, 1 and 2, show two soliton long wave with small humps and larger span due the conservation of wave energy. Conversely, the figures, 3 and 4, show multiple soliton wave with higher peaks and smaller span because the waves lose its energy quickly.

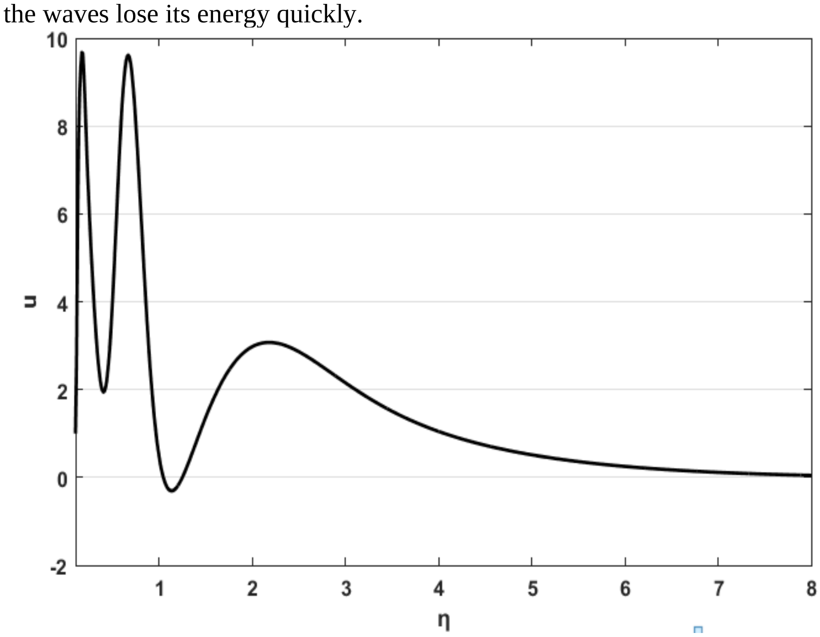

Figure 1: Evolution of the first wave for case 1

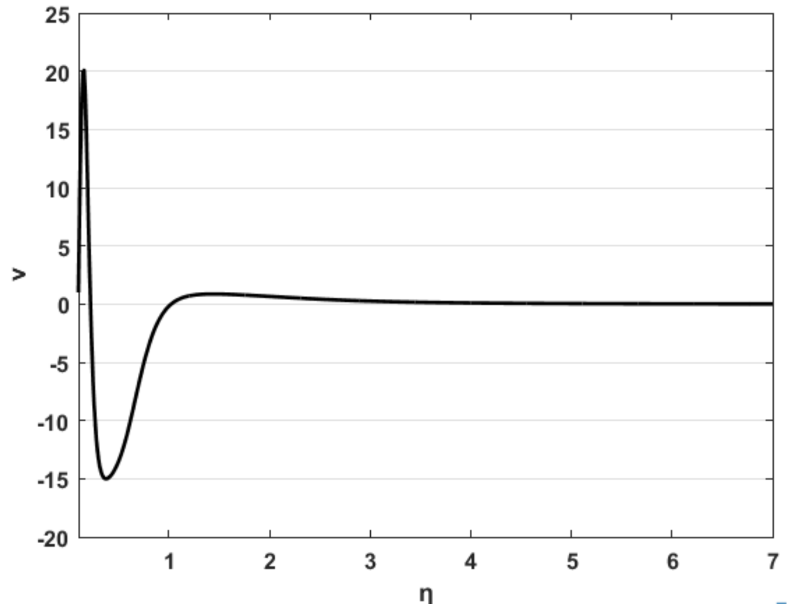

Figure 2: Evolution of the second wave case 1

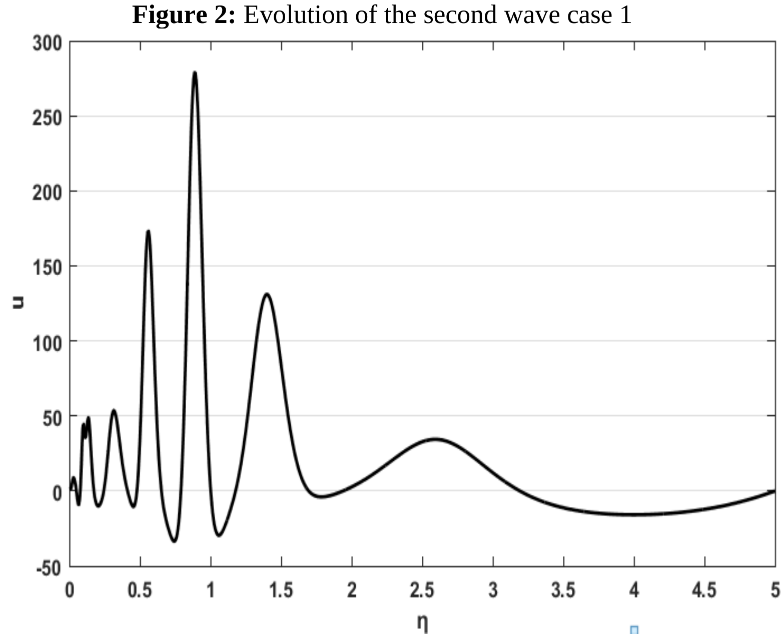

Figure 3: Evolution of the first wave for case 2

7. Conclusion

The interaction between two shallow water waves has been studied throughout Hirota-Satsuma evolution equation model. The similarity transformation led to two cases, $\eta = t\left(\alpha_1 x + \beta_1\right)^{-3}$ and $\eta = t$. Both cases resulted in two interacting waves. The first shows two solitons solution as shown in Figs. 1 and 2 while the second solution indicates multiple solitons wave. The two solitons solution has lower lumps, so it conserves its energy to vanish and dissipated slowly in the water pond. On the contrary, the multiple soliton solution indicates higher lumps, so it loses its energy and vanishes faster than the first case. Both cases represent interesting real simulations inside water ponds with shallow water. The interaction between the two waves affects their peaks and spans.

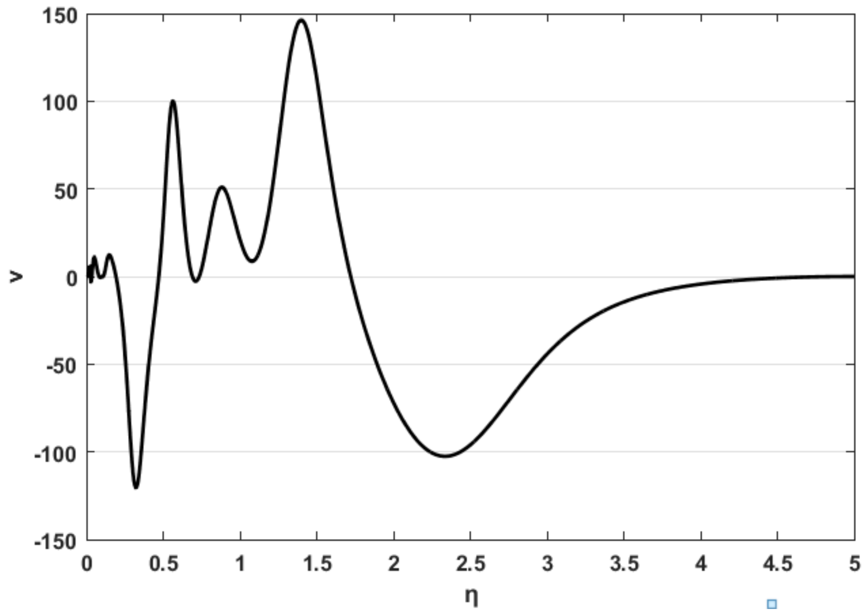

Figure 4: Evolution of the second wave case 2

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