



## Evolution of the Motion Around a Slowly Rotating Body

W. M. El-Mahy\*, M. K. Ahmed\*, M. A. Awad\*, Z. M. Hayman\*

**Abstract:** The motion of a satellite about a rotating triaxial body will be investigated, stressing on the case of slow rotation. The Hamiltonian of the problem will be formed including the zonal harmonic  $J_2$  and the leading tesseral harmonics  $C_{22}$  and  $S_{22}$ . The small parameter of the problem is the spin rate ( $\sigma$ ) of the primary. The solution proceeds through three canonical transformations to eliminate in succession; the short, intermediate and long-period terms. Thus secular and periodic terms are to be retained up to orders four and two respectively.

### 1. Introduction:

To understand the dynamics of a spacecraft or a natural particle around a celestial body (a planet, an asteroid or even a comet) it's convenient to take into account the spin rate ( $\sigma$ ) of the primary, since it is important for several applications especially for geodetic satellites and when dealing with communication satellites where there's commensurability between the satellite period and  $\sigma$ , and a case of resonance arises (Harris, 1994; Hudson and ostro, 1994; Pravec and Harris, 2000). Considering a slowly rotating earth like planet, the addition of tesseral harmonics is necessary. Scheeres (1994, 1998) analyzed the orbital dynamics about an asteroid and established that the major perturbations acting on the orbiter are due to the leading harmonics of the geopotential. In a subsequent work (scheeres, 2001) he tackled the problem of secular motion in a 2nd degree and order-gravity field with no rotation qualitatively. In this respect the main problem of artificial satellite theory is very useful; the major contributors to this subject were (Brower, 1959; Kozai, 1959; Brower and Clemence, 1961).

In this paper the gravitational force exerted by an earth like planet on an artificial satellite will be considered, the Hamiltonian of the problem will be formed, in terms of the Delaunay variables, with the earth's spin rate  $\sigma$  taken as a small parameter of  $O(1)$ . The planet's potential will be considered up to the leading zonal and leading tesseral harmonics, the paper ends with an outline of the perturbation technique which will be used subsequently, which is based on the Lie- Deprit - Kamel transform.

---

\* Cairo University

## 2. The Geopotential

The earth's gravitational potential is usually expressed by "Vinti's potential":

$$V = -\frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^n \left(\frac{R}{r}\right)^n P_n^m(\sin \delta) (c_{nm} \cos m\lambda + s_{nm} \sin m\lambda) \quad (1)$$

where R: the equatorial radius of the earth.  $\mu = GM_e$  is the product of the gravitational constant and the mass of the earth, and known as the earth's gravitational parameter,  $(r, \lambda, \delta)$  are the geocentric coordinates with  $\lambda$  being measured east of Greenwich,  $P_n^m(\sin \delta)$  are the associated legendre polynomials, and  $c_{nm}$  and  $s_{nm}$  are harmonic coefficients. The terms with  $m=0$  correspond to zonal harmonics, those with  $0 < m < n$  correspond to tesseral harmonics, while  $m=n$  correspond to sectorial harmonics,  $J_2$  measures the equatorial bulge of the earth and  $C_{22}, S_{22}$  measure the elliptical shape of the earth's equator. The coefficients  $C_{21}$  and  $S_{21}$  are vanishingly by small and since the origin is taken at the center of mass, the coefficients  $C_{10}, C_{11}$  and  $S_{21}$  will be zero; also both the tesseral and sectorial harmonics will be simply referred to as tesseral harmonics. With the previous considerations, and writing the zonal and tesseral harmonics separately, eqn. (1) will be:

$$v = -\frac{\mu}{r} + \sum_{n=2}^{\infty} Z'_n + \sum_{n=2}^{\infty} \sum_{m=2}^n T'_{nm} \quad (2)$$

where

$$Z'_n = \mu R^n J_n \frac{P_n(\sin \delta)}{r^{n+1}}, \quad J_n = -c_{n0} \quad (3)$$

$$T'_{nm} = -\mu R^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) \frac{P_n^m(\sin \delta)}{r^{n+1}} \quad (4)$$

Taking into account the orders considered only and using the associated legendre and the legendre polynomial formulas then:

$$P_2 = \frac{1}{4}[(3S^2 - 2) - 3S^2 \cos F_{22}] \quad (5)$$

$$P_2^2(\sin \delta) = 3 \cos^2 \delta \quad (6)$$

$$\text{Setting} \quad if + j\omega = F_{ij}, C = \cos I, S = \sin I \quad (7)$$

$$\text{and substituting} \quad \sin \delta = S \sin F_{11} \quad (8)$$

where I is the orbital inclination, F is the true anomaly and  $\omega$  is the argument of perigee, then using the trigonometric formulae developed by Garfinkel (1965) the factors  $(\cos^m \delta)$  can be cancelled out. If  $\Omega$  is the longitude of the node measured east ward from the (rotating) meridian of Greenwich, then from the Fig. 1.

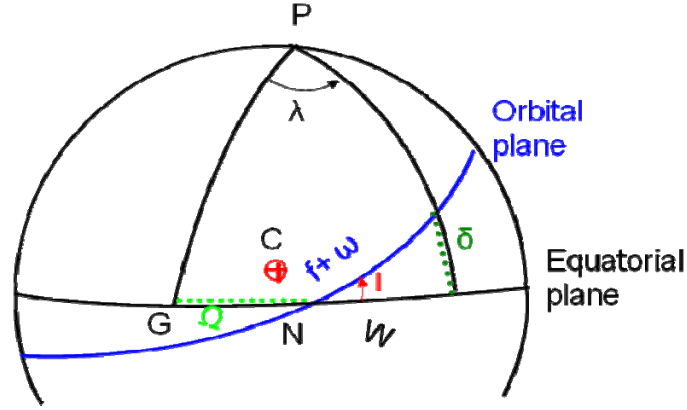


Fig. 1

$$\lambda = \Omega + W$$

$$\cos W = \frac{\cos F_{11}}{\cos \delta}, \sin W = \frac{C \sin F_{11}}{\cos \delta} \quad (9)$$

Setting  $C_1 = \cos F_{11}, S_1 = C \sin F_{11}$  (9\*)

$$\sin \lambda = \frac{C_1 \sin \Omega + S_1 \cos \Omega}{\cos \delta}$$

$$\cos \lambda = \frac{C_1 \cos \Omega - S_1 \sin \Omega}{\cos \delta} \quad (10)$$

From (10) and using De Moivre's theorem

$$\begin{aligned} \cos m\lambda + i \sin m\lambda &= (\cos \lambda + i \sin \lambda)^m \\ &= \frac{1}{\cos^m \delta} (\cos m\Omega + i \sin m\Omega)(C_1 + iS_1)^m \\ &= \frac{1}{\cos^m \delta} (\cos m\Omega + i \sin m\Omega) \sum_{r=0}^m \binom{m}{r} (iS_1)^r C_1^{m-r} \end{aligned}$$

Equating the real and imaginary parts,

$$\sin m\lambda = \frac{C_m \sin m\Omega + S_m \cos m\Omega}{\cos^m \delta}$$

$$\cos m\lambda = \frac{C_m \cos m\Omega - S_m \sin m\Omega}{\cos^m \delta} \quad (11)$$

where

$$C_m = \sum_{j \geq 0} \binom{m}{2j} (iS_1)^{2j} C_1^{m-2j}$$

$$S_m = \frac{1}{i} \sum_{j \geq 0} \binom{m}{2j+1} (iS_1)^{2j+1} C_1^{m-2j-1} \quad (12)$$

Then from (12):

$$\begin{aligned} C_2 &= C_1 C_1 - S_1 S_1 \\ S_2 &= S_1 C_1 - C_1 S_1 \end{aligned} \quad (13)$$

The general relation, which can be generalized to the recursive relation so from (9\*, 13):

$$C_2 = \frac{S_2}{2} + (1 - \frac{S_2}{2}) \cos F_{22}, S_2 = C \sin F_{22} \quad (14)$$

And the tesseral harmonics become in the form

$$T'_{nm} = \frac{-\mu R^n}{r^{n+1}} \frac{P_n^m(\sin \delta)}{\cos^m \delta} [(C_{nm} \cos m\Omega + S_{nm} \sin m\Omega)C_m + (S_{nm} \cos m\Omega - C_{nm} \sin m\Omega)S_m] \quad (15)$$

### 3. The Hamiltonian in terms of the Delaunay variables:

If the Delaunay set of canonical variables is considered

$$L_D = \sqrt{\mu a} \quad , \quad l_D = M \quad (\text{Mean anomaly}) \quad (16)$$

$$G_D = L_D \sqrt{1 - e^2} \quad , \quad g_D = \omega \quad (\text{argument of periapsis}) \quad (17)$$

$$H_D = CG_D \quad , \quad h_D = \Omega \quad (\text{Longitude of the node})$$

then the equations of motion become

$$\dot{y}_D = M \left( \frac{\partial \hat{\mathcal{H}}}{\partial y_D} \right)^T \quad , \quad y_D = \text{col}(l_D, \dots, H_D) \quad (18)$$

where M is the canonical matrix  $M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ , I is the identity matrix and  $\hat{\mathcal{H}}$  is expressed in

terms of the above set of canonical variables.  $\hat{\mathcal{H}}$  depends explicitly on time due to the measurement of  $h_D$  from the rotating meridian of Greenwich. The autonomous form of the equations is recovered by:

- (i) Adjoining to the above set a new pair of conjugate variables  $(k_D, K_D)$  where  $k_D = \nu t + \text{const.}$ , and augmenting the Hamiltonian such that:

$$\dot{k}_D = \nu = \frac{\partial \mathcal{H}_D}{\partial K_D} \quad , \quad \dot{K}_D = -\frac{\partial \mathcal{H}_D}{\partial k_D} \quad (19)$$

Since  $\hat{\mathcal{H}}$  does not depend on  $K_D$  so it can be taken as constant of integration. So that:

$$\mathcal{H}_D = \nu K_D + \hat{\mathcal{H}} \quad (20)$$

The variable  $K_D$  may be identified by means of the second of equations (19).

(ii) Performing a canonical transformation so that the new angular variables become:

$$l = l_D, \quad g = g_D, \quad h = h_D - \sigma t, \quad k = k_D \quad (21)$$

where  $\sigma$  is the angular speed of the earth. To find the new momenta and the new Hamiltonian:

$$(L_D dl + G_D dg + H_D dh_D + K_D dk) - [L dl + G dg + H (dh_D - \sigma dt) + K dt] + (\mathcal{H} - \mathcal{H}_D) dt = d\tau \quad (22)$$

where  $d\tau$  is the total differential of a function that is chosen to be zero. Equating the coefficients of earth differential to zero yields:

$$L = L_D, \quad G = G_D, \quad H = H_D, \quad K = K_D \quad (23)$$

and

$$\mathcal{H} = \mathcal{H}_D - \sigma H \quad (24)$$

So in terms of this set of variables the equations of motion become:

$$\dot{z} = M \mathcal{H}_z^T \quad \text{Where } z = \text{col}(l_D, \dots, K_D) \quad (25)$$

and

$$\mathcal{H} = \frac{-\mu^2}{2L^2} - \sigma H + \sum_n^2 Z'_n + \sum_n^2 \sum_m^n T'_{nm} \quad (26)$$

Considering  $\sigma$  as the small parameter of the problem, the orders of magnitude of the involved parameters are defined as:

$$\sigma \equiv o(1), \quad J_2 \equiv o(2), \quad C_{nm}, S_{nm} \equiv o(4) \quad (27)$$

The Hamiltonian can now be expressed as a power series of  $\sigma$

$$\mathcal{H} = \sum_{n=0}^4 \frac{\sigma^n}{n!} \mathcal{H}_n \quad (28)$$

where the component of (28) is retained up to 4<sup>th</sup> order in  $\sigma$ , i.e. up to  $J_2$  in zonal harmonics and up to  $C_{22}, S_{22}$  in tesseral harmonics, so the components will be as follows:

$$\mathcal{H}_0 = \frac{-\mu^2}{2L^2} \quad (29)$$

$$\mathcal{H}_1 = -H \quad (30)$$

$$\mathcal{H}_2 = \frac{A_2}{L^6} \phi^3 z_2 \quad (31)$$

$$\mathcal{H}_3 = 0 \quad (32)$$

$$\mathcal{H}_4 = \frac{\phi^3}{L^6} T_{22} \quad (33)$$

where  $A_2$  is a zero order constant, according to equations (26), (3) and (5) is given by:

$$A_2 = \frac{\mu^4 R^2 J_2}{2\sigma^2} \quad (34)$$

$$z_2 = (3S^2 - 2) - 3S^2 \cos F_{22} \quad (35)$$

and

$$T_{21} = 0 \quad C_{21}, S_{21} \rightarrow 0 \quad (36)$$

$$T_{22} = \gamma_{22} \left[ S^2 + (2 - S^2) \cos F_{22} \right] + 2\Gamma_{22} C \sin F_{22} \quad (37)$$

where

$$\gamma_{22} = A_{22} \cos 2h + B_{22} \sin 2h \quad (38)$$

$$\Gamma_{22} = B_{22} \cos 2h - A_{22} \sin 2h \quad (39)$$

and the  $A_{22}, B_{22}$  are zero order constants given by:

$$A_{22} = \frac{-36\mu^4 R^2}{\sigma^4} c_{22} \quad (40)$$

$$B_{22} = \frac{-36\mu^4 R^2}{\sigma^4} s_{22} \quad (41)$$

#### 4. The perturbation technique:

Now the perturbation technique will be outlined up to 4<sup>th</sup> order in the secular, 3<sup>rd</sup> order in the intermediate and 2<sup>nd</sup> order in the short periodic terms.

Let  $\varepsilon$  be the small parameter of the problem and let the considered system of differential equations be:

$$\dot{u} = \mathcal{H}_v^T, \quad \dot{U} = \mathcal{H}_u^T \quad (42)$$

where  $(u, U)$  is the six-vector of adopted canonical variables, the Hamiltonian is assumed expandable as:

$$\mathcal{H} = \mathcal{H}_0 + \sum \frac{\varepsilon^n}{n!} \mathcal{H}_n \quad (43)$$

and the system with  $\mathcal{H} = \mathcal{H}_0$  is assumed integrable with  $\mathcal{H}_0 = \mathcal{H}_0(U_1)$

What is required is to construct three canonical transformation  $(u, U; \varepsilon) \rightarrow (u', U')$ ,  $(u', U'; \varepsilon) \rightarrow (u'', U'')$  and  $(u'', U''; \varepsilon) \rightarrow (u''', U''')$  analytic in  $\varepsilon$  at  $\varepsilon = 0$  to eliminate in succession the short, intermediate and long period terms from the Hamiltonian such that  $U'''$  reduce to constants and  $u'''$  become linear functions of time, where the short period terms are those periodic in the mean anomaly,  $u_1 = l$ , the intermediate terms are those periodic in the longitude of the node,  $u_2 = h$ , and the long period terms are those periodic in the argument of perigee,  $u_3 = g$ . The transformed Hamiltonians and the corresponding generators will be assumed expandable as:

$$\begin{aligned}\mathcal{H}^*(-, u'_2, u'_3; U'; \varepsilon) &= \mathcal{H}_0^*(U'_1) + \sum_{n=1} \frac{\varepsilon^n}{n!} \mathcal{H}_n^*(-, u'_2, u'_3; U') \\ \mathcal{H}^{**}(-, u''_3; U''; \varepsilon) &= \mathcal{H}_0^{**}(U''_1) + \sum_{n=1} \frac{\varepsilon^n}{n!} \mathcal{H}_n^{**}(-, u''_3; U'')\end{aligned}\quad (44)$$

$$\begin{aligned}\mathcal{H}^{***}(-; U'''; \varepsilon) &= \mathcal{H}_0^{***}(U'''_1) + \sum_{n=1} \frac{\varepsilon^n}{n!} \mathcal{H}_n^{***}(-; U''') \\ w(u', U'; \varepsilon) &= \sum_{n=0} \frac{\varepsilon^n}{n!} w_{n+1}(u', U') \\ w^*(-, u''_2, U''; \varepsilon) &= \sum_{n=0} \frac{\varepsilon^n}{n!} w_{n+1}^*(-, u''_2, U'') \\ w^{**}(-, u''_3, U'''; \varepsilon) &= \sum_{n=0} \frac{\varepsilon^n}{n!} w_{n+1}^{**}(-, u''_3, U''')\end{aligned}\quad (45)$$

The following equations will be used during the process of elimination.

$$G_j = L_j - \sum_{m=0}^{j-2} \binom{j-1}{m} L_{m+1} G_{j-m-1} \quad (46)$$

$$\mathcal{H}_0^* = \mathcal{H}_0 \quad (47)$$

$$\mathcal{H}_n^* = \mathcal{H}_n + L_n \mathcal{H}_0 + \sum_{j=1}^{n-1} \left[ \binom{n-1}{j-1} L_j \mathcal{H}_{n-j} + \binom{n-1}{j} G_j \mathcal{H}_{n-j}^* \right], \quad n \geq 1 \quad (48)$$

## 5. Elimination of short period terms:

The basic identities are:

$$\begin{aligned}\mathcal{H}_0^* &= \mathcal{H}_0 \\ \mathcal{H}_n^* &= \tilde{\mathcal{H}}_n + (\mathcal{H}_0; w_n) \\ \tilde{\mathcal{H}}_n &= \mathcal{H}_n + \sum_{j=1}^{n-1} \left\{ \binom{n-1}{j-1} (\mathcal{H}_{n-j}; w_j) + \binom{n-1}{j} G_j \mathcal{H}_{n-j}^* \right\}\end{aligned}\quad (49)$$

Let  $u'_1$  be the fast variable in  $\mathcal{H}$ , choose  $\mathcal{H}_n^*$  to be the average of  $\tilde{\mathcal{H}}_n$  over  $u'_1$ ; i.e.

$$\begin{aligned}\mathcal{H}_n^* &= \langle \tilde{\mathcal{H}}_n \rangle_{u'_1} \\ P_n &= \tilde{\mathcal{H}}_n - \mathcal{H}_n^* = (w_n; \mathcal{H}_0)\end{aligned}\quad (50)$$

So that:

$$w_n = \left( \frac{\partial \mathcal{H}_0}{\partial U'_1} \right)^{-1} \int P_n du'_1 \quad (51)$$

where the previous equations are applied up to  $\mathcal{H}_4^*$ ,  $w_2$ .

## 6. Elements of the short period transformation and its inverse:

These are obtained from the equations for the vector transformation, namely:

$$\begin{aligned} u &= u' + \sum_{n=1}^3 \frac{\mathcal{E}^n}{n!} u'^{(n)} \\ U &= U' + \sum_{n=1}^3 \frac{\mathcal{E}^n}{n!} U'^{(n)} \end{aligned} \quad (52)$$

where

$$\begin{aligned} u'^{(n)} &= \frac{\partial w_n}{\partial U'} + \sum_{j=1}^{n-1} \binom{n-1}{j} G_j u'^{(n-j)}, n \geq 1 \\ U'^{(n)} &= -\frac{\partial w_n}{\partial u'} + \sum_{j=1}^{n-1} \binom{n-1}{j} G_j U'^{(n-j)}, n \geq 1 \end{aligned} \quad (53)$$

And for the inverse transformation:

$$\begin{aligned} u' &= u + \sum_{n=1}^3 \frac{\mathcal{E}^n}{n!} u^{(n)}(u, U) \\ U' &= U + \sum_{n=1}^3 \frac{\mathcal{E}^n}{n!} U^{(n)}(u, U) \end{aligned} \quad (54)$$

where

$$\begin{aligned} u^{(n)} &= -u'^{(n)} + \sum_{j=1}^{n-1} \binom{n}{j} G_j u'^{(n-j)} \\ U^{(n)} &= -U'^{(n)} + \sum_{j=1}^{n-1} \binom{n}{j} G_j U'^{(n-j)} \end{aligned}, n \geq 1 \quad (55)$$

## 7. The intermediate transformation:

The procedure is essentially similar to that at the short period transformation with the averages taken over  $u_2''$ , so:

$$\begin{aligned} \mathcal{H}_0^{**} &= \mathcal{H}_0^* \\ \mathcal{H}_n^{**} &= \mathcal{H}_n^* + \sum_{j=1}^{n-1} \left\{ \binom{n-1}{j-1} (\mathcal{H}_{n-j}^*; w_j^*) + \binom{n-1}{j} G_j^* \mathcal{H}_{n-j}^{**} \right\} + (\mathcal{H}_0^*; w_n^*) \quad , n \geq 1 \end{aligned} \quad (56)$$

Since  $\mathcal{H}_n^{**}$  and  $\mathcal{H}_n^*$  are independent of  $u_2''$ ,  $w_n^*$  may be chosen also independent of  $u_1''$ , then the last term vanishes and the last equation reduces to

$$\mathcal{H}_n^{**} = \mathcal{H}_n^* + \sum_{j=1}^{n-1} \left\{ \binom{n-1}{j-1} (\mathcal{H}_{n-j}^*; w_j^*) + \binom{n-1}{j} G_j^* \mathcal{H}_{n-j}^{**} \right\} \quad (57)$$



The equations involved may be outlined as before with the change  $\mathcal{H} \rightarrow \mathcal{H}^*$ ,  $\mathcal{H}^* \rightarrow \mathcal{H}^{**}$ ,  $w \rightarrow w^*$ ,  $u', U' \rightarrow u'', U''$

Order (1)

$$\mathcal{H}_1^{**} = \mathcal{H}_1^* + (\mathcal{H}_0^*; w_1^*) = \mathcal{H}_1^* \quad (58)$$

where it is noted that  $\mathcal{H}_{0(U'_1=U'_1)}^* = \mathcal{H}_0^*(U'')$  and  $w^*$  is independent at  $u''$  so  $(\mathcal{H}_0^*; w_1^*) = 0$

Order (2)

$$\begin{aligned} G_1 &= L_1^* \\ \mathcal{H}_2^{**} &= \mathcal{H}_2^* + (\mathcal{H}_1^*; w_1^*) + (\mathcal{H}_1^{**}; w_1^*) = \mathcal{H}_2^* + 2(\mathcal{H}_1^*; w_1^*) \end{aligned} \quad (59)$$

Choosing,

$$\begin{aligned} \mathcal{H}_2^{**} &= \langle \mathcal{H}_2^* \rangle_{u_2''} \\ P_2^* &= \mathcal{H}_2^* - \mathcal{H}_2^{**} = 2(w_1^*; \mathcal{H}_1^*) \end{aligned} \quad (60)$$

Let

$$P_n^* = \sum_{i,j} c_n^{ij} \cos(iu_2'' + ju_3'' + \alpha_n^{ij}) \quad (61)$$

where  $c_n$  are functions of  $U_1''$  and  $\alpha_n^{ij}$  are numerical constants to account for the phase then

$$w_1^* = \frac{1}{2} \sum_{i,j} \frac{c_2^{ij} \sin(iu_2'' + ju_3'' + \alpha_2^{ij})}{i \frac{\partial \mathcal{H}_1^*}{\partial u_2''} + j \frac{\partial \mathcal{H}_1^*}{\partial u_3''}} \quad (62)$$

Similarly order (3), may be found elements of the intermediate transformation are obtained by the same equations used for the short period transformation and the use of the interchanges used previously. The long period transformation and its elements are found in a similar manner.

## 8. Conclusion:

The present work aims at formulating the problem of the motion of an artificial satellite around a slowly rotating planet, this problem has many applications particularly for missions launched to study the potential harmonics of the earth(or any other planet),and for communication satellites. The formulation is developed in a simple canonical form expressed in terms of a set of Delaunay elements modified to allow for the appearance of the independent variable in the Hamiltonian, this enables using very powerful tools for solving for the motion using canonical perturbation approaches such as the methods of Von Ziepel's and those based on a Lie series and transform, which will be effected in a subsequent work.

## 9. References:

- [1] D.J.SCHEERES and W.HU, *Secular Motion in 2<sup>nd</sup> Degree and Order-Gravity Field with no Rotation*, Celestial Mechanics and Dynamical Astronomy, 2001.
- [2] Harris, A.W., *Tumbling asteroids*, Icarus, 1994.
- [3] Hudson, R.S. and Ostro, S.J., *Shape and non-principle axis spin state of asteroids Toutatis*, Science, 1994.
- [4] Pravec, P. and Harris, A.W., *Fast and slowly rotating asteroids*, Icarus, 2000.
- [5] AHMED ALY KAMEL, *Expansion Formulae in Canonical Transformations Depending on a Small Parameter*, Stanford University, U.S.A. 1969.
- [6] ANDRE DEPRIT, *Canonical Transformations Depending on a Small Parameter*, Seattle, U.S.A., 1968.
- [7] Mustafa Kamal, *PhD. Artificial Satellite Theory including the gravitational effects up to S<sub>55</sub>*, Cairo University, 1982.
- [8] Scheeres D.J., Ostro, S.J., Hunson, R.S., DeJong, E.M. and Suzuki, S., *Dynamics of orbit close to asteroid*, Icarus, 1998.
- [9] Scheeres D.J., *Dynamics about uniformly rotating triaxial ellipsoids*, Icarus, 1994.