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ON THE NUMERICAL SOLUTION OF THE INVISCID THREE DIMENSIONAL NON CIRCULAR NOZZLES WITH STRUCTURED NON-ORTHOGONAL GRIDS

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ABSTRACT
A method is presented for solving three dimensional inviscid internal flows, with application to three dimensional noncircular nozzles. Spatial discretization is made on the finite difference formulation of Euler's equations. The convective terms is differenced using the Van Leer flux vector splitting technique. Time is advanced using the implicit Euler backward time stepping. The resulting system of algebraic equations is solved using ADI method. New numerical treatment of the flow boundary conditions is presented and applied to handle the complicated mixed boundary conditions. Application to solve flow through nozzles shows good agreement with analytical results.

NOMENCLATURE

A,B,C	Flux Jacobian matrices in ξ,η,ζ directions.
CFL	Courant, Fridrichs, and Lewys number.
E,F,G	Inviscid flux vectors in ξ,η,ζ directions.
e_{tt}	Total energy.
a	Speed of sound.
γ	Specific heat ratio =1.4 for air
ρ	Staticdensity.
P	Static pressure.
u,v,w	Cartesian velocity components in (x, y, z) directions
U,V,W	Contravariant velocity components in ξ,η,ζ directions.
J	Jacobian.
M	Mach number.
Q	Conservative Variables Vector.
t	Time.
T	Temperature.
x,y,z	Physical plane coordinates.
ξ,η,ζ	Computational plane coordinates.

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Δt	Time step.
ΔQ	Increment in conservative variable vector.
λ	Eigenvalue.
Λ	Eigenvalues diagonal matrix.
$\xi_t, \xi_x, \xi_y, \xi_z$	$\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \xi}{\partial z},$
$\eta_t, \eta_x, \eta_y, \eta_z$	$\frac{\partial \eta}{\partial t}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}, \frac{\partial \eta}{\partial z}$
$\zeta_t, \zeta_x, \zeta_y, \zeta_z$	$\frac{\partial \zeta}{\partial t}, \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, \frac{\partial \zeta}{\partial z}$
$\Delta\xi, \Delta\eta, \Delta\zeta$	Increments in ξ, η, ζ directions.

INTRODUCTION

The procedure of the time marching technique consists of writing the unsteady Euler's equations in finite difference form and then solving it as a function of time. The steady state solution is then computed as the asymptotic limit, of an unsteady solution. That is obtained after a large calculation time. The solution approach adopts a fully implicit difference scheme for the numerical solution of the gas dynamic equations. The Douglas-Rashford method of stabilizing correction type is used as an ADI sequence. To obtain an economical scheme, the splitting operator matrices are chosen so that at each fractional step all the unknown variables are determined independently of one another by scalar pivotal condensations.

Van Leer [1] introduced an alternate flux splitting with continuously differentiable flux contributions that leads to smoother solutions at sonic points. In addition, the splitting is designed so that the shock structures can be realized with no more than two interior zones [2], THE Van leer flux vector splitting is superiour than the Jamson's strategy for flux splittings introduced in ref {3}.

Some points in nozzle have many physical boundary conditions such as , tangency to surface plus out flow or inflow , this motivate the need of developing new boundary condition treatment for the finite difference solution of such flow capable of overcoming such problem without many approximations.

A new boundary condition treatment is presented to handle the mixed boundary conditions occur at the same point in the physical domain.

The objective of this study is;

1. Developing a new nonreflecting boundary condition treatment for the mixed boundary conditions occurred at the same point, for the finite difference solution of the flow field.
2. Applying the constructed flow solver to solve the flow problem of nozzle flow in three dimensions.

GOVERNING EQUATIONS

Euler's model, which represents the conservation of mass and balance of momentum and energy, is used. The flow field variables are nondimensionalized as follows:

$$\rho=\frac{\tilde{\rho}}{\rho_o}\quad u=\frac{\tilde{u}}{a_o}\quad v=\frac{\tilde{v}}{a_o}\quad w=\frac{\tilde{w}}{a_o}\quad p=\frac{\tilde{p}}{\rho_o a_o^2}\quad e_{tt}=\frac{\tilde{e}_{tt}}{\rho_o a^2{}_o}\quad a=\frac{\tilde{a}}{a_o}\quad x=\frac{\tilde{x}}{l_r}$$

$$y=\frac{\tilde{y}}{l_r}\quad z=\frac{\tilde{z}}{l_r}\quad t=\frac{\tilde{t}*a_o}{l_r}$$

The strong conservation form of 3-D Euler equations in the generalized coordinates can be written in non-dimensional form as:

$$\frac{\partial Q}{\partial t}+\frac{\partial E}{\partial \xi}+\frac{\partial F}{\partial \eta}+\frac{\partial G}{\partial \zeta}=0$$

$$Q=\frac{1}{J}\begin{bmatrix}\rho\\ \rho.u\\ \rho.v\\ \rho.w\\ \rho.e_t\end{bmatrix},E=\frac{1}{J}\begin{bmatrix}\rho.U\\ \rho.u.U+p.\xi_x\\ \rho.v.U+p.\xi_y\\ \rho.w.U+p.\xi_z\\ U.(\rho.e_t+p)\end{bmatrix},F=\frac{1}{J}\begin{bmatrix}\rho.V\\ \rho.u.V+p.\eta_x\\ \rho.v.V+p.\eta_y\\ \rho.w.V+p.\eta_z\\ V.(\rho.e_t+p)\end{bmatrix},G=\frac{1}{J}\begin{bmatrix}\rho.W\\ \rho.u.W+p.\zeta_x\\ \rho.v.W+p.\zeta_y\\ \rho.w.W+p.\zeta_z\\ W.(\rho.e_t+p)\end{bmatrix} \quad (1)$$

The contravariant velocity components (U, V, W) in equations (2) are as follows:

$$U=\xi_x.u+\xi_y.v+\xi_z.w,\ V=\eta_x.u+\eta_y.v+\eta_z.w,\ \ W=\zeta_x.u+\zeta_y.v+\zeta_z.w \quad (2)$$

NUMERICAL PROCEDURE

The governing equations are descretised on finite difference formulation where the spatial derivatives are evaluated using Van Leer flux vector splitting technique.
The descretised flow equations suitable for implicit time marching can be written as;

$$\left[I+\Delta t*\left(\frac{\partial A}{\partial \xi}+\frac{\partial B}{\partial \eta}+\frac{\partial C}{\partial \zeta}\right)\right]\Delta Q=\Delta t*\left(R-\frac{\partial E}{\partial \xi}-\frac{\partial F}{\partial \eta}-\frac{\partial G}{\partial \zeta}\right)^n \quad (3)$$

Where, $$\frac{\Delta Q}{\Delta t}+\left[\frac{\partial E}{\partial \xi}+\frac{\partial F}{\partial \eta}+\frac{\partial G}{\partial \zeta}\right]^{n+1}=R \quad (Note: Q^{n+1}=Q^n+\Delta Q) \quad (4)$$

The generalized fluxes $\hat{E},\hat{F}$ and $\hat{G}$ are split according to Van Leer flux-vector splitting, For example E is split according to the contravariant Mach number

$$\delta_\varsigma E=\delta_\varsigma^- E^+ +\delta_\varsigma^+ E^- \quad (5)$$

as, $$M_\xi=\frac{\bar{U}}{a},\qquad where\ \bar{U}=\frac{U}{|\nabla\xi|}\quad and\quad U=\xi_x u+\xi_y v+\xi_z w+\xi_t \quad (6)$$

For locally supersonic flow, where $|M_\xi|\geq 1$,

E^+=E & E^-=0 $(M_\xi\geq +1)$, E^-=E & E^+=0 $(M_\xi\leq -1)$ (7)

and for locally subsonic flow, where $|M_\xi|<1$, for 3-D flow

$$E^{\pm} = \frac{|\nabla \xi|}{J} \begin{bmatrix} f^{\pm}_{mass} \\ f^{\pm}_{mass}\left[\hat{\xi}_x \left(-\bar{U} \pm a \right) / \gamma + u \right] \\ f^{\pm}_{mass}\left[\hat{\xi}_y \left(-\bar{U} \pm a \right) / \gamma + v \right] \\ f^{\pm}_{mass}\left[\hat{\xi}_z \left(-\bar{U} \pm a \right) / \gamma + w \right] \\ f^{\pm}_{energy} \end{bmatrix} \quad (8)$$

Where, $\xi_x = \frac{\xi_x}{|\nabla \xi|}, \quad \xi_y = \frac{\xi_y}{|\nabla \xi|}, \quad \xi_z = \frac{\xi_z}{|\nabla \xi|}$ (9)

and

$$f^{\pm}_{mass} = \pm \frac{\rho a}{4} \left(M_{\xi} \pm 1 \right)^2$$

$$f^{\pm}_{energy} = f^{\pm}_{mass} \left[\frac{(1-\gamma)\bar{U}^2 \pm 2(\gamma-1)\bar{U}a + 2a^2}{(\gamma^2 - 1)} + \frac{(u^2 + v^2 + w^2)}{2} - \frac{\xi_t}{\gamma}\left(-\bar{U} \pm 2a \right) \right] \quad (10)$$

And, $\xi_t = \frac{\xi_t}{|\nabla \xi|}$ (11)

THE SOLUTION PROCEDURE

Appling the finite difference technique, the descretised flow equations will be;

$$\left[I + \Delta t \left(\delta^-_{\xi} A^+ + \delta^+_{\xi} A^- + \delta^-_{\eta} B^+ + \delta^+_{\eta} B^- + \delta^-_{\zeta} C^+ + \delta^+_{\zeta} C^- \right) \right] \Delta Q = -\Delta t * RHS^n$$
$$where,\ RHS^n = \delta^-_{\xi} E^+ + \delta^+_{\xi} E^- + \delta^-_{\eta} F^+ + \delta^+_{\eta} F^- + \delta^-_{\zeta} G^+ + \delta^+_{\zeta} G^- - R \quad (12)$$

the solution is obtained using ADI method and the solution after each time step will be,

$$Q^{n+1} = Q^n + \Delta Q \quad (13)$$

Steady state solution is reached with $\Delta Q \to 0$ hence the results are 2nd order accurate in space.

New Boundary Condition Implementation Method in 3-D Inviscid Flows:

In the present subsection the new method of dealing with the flow boundary conditions and its implementation in the (FVS) for two-dimensional inviscid compressible flow is presented.

1. Tangency to surface (η=cons. Or ξ = cons)
The boundary condition is $V = 0$ (η=cons) where V is the contravariant velocity component in the η direction. Or W=0 (ξ = cons) where W is the contravariant velocity component in the ξ direction, and both can be expressed as follows:

$$\begin{aligned} &\eta_x * \rho u + \eta_y * \rho v + \eta_z * \rho w = 0 \\ or, \quad &\eta_x * Q_2 + \eta_y * Q_3 + \eta_z * Q_4 = 0 \\ or, \quad &\xi_x * \rho u + \xi_y * \rho v + \xi_z * \rho w = 0 \\ or, \quad &\xi_x * Q_2 + \xi_y * Q_3 + \xi_z * Q_4 = 0 \end{aligned} \qquad (14)$$

It is required to eliminate a momentum equation and the rule is to eliminate the momentum equation in the direction at which η or ξ have large gradient. The above boundary equation (14) can be used to get the relations between the flow variables. The modification of fluxes and Jacobian matrices can be summarized as follows,

$$[A]*\{\Delta Q\} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} * \begin{Bmatrix} \Delta Q1 \\ \Delta Q2 \\ \Delta Q3 \\ \Delta Q4 \\ \Delta Q5 \end{Bmatrix} \qquad (15)$$

After modification, this becomes

$$[A]*\{\Delta Q\} = \begin{bmatrix} A_{11} & A_{1i} + A_{1k} * \dfrac{\partial Q_k}{\partial Q_i} & A_{14} + A_{1k} * \dfrac{\partial Q_k}{\partial Q_4} & A_{15} \\ A_{i1} & A_{i2} + A_{ik} * \dfrac{\partial Q_k}{\partial Q_i} & A_{i4} + A_{ik} * \dfrac{\partial Q_k}{\partial Q_4} & A_{i5} \\ & & & \\ A_{j1} & A_{ji} + A_{jk} * \dfrac{\partial Q_k}{\partial Q_i} & A_{jj} + A_{jk} * \dfrac{\partial Q_k}{\partial Q_j} & A_{j5} \\ A_{51} & A_{5i} + A_{5k} * \dfrac{\partial Q_k}{\partial Q_i} & A_{5j} + A_{53} * \dfrac{\partial Q_k}{\partial Q_j} & A_{55} \end{bmatrix} * \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_i \\ \Delta Q_j \\ \Delta Q_5 \end{Bmatrix} \qquad (16)$$

Where if $\eta_x > \eta_y, \eta_z$ then i = 3, j = 4, k = 2 and if $\eta_y > \eta_x, \eta_z$ then i = 2, j = 4, k = 3 and if $\eta_z > \eta_x, \eta_y$ then i = 2, j = 3, k = 4.
The modified fluxes are,

$$E = \begin{bmatrix} E_1 \\ E_i \\ E_j \\ E_5 \end{bmatrix} \quad (17). \quad F = \begin{bmatrix} F_1 \\ F_i \\ F_j \\ F_5 \end{bmatrix} \quad (18). \quad G = \begin{bmatrix} G_1 \\ G_i \\ G_j \\ G_5 \end{bmatrix} \qquad (19)$$

The above modification is implemented in the three steps of ADI splitting and produces a standard block (4*4) tri-diagonal system, which can be solved to get $\Delta Q_1, \Delta Q_i, \Delta Q_j, \Delta Q_5$ and the reminder $\Delta Q's$ are computed from the calculated $\Delta Q's$.

2. Inflow Boundary Condition

The boundary conditions are, Po, To are specified at the inlet and the flow direction are known and given by the following relations,

$$v = u * \tan(\theta_{uv}), \quad w = u * \tan(\theta_{uw}) \tag{20}$$

where $\theta_{uv}.and.\theta_{uw}$ are the flow direction angles. It is required to eliminate two momentum equations (y and z), and continuity and energy. The above boundary equation can be used to get the relations between the flow variables. the modification of fluxes and Jacobian matrices can be summarized as follows,

$$[A]*\{\Delta Q\} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} * \begin{Bmatrix} \Delta Q1 \\ \Delta Q2 \\ \Delta Q3 \\ \Delta Q4 \\ \Delta Q5 \end{Bmatrix} \tag{21}$$

After modification, this becomes

$$[A]*\{\Delta Q\} = \left[A_{22} + A_{21} * \frac{\partial Q_1}{\partial Q_2} + A_{23} * \frac{\partial Q_3}{\partial Q_2} + A_{24} * \frac{\partial Q_4}{\partial Q_2} + A_{25} * \frac{\partial Q_5}{\partial Q_2} \right] * \{\Delta Q_2\} \tag{22}$$

and the modified fluxes are,

$[E] = [E_2]$ (23). $[F] = [F_2]$ (24). $[G] = [G_2]$ (25)

The above modification is implemented in the three steps of ADI splitting and produces a standard scalar tri-diagonal system, which can be solved to get $\Delta Q's$ listed above and the reminder $\Delta Q's$ are computed from the calculated ones.

3. Outflow Boundary Condition:

The boundary condition is that the back pressure, P_b, is specified at the outlet .We need to eliminate energy equation. The above boundary equation can be used to get the relations between the flow variables. the modification of fluxes and Jacobian matrices can be summarized as follows,

$$[A]*\{\Delta Q\} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} * \begin{Bmatrix} \Delta Q1 \\ \Delta Q2 \\ \Delta Q3 \\ \Delta Q4 \\ \Delta Q5 \end{Bmatrix} \tag{26}$$

After modification, this becomes

$$[A]*\{\Delta Q\} = \begin{bmatrix} A_{11}+A_{15}*\frac{\partial Q_5}{\partial Q_1} & A_{12}+A_{15}*\frac{\partial Q_5}{\partial Q_2} & A_{13}+A_{15}*\frac{\partial Q_5}{\partial Q_3} & A_{14}+A_{15}*\frac{\partial Q_5}{\partial Q_4} \\ A_{21}+A_{25}*\frac{\partial Q_5}{\partial Q_1} & A_{22}+A_{25}*\frac{\partial Q_5}{\partial Q_2} & A_{23}+A_{25}*\frac{\partial Q_5}{\partial Q_3} & A_{24}+A_{25}*\frac{\partial Q_5}{\partial Q_4} \\ A_{31}+A_{35}*\frac{\partial Q_5}{\partial Q_1} & A_{32}+A_{35}*\frac{\partial Q_5}{\partial Q_2} & A_{33}+A_{35}*\frac{\partial Q_5}{\partial Q_3} & A_{34}+A_{35}*\frac{\partial Q_5}{\partial Q4} \\ A_{41}+A_{45}*\frac{\partial Q_5}{\partial Q_1} & A_{42}+A_{45}*\frac{\partial Q_5}{\partial Q_2} & A_{43}+A_{45}*\frac{\partial Q_5}{\partial Q_3} & A_{44}+A_{45}*\frac{\partial Q_5}{\partial Q_4} \end{bmatrix} * \begin{Bmatrix} \Delta Q_1 \\ \Delta Q_2 \\ \Delta Q_3 \\ \Delta Q_4 \end{Bmatrix} \qquad (27)$$

The modified fluxes are

$$E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{bmatrix} \quad (28). \qquad F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \quad (29). \qquad G = \begin{bmatrix} G_1 \\ G_2 \\ G_3 \\ G_4 \end{bmatrix} \quad (30).$$

The above modification is implemented in the three steps of ADI splitting and produces a standard scalar tri-diagonal system, which can be solved to get $\Delta Q's$ listed above and the reminder $\Delta Q's$ are computed from the calculated $\Delta Q's$.

APPLICATIONS

The three-dimensional Euler's flow solver has been constructed to solve the subsonic and transonic flow through a three-dimensional duct. The transonic test cases show the success of the inflow and out flow boundary conditions and at each test case presents a success in tangency boundary condition implementation and mixed boundary conditions (tangency + inflow, tangency + out flow… etc). The transonic test case shows a sharp presentation of the shock surface. The 90^o-bend duct shows the ability of the code to solve the flow field with large metric gradients. The results obtained using a very coarse grid could be executed on typical PC platforms with moderate computation time.

Test case 1: Nozzle (Transonic)

In this test case, the 3-D Euler solver developed is tested to deal with transonic flow through a convergent-divergent nozzle with geometry variation in the (y & z)-direction. The solution is obtained using two grid systems, coarse grid (21*5*5), and fine grid (42*10*10). The comparison of the results shows that there is a small difference between results but with great computation time increasing for the fine grid case. Results are shown in following figs. A sharp resolution of the shock surface is obtained. The nozzle geometry and back - pressure are given below:, the results compares well with the quasi one dimensional analytical solution which gives exit Mach number =.22, while the results from 3-D Euler code gives exit Mach number=.207, the developed boundary condition treatment gives the ability to increase the normal grid size without loosing to much accuracy but will eventually decrease the amount of computer processing time and memory, which is important in the early stage of the aerodynamic shape design.

$$\frac{P_b}{P_{o1}} = 0.65 \quad \& \quad z_{u,l} = \pm\begin{Bmatrix} 1 \\ .5*\left(1.5+.5*\cos\left((x-.5)*\frac{\pi}{2}\right)\right) \\ 1 \end{Bmatrix} \begin{matrix} 0 \le x \le .5 \\ .5 < x \le 4.5, \\ 4.5 < x \le 5 \end{matrix}$$
$$y_{u,l} = \pm\begin{Bmatrix} 1 \\ .5*\left(1.5+.5*\cos\left((x-.5)*\frac{\pi}{2}\right)\right) \\ 1 \end{Bmatrix} \begin{matrix} 0 \le x \le .5 \\ .5 < x \le 4.5 \\ 4.5 < x \le 5 \end{matrix} \qquad (31)$$

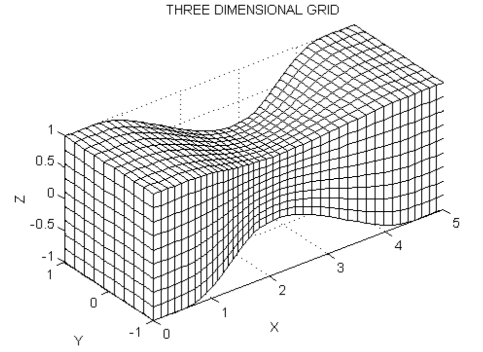

Figure (1) Physical Domain Grid (21*5*5)

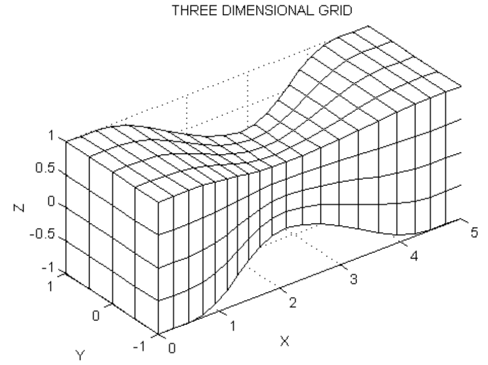

Figure (2) Physical Domain Grid (42*10*10)

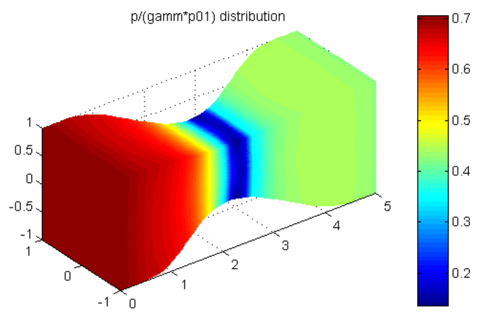

Figure (3) Pressure Distribution (42*10*10) grid

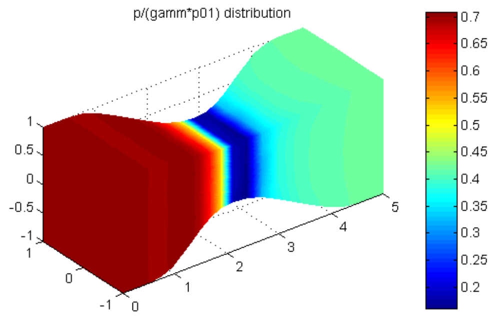

Figure (4) Pressure Distribution (21*5*5) grid

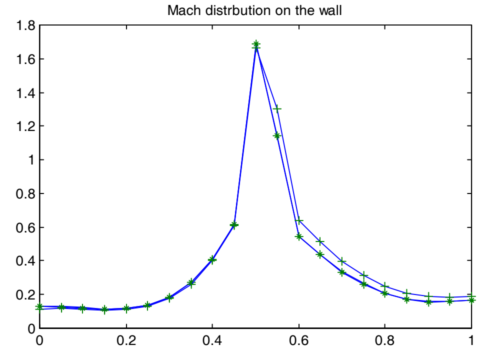

Figure (5) Mach distribution along the wall (21*5*5) grid - (“*” & “.” for edges, “+” for center)

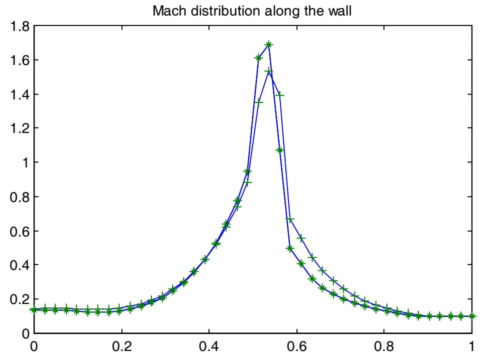

Figure (6) Mach distribution along the wall (42*10*10) grid - (“*”&“.”edges, “+”center)

Test Case 2: uniform 90^o bend duct (Subsonic)

The solution of the three-dimensional flow through uniform 90^o-bend duct is obtained for Pb/Po1 = 0.8,this test case was carried out for testing the rigidity of the boundary condition treatment (tangency surface) for dealing with high gradient tangency surfaces and the results show that the flow can be treated as a two-dimensional flow. Results are shown in the following figs.

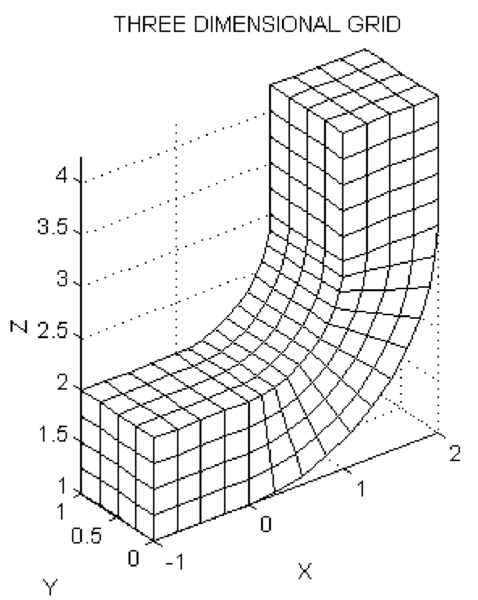

Figure (7) Physical Domain Grid

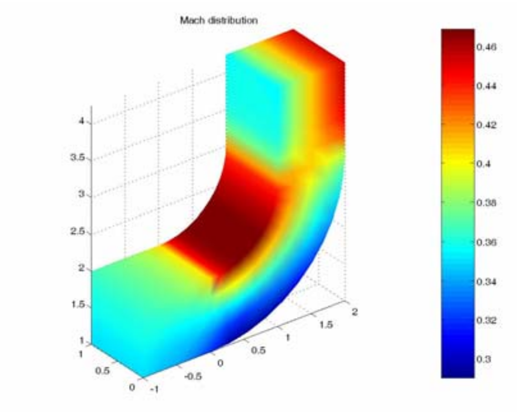

Figure (8) Mach Distribution

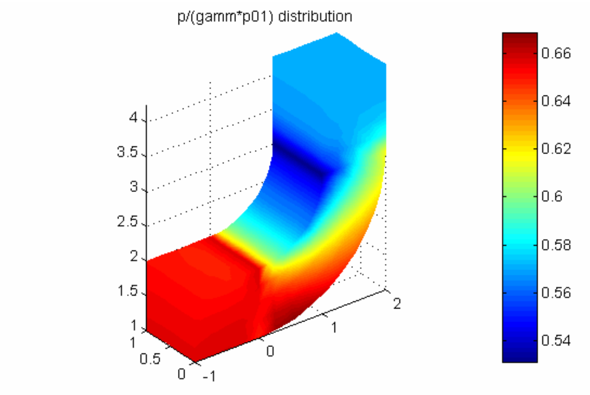

Figure (9) Pressure Distribution

CONCLUSION

The three-dimensional Euler's flow solver has succeeded to solve the subsonic and transonic flow through a three-dimensional duct. The transonic test cases show the success of the inflow and out flow boundary conditions and at each test case presents a success in tangency boundary condition implementation and mixed boundary conditions (tangency + inflow, tangency + out flow… etc). The transonic test case shows a sharp presentation of the shock surface. The 90^{o}-bend duct shows the ability of the code to solve the flow field with large metric gradients. The results obtained using a very coarse grid could be executed on typical PC platforms with moderate computation time.

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