Review on the applications of 2nd order linear P.D.E

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Abstract*– PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics by adding a few statistics to PDE it can be used in weather forecasting, prediction of crime places, disasters, how universe behave ……. Etc. second order linear PDEs can be classified according to the characteristic equation into 3 types hyperbolic, parabolic and elliptic. Hyperbolic equations have two distinct families of (real) characteristic curves, parabolic equations have a single family of characteristic curves, and the elliptic equations have none. All the three types of equations can be reduced to its first canonical form finding the general solution or the second canonical form similar to 3 basic PDE models.

Hyperbolic equations reduce to a form coinciding with the wave equation in the leading terms, the parabolic equations reduce to a form modeled by the heat equation, and the Laplace's equation models the canonical form of elliptic equations. Thus, the wave, heat and Laplace's equations serve as basic canonical models for all second order linear PDEs.

Keywords-- Partial differential equation – parabolic equation – hyperbolic equation – elliptical equation – canonical form.

I. Introduction

A PDE is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. Usually one of these deals with time t and the remaining with space. PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics.

The theory of partial differential equations of the second order is more complicated than the equations of the first order, and it is much more typical of the subject as a whole. Within the context, considerably better results can be achieved for equations of the second order in two independent variables than for equations in space of higher dimensions. Linear equations are the easiest to handle. In general, a second order linear partial differential equation is of the form (1)

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y)$$

where A, B, C, D, E, F and G are in general functions of x and y but they may be constants. The subscripts are defined as partial derivatives where $u_x = \frac{\partial u}{\partial x}$

II. Canonical form

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + f = 0$$

Represents hyperbola, parabola, or ellipse accordingly as B^2-4AC is positive, zero, or negative.

Classifications of PDE are:

(i) Hyperbolic if $B^2-4AC > 0$

(ii) Parabolic if $B^2-4AC = 0$

(iii) Elliptic if $B^2-4AC < 0$

The classification of second-order equations is based upon the possibility of reducing equation by coordinate transformation to canonical or standard form at a point. An equation is said to be hyperbolic, parabolic, or elliptic at a point (x_0, y_0) accordingly as; $B^2(x_0,y_0)-4A(x_0,y_0)C(x_0,y_0)$ (2)

Is positive, zero, or negative. If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic. In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, in the case of several independent variables, it is not, in general, possible to find such a transformation

To transform equation (1) to a canonical form we make a change of independent variables. Let the new variables be;

$\varepsilon = \varepsilon(x, y)$, $\eta = \eta(x, y)$

Assuming that ε and η are twice continuously differentiable and that the Jacobian;

$$\begin{vmatrix} \varepsilon_x & \varepsilon_y \\ \eta_x & \eta_y \end{vmatrix}$$

is nonzero in the region under consideration, then x and y can be determined uniquely. Let x and y be twice continuously differentiable functions of ε and η Then we have

$$u_x = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\varepsilon \varepsilon_x + u_\eta \eta_x \quad ,$$

$$u_y = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\varepsilon \varepsilon_y + u_\eta \eta_y$$

$u_{xx} = \frac{\partial u_x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\delta u_x}{\delta \eta}\frac{\delta \eta}{\delta x} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x +$
$u_{\eta\eta}{\eta_x}^2 + u_\varepsilon \varepsilon_{xx} + u_\eta \eta_{xx}$
$u_{yy} = \frac{\partial u_y}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y +$
$u_{\eta\eta}{\eta_y}^2 + u_\varepsilon \varepsilon_{yy} + u_\eta \eta_{yy}$
$u_{xy} = \frac{\partial u_x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\varepsilon\varepsilon}\varepsilon_x\varepsilon_y + u_{\eta\eta}\eta_x\eta_y +$
$u_\varepsilon \varepsilon_{xy} + u_\eta \eta_{xy} + u_{\varepsilon\eta}\left(\varepsilon_x\eta_y + \varepsilon_y\eta_x\right)$
$A^*(x,y)u_{xx} + B^*(x,y)u_{xy} + C^*(x,y)u_{yy} +$
$D^*(x,y)u_x + E^*(x,y)u_y + F^*(x,y)u = G^*(x,y)$

Where

$A^* = A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2$
$B^* = 2A\varepsilon_x\eta_x + B\left(\varepsilon_x\eta_y + \varepsilon_y\eta_x\right) + 2C\varepsilon_y\eta_y$
$C^* = A{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2$, $D^* = A\varepsilon_{xx} +$
$B\varepsilon_{xy} + C\varepsilon_{yy} + D\varepsilon_x + E\varepsilon_y$
$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x +$
$E\eta_y$, $F^* = F$, $G^* = G$

The resulting equation (3) is in the same form as the original equation (1) under the general transformation. The nature of the equation remains constant if the Jacobian does not vanish.

$B^{*2} - 4A^*C^* = J^2(B^2 - 4AC)$ And $J^2 \neq 0$,

we shall assume that the equation under consideration is of the single type in a given domain. The classification of equation (1) depends on the coefficients A(x,y), B (x,y), and C (x,y) at a given point (x,y) so equation (1) rewritten as ;

$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} =$
$H(x,y,u,u_x,u_y)$

Where A,B,C$\neq$0

And equation (3) rewritten as ;

$A^*(x,y)u_{\varepsilon\varepsilon} + B^*(x,y)u_{\varepsilon\eta} + C^*(x,y)u_{\eta\eta} =$
$H\left(\varepsilon,\eta,u,u_\varepsilon,u_\eta\right)$

Where A^*, C^* =0

$A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2 = 0$,
$A{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2 = 0$

Since the 2 equation from the same type we can rewrite them ;

$A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2 = 0$ where $\boldsymbol{\varepsilon}$ stands for the 2 functions $\boldsymbol{\varepsilon}$, η

Dividing by ${\varepsilon_y}^2$ $A(\frac{\varepsilon_x}{\varepsilon_y})^2 + B\frac{\varepsilon_x}{\varepsilon_y} + C = 0$

As we studied in partial differentiation ; $\frac{dy}{dx} = -\frac{\varepsilon_x}{\varepsilon_y}$

$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$; therfore two roots are

$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2-4AC}}{2A}$

These equations, which are known as the characteristic equations, are ordinary differential equations for families of curves in the xy-plane along which $\boldsymbol{\varepsilon}$ = constant and η = constant. The integrals of equations are called the characteristic curves. Since the equations are first order ordinary differential equations, the solutions may be written as;

φ1(x,y)=c1 φ2(x,y)=c2 with c1 and c2 as constants.

Hence the transformations $\boldsymbol{\varepsilon}$=φ1(x,y), η =φ2(x,y) will transform equation (4) to a canonical form.

We show that the characteristic of any hyperbolic PDE can be transformed as;

*$B^2 - 4AC > 0$ so we have 2 real different characteristic integration yields reduced into first canonical form $u_{\varepsilon\eta} = H\left(\varepsilon,\eta,u,u_\varepsilon,u_\eta\right)$, $B^* \neq 0$

let we have new independent variable α,β where

$\alpha = \varepsilon + \eta$, $\beta = \varepsilon - \eta$ to elimnate $u_{\varepsilon\eta}$ by setting $B^*(x,y)u_{\varepsilon\eta}$=o where

$u_{\varepsilon\eta} \neq 0$ so $B^* = 0$ $A^* = -C^*$

$B^* = 2A\varepsilon_x\eta_x + B\left(\varepsilon_x\eta_y + \varepsilon_y\eta_x\right) + 2C\varepsilon_y\eta_y =$
0

which is transformed into second canonical form

$u_{\alpha\alpha} - u_{\beta\beta} = H(\alpha,\beta,u,u_\alpha,u_\beta)$

similar to wave equation to be modeled

*$B^2 - 4AC = 0$

so we have 2 real repeated characteristic integration yields reduced into canonical form

$u_{\varepsilon\varepsilon} = H\left(\varepsilon,\eta,u,u_\varepsilon,u_\eta\right)$

*$B^2 - 4AC < 0$

so we have no real characteristic but it has complex solution which is analytic along some neighbourhood domain can be reduced into first canonical form $u_{\varepsilon\eta} = H\left(\varepsilon,\eta,u,u_\varepsilon,u_\eta\right)$ where ε =α+iβ , η =α-iβ are two conjugate functions where

$\alpha = \frac{1}{2}(\varepsilon + \eta)$, $\beta = \frac{1}{2i}(\varepsilon - \eta)$ which can be used to to elimnate $u_{\varepsilon\eta}$ by setting $B^*(x,y)u_{\varepsilon\eta}$=o where

$u_{\varepsilon\eta} \neq 0$ so $B^* = 0$ $A^* = C^*$

$B^* = 2A\varepsilon_x\eta_x + B\left(\varepsilon_x\eta_y + \varepsilon_y\eta_x\right) + 2C\varepsilon_y\eta_y =$
0

Which is transformed into second canonical form

$u_{\alpha\alpha} + u_{\beta\beta} = H(\alpha,\beta,u,u_\alpha,u_\beta)$

similar to la place equation to be modeled

III- HYPERBOLIC EQUATIONS

1-Hemogeneuse wave equation

$u_{tt} - c^2 u_{xx} = 0$, $B^2 - 4AC = 4c^2 > 0$, $A\lambda^2 - B\lambda + c = 0$ $\frac{dt}{dx} = \frac{\pm 1}{c}$ sepration of variables and integrate we get $\varepsilon = x + ct$, $\eta = x - ct$

$u_{xx} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon \varepsilon_{xx} + u_\eta \eta_{xx}$

$u_{tt} = u_{\varepsilon\varepsilon}{\varepsilon_t}^2 + 2u_{\varepsilon\eta}\varepsilon_t\eta_t + u_{\eta\eta}{\eta_t}^2 + u_\varepsilon \varepsilon_{tt} + u_\eta \eta_{tt}$

Then substate in original p.d.e

$-4c^2 u_{\varepsilon\eta} = 0$, $c \neq 0$, $u_{\varepsilon\eta} = 0$ the integrate w.r.t η

$u_\varepsilon = f(\varepsilon)$ then integrate w.r.t ε $u = f(\varepsilon) + g(\eta) = f(x+ct) + g(x-ct)$

Using initial and boundary conditions

$f(x+ct)$, $g(x-ct)$ can be determined

$u_{(x,0)} = p(x)$, $u_{t_{(x,0)}} = v(x)$, $u_{(0,t)} = u_{(l,t)} = 0$, $t > 0$

$f(x) + g(x) = p(x)$, $f^{\backslash}(x) - g^{\backslash}(x) = \frac{1}{c}v(x)$ by integration w.r.t x

$f(x) - g(x) = \frac{1}{c}\int_{x_0}^{x} v(x)\,dx + f(x_0) - g(x_0)$ solving for $f(x)$, $g(x)$

$$f(x+ct) = \frac{1}{2}p(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} v(x)\ dx + \frac{1}{2}\left(f(x_0) - g(x_0)\right)$$

$$g(x-ct) = \frac{1}{2}p(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} v(x)\ dx - \frac{1}{2}\left(f(x_0) - g(x_0)\right)$$

By adding

$$u_{(x,t)} = \frac{1}{2}\left(p(x+ct) + p(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)\,dx \quad (\textbf{De alembart})$$

Where

$t > 0$, $0 \le x + ct \le l$, $0 \le x - ct \le l$, $t \le \frac{l-x}{c}$, $t \le \frac{x}{c}$

$u_{(0,t)} = u_{(l,t)} = 0$, $t > 0$, $f(ct) = -g(-ct)$, $-ct \le 0$

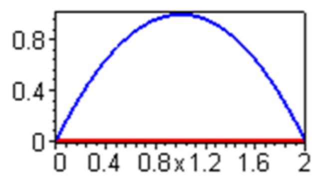
$f(l+ct) = -g\left(2l - (l+ct)\right)$, $l + ct \ge l$ f, g odd periodic

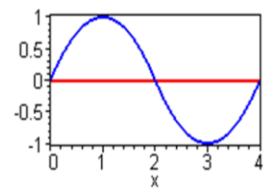
$-g\left(2l-(l+ct)\right) = -\frac{1}{2}p\left(2l-(l+ct)\right) + \frac{1}{2c}\int_0^{2l-(l+ct)} v(x)\ dx + \frac{1}{2}\left(f(0) - g(0)\right)$

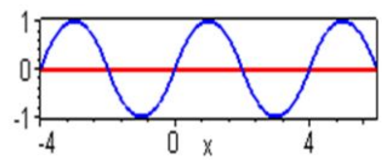
$l \le x + ct \le 2l$ extended to the right

$l + ct = -(l - ct)$

$f\left(-(l-ct)\right) = \frac{1}{2}p\left(-(l-ct)\right) + \frac{1}{2c}\int_0^{-(l-ct)} v(x)\ dx + \frac{1}{2}\left(f(0) - g(0)\right)$,







$-2l \le x - ct \le -l$ extended to the left Let a string of length 2 units as shown

Fig. A,B,C string extended

Physical application

1-Motion of stretched string in musical instruments such as guitar, piano …. etc. described by

$\boldsymbol{u_{tt} - c^2 u_{xx} = 0}$

where $c^2 = \frac{T}{\rho}$ T horizantel component of tension force , ρ mass per unit length

Suppose such string placed on x-axis

1- Damping forces are neglected such as air resistance
2- Weight of string is also neglected
3- Tension force is tangential to string curve

Initial position function

$u_{(x,0)} = p(x) = \sin\left(\frac{nx\pi}{l}\right) \quad l \text{ is length of string}$

Initial velocity function $u_{t_{(x,0)}} = v(x) = 0$
(initially at rest)

Boundaries $u_{(0,t)} = u_{(l,t)} = 0 \quad , \quad t > 0$

applying de Alembert formula

$$u_{(x,t)} = \frac{1}{2}\big(p(x+ct) + p(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} v(x)\,dx$$

$$u_{(x,t)} = \frac{1}{2}\left(\sin\left(\frac{n\pi}{l}(x+ct)\right) + \sin\left(\frac{n\pi}{l}(x-ct)\right)\right) \qquad let \quad n = 1\,, \quad l = c = 2$$

$$u_{(x,t)} = \frac{1}{2}\left(\sin\left(\frac{\pi}{2}(x+2t)\right) + \sin\left(\frac{\pi}{2}(x-2t)\right)\right) = \sin\left(\frac{\pi x}{2}\right)\cos(\pi t)$$

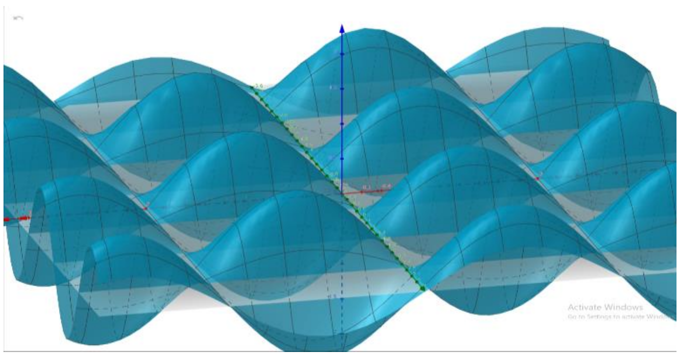


Fig. D 1-physical application

2- longitudinal waves travelling along thin Rod with youngs Y modulus and mass density ρ where the constant $c^2 = \frac{Y}{\rho}$ is phase velocity where c is specific for each material

Table: Calculated and measured longitudinal wave speeds in thin rods made up of common metals. Sources: Haynes and Lide 2011c, Wikipedia contributors 2012.

Metal	$Y\,(\mathrm{N\,m^{-2}})$	$\rho\,(\mathrm{kg\,m^{-3}})$	$\sqrt{Y/\rho}\,(\mathrm{m\,s^{-1}})$	$v\,(\mathrm{m\,s^{-1}})$
Aluminium	7.0×10^{10}	2.7×10^{3}	5100	5000
Copper	1.2×10^{11}	8.9×10^{3}	3600	3800
Lead	1.6×10^{10}	1.1×10^{4}	1100	1100
Nickel	2.0×10^{11}	8.9×10^{3}	4700	4900
Silver	8.3×10^{10}	1.1×10^{4}	2800	2700
Tin	5.0×10^{10}	7.4×10^{3}	2600	2700
Zinc	1.1×10^{11}	7.1×10^{3}	3900	3900

Fig. E constant for materials

3-high frequancy AC submarine cable where the cable is made such that ressistance R and leakage of conductance G is also neglected where general telegraph equation

$i_{xx} = LCi_{tt} + (RC+GL)i_t + RGi \quad , \quad R = G = 0 \quad ,$

$L \text{ inductance} \;, \quad C \text{ capictance}$

$$i_{tt} - \frac{1}{LC}\, i_{xx} = 0 \quad high\ freq\ AC\ similar\ to\ wave\ equation$$

$$i(x,t) = f\left(x + \frac{t}{\sqrt{LC}}\right) + g\left(x - \frac{t}{\sqrt{LC}}\right) \quad , \quad V(x,t) = f\left(x + \frac{t}{\sqrt{LC}}\right) + g\left(x - \frac{t}{\sqrt{LC}}\right)$$

2- Non-hemogeneuse wave equation

$$\boldsymbol{u_{tt} - c^2 u_{xx} = h^*(x,t)} \quad , \quad \boldsymbol{u_{(x,0)} = p(x)} \quad , \quad \boldsymbol{u_{t_{(x,0)}} = v^*(x)} \qquad let\ y = ct$$

$$u_{xx} - u_{yy} = h(x,y) \quad , \quad u_{(x,0)} = p(x) \quad , \quad u_{y_{(x,0)}} = v(x) = \frac{v^*(x)}{c} \quad , \quad h(x,y) = -\frac{h^*(x,t)}{c^2}$$

Let 2 characteristic curves ε , η whereas ε , η are constants so the 2 characteristic lines intersect at point p_0 and x axis at p_1, p_2 where the area is regioned by 3 lines B_0, B_1, B_2 as shown

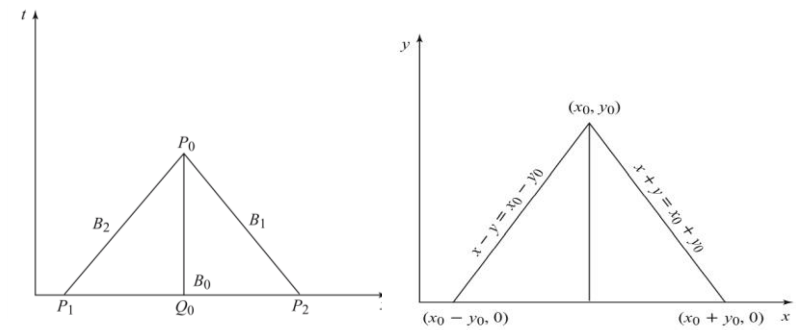


Fig. F, G

As the area represents a double integral along the interior surface of triangle to get u where

$\iint u_{xx} - u_{yy}\, dA = \iint h(x,y)\, dA$ applying greens theorem

$$\iint (Q_x - P_y)\, dxdy = \oint P dx + Q dy \quad ,$$

$the\ closed\ integral\ along\ B \quad , \ as\ dA = dxdy$

$$\iint u_{xx} - u_{yy}\, dA = \oint u_y dx + u_x dy \quad along\ B_0, B_1,$$

$$\oint u_y dx + u_x dy \quad , \ along\ B_0 \ , \ where\ y = 0\ , dy = 0$$

$$\oint u_y dx + u_x dy = \int_{x_o - y_o}^{x_o + y_o} u_y\, dx$$

$$\oint u_y dx + u_x dy \quad , \quad along\ B_1 : x = x_o + (y_o - y) \quad , \quad dx = -dy$$

$$\oint u_y dx + u_x dy = \oint -u_y dy - u_x dx = \int_0^{y_o} -u_y dy \int_{x_o+y_o}^{x_o} -u_x dx = u_{(x_o+y_o,0)} - u_{(x_o,y_o)}$$

$$\oint u_y dx + u_x dy \quad , \quad along\ B_2: x = x_o - (y_o - y) \quad , \quad dx = dy$$

$$\oint u_y dx + u_x dy = \oint u_y dy + u_x dx = \int_{y_o}^{0} u_y dy + \int_{x_o}^{x_o-y_o} u_x dx = u_{(x_o-y_o,0)} - u_{(x_o,y_o)}$$

$$\iint u_{xx} - u_{yy}\, dA = u_{(x_o-y_o,0)} + u_{(x_o+y_o,0)} - 2u_{(x_o,y_o)} + \int_{x_o-y_o}^{x_o+y_o} u_y\, dx$$

$$\iint h(x,y)\ dA = u_{(x_o-y_o,0)} + u_{(x_o+y_o,0)} - 2u_{(x_o,y_o)} + \int_{x_o-y_o}^{x_o+y_o} u_y\, dx$$

$$u_{(x_o,y_o)} = \frac{1}{2}\left(u_{(x_o-y_o,0)} + u_{(x_o+y_o,0)}\right) + \frac{1}{2}\int_{x_o-y_o}^{x_o+y_o} u_y\, dx - \frac{1}{2}\iint h(x,y)\ dxdy$$

$$u_{(x,0)} = p(x) \quad , \quad u_{y_{(x,0)}} = v(x)$$

$$u_{(x,y)} = \frac{1}{2}(P(x-y) + P(x+y)) + \frac{1}{2}\int_{x_o-y_o}^{x_o+y_o} v(x)\, dx - \frac{1}{2}\int_0^{y_o}\int_{x_o-(y_o-y)}^{x_o+(y_o-y)} h(x,y) dxdy$$

Physical application

1-Motion of stretched string in musical instruments such as guitar, piano …. etc. described by

$$\boldsymbol{u_{tt} - c^2 u_{xx} = h^*(x,t)}$$

$where\ c^2 = \frac{T}{\rho}\ \ T\ horizantel\ component\ of\ tension\ force,$

$\rho\ mass\ per\ unit\ length,$

$h^*(x,y)\ are\ damping\ forces\ due\ to\ weight\ of\ string\ or\ air$

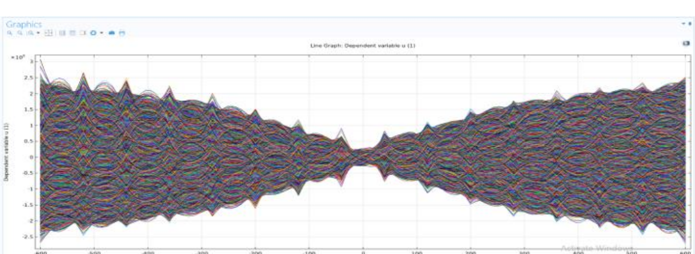
Resistance Suppose a such string placed on x-axis

Initial position function

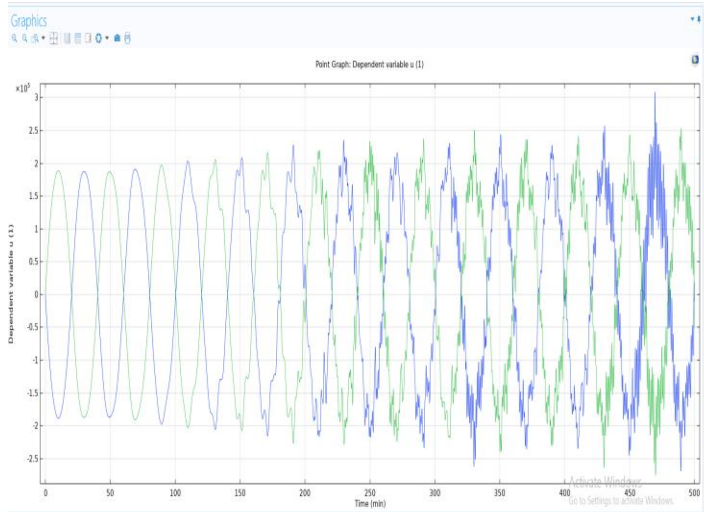
$$u_{(x,0)} = p(x) = \sin(x) \quad , \quad Damping\ of \quad h^*(x,t) = xe^{-t}$$

Initial velocity function

$$u_{t_{(x,0)}} = v(x) = \cos(x) \qquad , \quad c = 1$$



$$u_{tt} - u_{xx} = xe^{-t} \quad , \ applying\ the\ previous\ formula \quad u_{xx} - u_{yy} = -xe^{-y}$$



$$u_{(x,y)} = \frac{1}{2}(P(x-y) + P(x+y)) + \frac{1}{2}\int_{x_o-y_o}^{x_o+y_o} v(x)\, dx - \frac{1}{2}\int_0^{y_o}\int_{x_o-(y_o-y)}^{x_o+(y_o-y)} h(x,y) dxdy$$

$$u_{(x,y)} = \frac{1}{2}(\sin(x+y) + \sin(x-y)) + \frac{1}{2}\int_{x_o-y_o}^{x_o+y_o} \cos(x)\ dx + \frac{1}{2}\int_0^{y_o}\int_{x_o-(y_o-y)}^{x_o+(y_o-y)} xe^{-y}\ dx\ dy$$

$$u_{(x,y)} = \sin(x+t) + x(t + e^{-t} - 1)$$

Fig. H u over x

Fig. I u over time

Initial position function

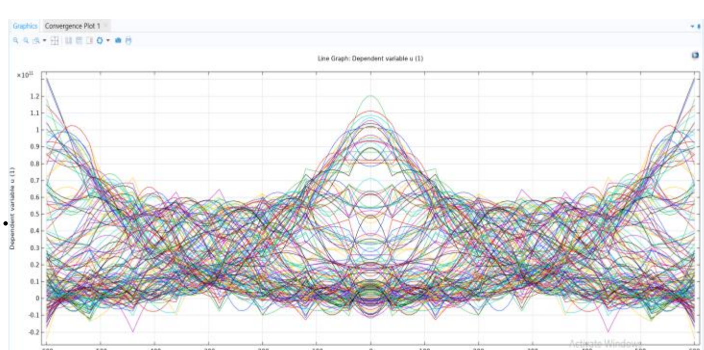
$$u_{(x,0)} = p(x) = x^4 \quad , \quad Damping\ of \quad h^*(x,t) = xe^{-t}$$

Initial velocity function

$$u_{t_{(x,0)}} = v(x) = \cos(x) \qquad , \quad c = 1$$

$$u_{tt} - u_{xx} = xe^{-t} \quad , \ applying\ the\ previous\ formula \quad u_{xx} - u_{yy} = -xe^{-y}$$

$$u_{(x,y)} = \frac{1}{2}(P(x-y) + P(x+y)) + \frac{1}{2}\int_{x_o-y_o}^{x_o+y_o} v(x)\, dx - \frac{1}{2}\int_0^{y_o}\int_{x_o-(y_o-y)}^{x_o+(y_o-y)} h(x,y) dxdy$$



$$u_{(x,y)} = \frac{1}{2}((x-y)^4 + (x+y)^4) + \frac{1}{2}\int_{x_o-y_o}^{x_o+y_o} cos(x)\,dx + \frac{1}{2}\int_0^{y_o}\int_{x_o-(y_o-y)}^{x_o+(y_o-y)} xe^{-y}\,dx\,dy$$

$$u_{(x,y)} = \frac{1}{2}((x-y)^4 + (x+y)^4) + \frac{1}{2}(\sin(x+y) - \sin(x-y) + x(y + e^{-y} - 1)$$

$$u_{(x,y)} = x^4 + t^4 + 6x^2t^2 + \cos(x)\sin(t) + x(t + e^{-t} - 1)$$

Fig. J u over x

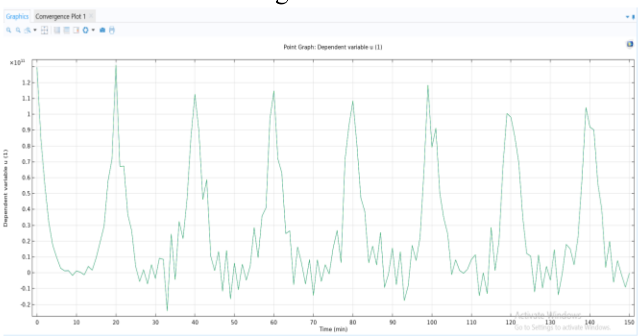


Fig. K u over time

IV-ELLIPTICAL EQUATION

1 – la place equ. $u_{xx} + u_{yy} = 0$ $A = 1$ $B = 0$ $C = 1$ $B^2 - 4AC = -4 < 0$

$A\lambda^2 - B\lambda + c = 0$ $\lambda^2 + 1 = 0$ $\lambda = \pm i$ $\frac{dy}{dx} = i$ $\frac{dy}{dx} = -i$

$\int dy = \int i\,dx$ $iy = -x + c$ $x + iy = c1$ let $\varepsilon = x + iy$

$\int dy = \int -i\,dx$ $iy = x + c$ $x - iy = c2$ let $\eta = x - iy$

$\varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yy} = 0$ $\varepsilon_y = i$ $\varepsilon_x = 1$

$\eta_{xx} = \eta_{yy} = \eta_{xy} = 0$ $\eta_x = 1$ $\eta_y = -i$

$u_{xx} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + u_\eta\eta_{xx} = -u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} - u_{\eta\eta}$

$u_{yy} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\varepsilon\varepsilon_{yy} + u_\eta\eta_{yy} = u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} + u_{\eta\eta}$

$u_{xx} + u_{yy} = 4u_{\varepsilon\eta}$ so the canonical form is $4u_{\varepsilon\eta} = 0$

$u_\varepsilon = g(\varepsilon)$, $u = f(\varepsilon) + g(\eta) = f(x+iy) + g(x - iy) = f(z) + g(\bar{z}) = u_{(x,y)} + iv_{(x,y)}$

General method for particular solution

$u = u_{(x,y)} + iv_{(x,y)}$ u, v should be analytic and harmonic function

$u_x = v_y$, $u_y = -v_x$ *where u, v are real function of real variables*

Then the Real and Imaginary part of u each represents a solution for Laplace P.D.E or any combination of them

As Laplace equation is symmetric so the solution should be radial so we can set $u = v(r)$

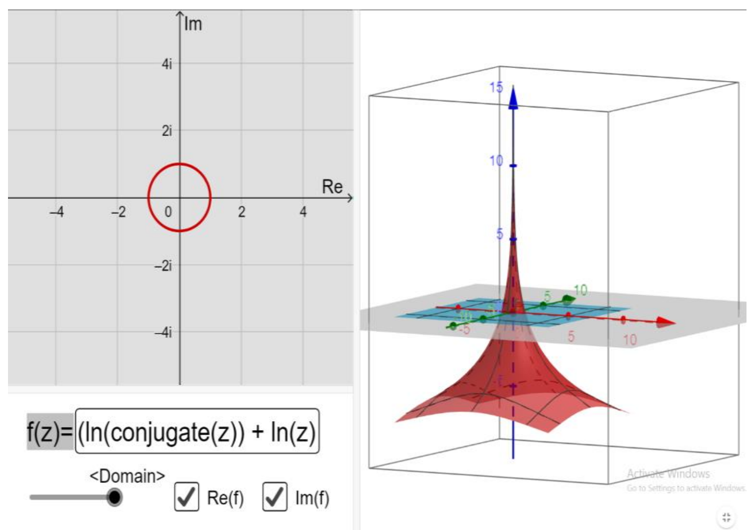
$r = \sqrt{x^2 + y^2}$, $u_x = v_r r_x = \frac{x}{r} v_r$, $u_{xx} = \frac{v_r}{r} + \frac{x^2}{r^2} v_{rr} - \frac{x^2}{r^3} v_r$

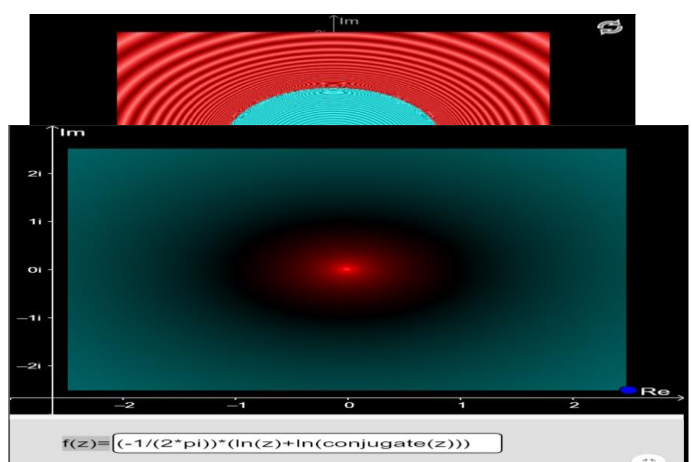
$u_y = v_r r_y = \frac{y}{r} v_r$, $u_{xx} = \frac{v_r}{r} + \frac{y^2}{r^2} v_{rr} - \frac{y^2}{r^3} v_r$, $u_{xx} + u_{yy} = \frac{1}{r} v_r + v_{rr}$

By this method the P.D.E reduced into ode

where $\frac{1}{r} v_r + v_{rr} = 0$ solving for v by integration

$v = c\ln\left(\frac{1}{r}\right) + c^*$ *which is the fundmental sol. where* $c = \frac{1}{\pi}$, $c^* = 0$





$u = v = \frac{-1}{2\pi} ln(x^2 + y^2) = \frac{-1}{2\pi}(\ln(x+iy) + \ln(x-iy))$ *from general solution*

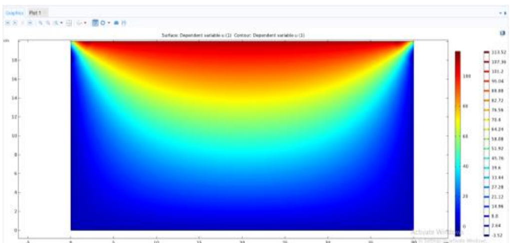


Fig. L,M,N contour and poles of u

Physical application

1-Electrostatic potentaial charge in free region where the potential in the rectangle whose upper side is kept at potential 110 V and whose other sides are grounded.

$0 \le x \le 40 \ , \ 0 \le y \le 20 \ , \ la\ place\ equ. \quad u_{xx} + u_{yy} = 0\ (cartesian)$

where u is the potential

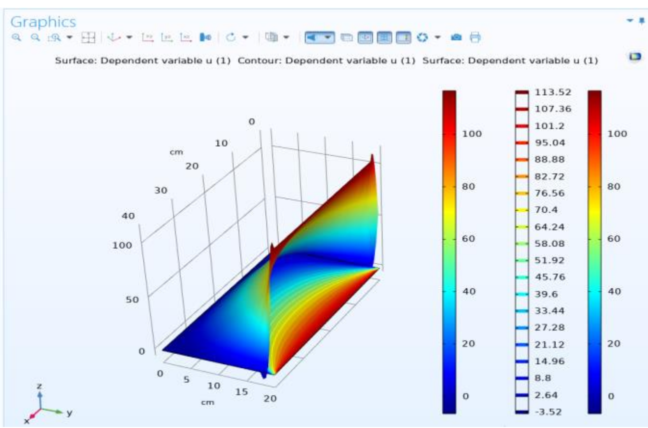


Fig. O,P repreasentation of u

2-The potential flow of an ideal incompressible fluid about a circular cylinder of radius R with a constant incident velocity v

$la\ place\ equation \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0 \ (cylinderical)$

$u = f(\varepsilon) + g(\eta) = f(\ln(r) - i\theta) + g(\ln(r) + i\theta) = Ar^n e^{in\theta} + \frac{B}{r^n} e^{-in\theta}$

$Re = \left(Ar^n + \frac{B}{r^n}\right)\cos(n\theta) \ , \ Im = \left(Ar^n - \frac{B}{r^n}\right)\sin(n\theta)$ *by multiplying Re, Im we get*

$u = \left(Ar^n + \frac{B}{r^n}\right)(C\cos(n\theta) + D\sin(n\theta)) \qquad n = 1,2,3,4,5, \ldots$

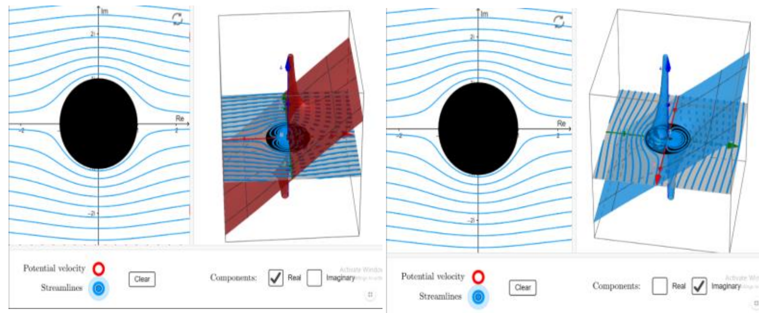
We are gonna to solve this pde twice with different intial and boundaries once for stream lines then for velocity potential.

$u_{(R,\theta)} = 0 \ , \ r = R \ , \ r \to \infty \ , \ u \to vrsin(\theta)$ *from IC, BC*

$Ar^n(C\cos(n\theta) + D\sin(n\theta)) = vrsin(\theta)$ *comparing coffecient* $n = 1, c = 0, AD = v$

$\left(AR + \frac{B}{R}\right)(D\sin(\theta)) = 0 \ , \ B = -AR^2$, *subistute in original so*

$u = \left(Ar^n + \frac{B}{r^n}\right)(C\cos(n\theta) + D\sin(n\theta)) = \left(Ar - \frac{AR^2}{r}\right) D\sin\theta$



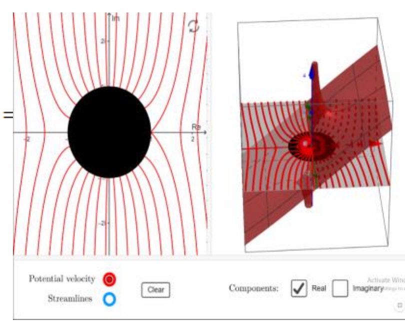
$u = v\left(r - \frac{R^2}{r}\right)\sin(\theta)$ *where u is stream function*

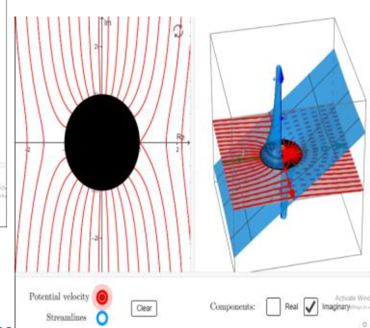
Fig. Q,R stream function

Solving the same pde again for velocity potential where

$u_{(R,\theta)} = 2vRcos(\theta) \ , \ r = R \ , \ r \to \infty \ , \ u \to vrcos(\theta)$

comparing coffecients to get $D = 0 \ , \ n = 1 \ , \ AC = v \ , \ B = AR^2$ *then substute*





$u_{(r,\theta)} = v\left(r + \frac{R^2}{r}\right)\cos(\theta)$ *where u is velocity potential*

Fig. S, T velocity potential

By adding stream lines and velocity potential to get the potential flow

$$U = v\left(r - \frac{R^2}{r}\right)\sin(\theta) + v\left(r + \frac{R^2}{r}\right)\cos(\theta)$$

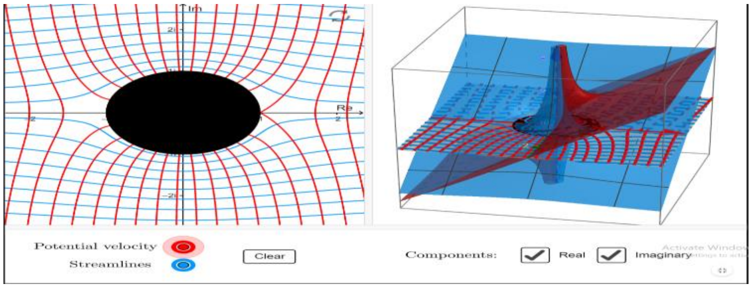


Fig. U stream function and velocity potential

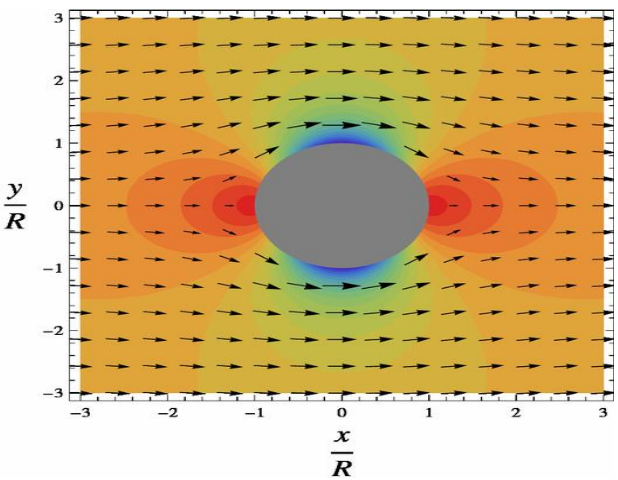


Fig. V vector field of stream function and velocity potential

V-PARABOLIC EQUATION

$1 - diffusion\ equation \quad u_t = ku_{xx} \quad , \quad |x| < \infty \quad , \quad t > 0$

$let\ a\ solution\ of \quad t = x^2 \quad satisfying\ the\ pde\ where \quad \frac{x}{\sqrt{t}} = 1 \quad let \quad \varepsilon = \frac{x}{\sqrt{t}}$

$suppose\ u = t^{-\alpha} f\left(\frac{x}{\sqrt{t}}\right)$ such that the total energy is conserved and preserved

$u_t = \frac{-t^{-\alpha-1}}{2}\varepsilon f^{\backslash}(\varepsilon) - \alpha t^{-\alpha-1} f(\varepsilon) \quad , \quad u_{xx} = t^{-\alpha-1} f^{\backslash\backslash}(\varepsilon) \quad substute\ in\ pde$

$kf^{\backslash\backslash}(\varepsilon) + \frac{\varepsilon}{2} f^{\backslash}(\varepsilon) + \alpha f(\varepsilon) = 0 \quad , \quad take \quad \alpha = \frac{1}{2} \quad , \quad kf^{\backslash\backslash}(\varepsilon) + \frac{1}{2}\left(\varepsilon f^{\backslash}(\varepsilon) + f(\varepsilon)\right) = 0$

$kf^{\backslash\backslash}(\varepsilon) + \frac{1}{2}\left(\varepsilon f(\varepsilon)\right)^{\backslash} = 0 \quad , \quad \left(kf^{\backslash}(\varepsilon) + \frac{1}{2}\varepsilon f(\varepsilon)\right)^{\backslash} = 0 \quad by\ integration\ we\ get$

$kf^{\backslash}(\varepsilon) + \frac{1}{2}\varepsilon f(\varepsilon) = c \quad where\ this\ ode\ has\ infinite\ solutions\ we\ take \quad , \quad c = 0$

$kf^{\backslash}(\varepsilon) + \frac{1}{2}\varepsilon f(\varepsilon) = 0 \quad which\ is\ called\ fundmental\ solution\ by\ integration\ ag$

$f(\varepsilon) = Ae^{-\frac{\varepsilon^2}{4kt}} \quad where\ A\ is\ integration\ constant\ can\ be\ determined\ by$

$\because\ total\ energy\ is\ conserved \quad \therefore \int_{-\infty}^{\infty} u(x,t)\,dx = 1 \quad , \quad A = \frac{1}{\sqrt{4\pi k}}$

$u = t^{-\alpha} f(\varepsilon) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \quad which\ is\ a\ fundemental\ solution\ of\ pde$

$$let \quad k = \frac{1}{4} \quad , \quad u = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

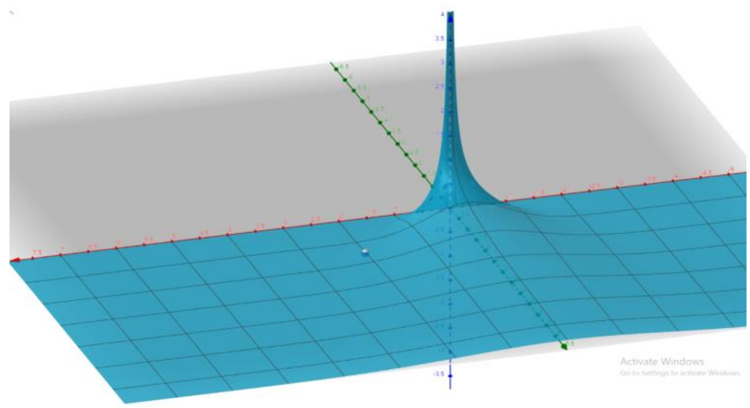


Fig. W fundamental solution

Physical application

1- low frequency AC submarine cable where the cable is made such that inductance L and leakage of conductance G are neglected where general telegraph equation

$i_{xx} = LCi_{tt} + (RC + GL)i_t + RGi \quad , \quad l = G = 0 \quad , R\ ressistance \ , \quad C\ capictance$

$$i_{tt} - \frac{1}{RC} i_{xx} = 0 \quad low\ freq\ AC\ similar\ to\ heat\ equation$$

$$i(x,t) = \frac{1}{\sqrt{\frac{4\pi t}{RC}}} e^{\frac{-x^2}{\frac{4t}{RC}}}$$

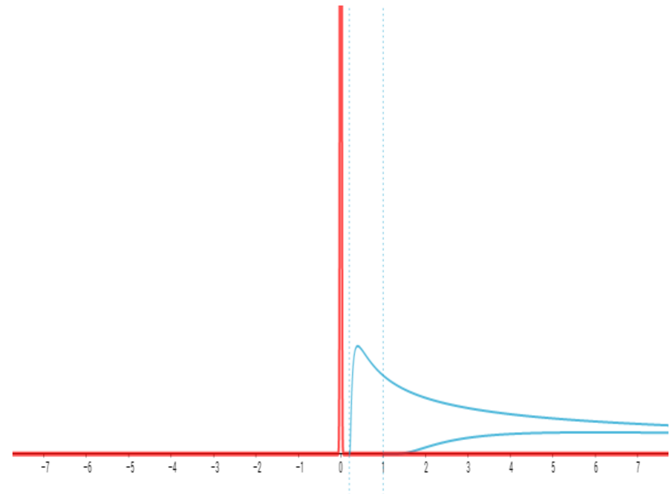


Fig. X current of low frequency submarine

VI-CONCLUSION

The second-order linear PDEs can be classified into three types, which are invariant under changes of variables. The types are determined by discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. Hyperbolic equations have two distinct families of (real) characteristic curves, parabolic equations have a single family of characteristic curves, and the elliptic equations have none. All the three types of equations can be reduced to canonical forms to be modeled allowing the analysis of physical phenomena to predict the variance over time.

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