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SIMPLIFIED VARIATIONAL APPROACH FOR ANALYSIS OF THICK ORTHOTROPIC LAMINATED

PLATES: 2- SIMPLE SUPPORT

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ABSTRACT

A simplified variational approach, stress-based, for the analysis of symmetric cross-ply laminate was developed in Part 1 of this work. It was also tested for the 1-D problem, orthotropic plate in a cylindrical bending, solved exactly by Pagano. This simplified approach is extended here to a two-dimensional structure. The accuracy of the present approach is examined by applying it to the case of rectangular laminated plate with simple support for which the elasticity solution was obtained [1,2]. The present approach gives results for multi-layered laminate with small span-to-thickness ratios that compare well with those from elasticity solutions and other known theories as well.

KEY WORDS

Laminate, Orthotropic, Plate, Simple Support, ESL, Variational, and Stress-based

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INTRODUCTION

As the technology of composites advances, laminated composites are used for thick and moderately thick structures. Also, laminated plates made of advanced composite materials, whose elastic to shear modulus ratios are very large, are susceptible to thickness effects because their effective transverse shear moduli are significantly smaller than the effective elastic moduli.

The classical laminated plate theory, CPT, is inadequate for these types of laminates since it ignores the transverse shear effects. Thus, the shear deformation theories were emerged. Shear deformation theories, FSDT with pre-assumed continuous displacement field across the thickness require a shear correction factor and suffer from locking problem. Thus the higher order, HSDT were developed.

The continuity of the displacement field and its derivatives through the laminate thickness is in contradiction with the continuity of transverse stresses. Thus, the transverse stresses predicted are doubled value when using constitutive relations. To remove these discrepancies in ESL, the Layer Wise models, LWM were introduced which give excellent results for both global and local distributions of displacements and stress. However, LWM's suffers from a numerical crisis if the layer number becomes large.

A stress-based approach developed by the authors in a previous work [3] was tested by the bench-mark problem for the plate in a cylindrical bending solved exactly by Pagano. The presented simplified approach gave good results for out-of-plane displacement and the transverse stresses. This approach is now to be applied for the 2-D problem. Specifically, the problem of a symmetrically bidirectional laminate with pinned edges under static bending forces is considered. For the sake of brevity, the details of the derivation of equilibrium equations, variational analysis and continuity conditions are omitted, see [3]. For the sake of continuity, only relevant equations are provided.
To assess the proposed approach, the problem of bending of a 3, 5, 7 and 9-layered, symmetrically laminated rectangular and square plates simply-supported on all edges are investigated. Numerical results are given and compared with those resulting from the elasticity solution, ES, classical lamination theory, CPT, shear deformation theory (first order, FSDT and higher order, HODT) and layer wise, LWM, as well.

GOVERNING EQUATIONS

Consider a laminated plate composed of an arbitrary number of orthotropic layers such that the various axes of elastic symmetry are parallel to the plate axes. The simple support boundary conditions are expressed as

$$\begin{aligned} @\, x_1 &= 0, a; \quad \sigma_{11} = u_2 = u_3 = 0, \\ @\, x_2 &= 0, b; \quad \sigma_{22} = u_1 = u_3 = 0 \end{aligned} \tag{1}$$

Where *a* and *b* denote the length and the width of the plate, respectively. A transverse normal loading, $-q_o \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}$, is applied to the top surface, while the lower surface is traction-free.

Since the stress resultants must satisfy the following equilibrium equation[3];

$$\begin{aligned} &N_{\alpha\beta,\beta} = 0 \\ &M_{\alpha\beta,\beta} - S_\alpha = 0 \\ &S_{\alpha,\alpha} + q = 0 \end{aligned} \qquad (2)$$

And using the following constitutive relations for orthotropic laminate[3]

$$\begin{aligned} &N_{\alpha\beta} = h(\bar{Q}_{\alpha\beta\gamma\delta}u^o_{\gamma,\delta} + \bar{B}_{\alpha\beta}p) \\ &M_{\alpha\beta} = \frac{h^3}{12}(\bar{Q}_{\alpha\beta\gamma\delta}\psi^{\gamma,\delta} + \frac{6}{5h}\bar{B}_{\alpha\beta}q) \\ &S_\alpha = h\bar{d}_{\alpha\beta}(\psi_\beta + w_{,\beta}) \end{aligned} \qquad (3)$$

where, $\alpha,\beta = 1,2$, $\bar{d}_{\alpha\beta} = 5\bar{C}_{\alpha 3\beta 3}/6$, and $\bar{B}_{\alpha\beta} = \bar{C}_{\alpha\beta 33}/\bar{C}_{3333}$.
The total load q and the mean extensional load p may be written as

$$\begin{aligned} &q = -q_o \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b} \\ &p = -\frac{1}{2}q_o \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b} \end{aligned} \qquad (4)$$

The stress resultants are chosen to be in the following form

$$\begin{aligned} &S_1 = S_1^\bullet \cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b},\ S_2 = S_2^\bullet \sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{b}, \\ &M_{11} = M_{11}^\bullet \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b},\ M_{22} = M_{22}^\bullet \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b}, M_{12} = M_{12}^\bullet \cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{b} = M_{21} \end{aligned} \qquad (5)$$

And the generalized displacements are also chosen as

$$w = w^\bullet \sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b},\ \psi_1 = \psi_1^\bullet \cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{b},\ \psi_2 = \psi_2^\bullet \sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{b} \qquad (6)$$

Where; ($S_1^\bullet, S_2^\bullet, M_{11}^\bullet, M_{22}^\bullet, M_{12}^\bullet, \psi_1^\bullet, \psi_2^\bullet$ and $w^\bullet$) are constants to be determined using the equilibrium equations (2), the constitutive equations (3), and satisfying boundary conditions at $x_1 = 0, a$ and $x_2 = 0, b$ (simple support). Since a bidirectional orthotropic laminate is assumed, we get the following 8 equations

$$S_{1,1} + S_{2,2} = -q \qquad (7)$$

$$M_{11,1} + M_{12,2} - S_1 = 0 \qquad (8)$$

$$M_{21,1} + M_{22,2} - S_2 = 0 \qquad (9)$$

$$M_{11} = \frac{h^3}{12}(\bar{Q}_{1111}\psi_{1,1} + \bar{Q}_{1122}\psi_{2,2}) + \frac{h^2}{10}\bar{B}_{11}q \quad (10)$$

$$M_{22} = \frac{h^3}{12}(\bar{Q}_{2222}\psi_{2,2} + \bar{Q}_{2211}\psi_{1,1}) + \frac{h^2}{10}\bar{B}_{22}q \quad (11)$$

$$M_{12} = \frac{h^3}{12}\bar{Q}_{1212}(\psi_{1,2} + \psi_{2,1}) + \frac{h^2}{10}\bar{B}_{12}q \quad (12)$$

$$S_1 = h\,\bar{d}_{11}(\psi_1 + w_{,1}) \quad (13)$$

$$S_2 = h\,\bar{d}_{22}(\psi_2 + w_{,2}) \quad (14)$$

Solving these 8 equations leads to the determination of the both stress resultants ($S_1^\bullet, S_2^\bullet, M_{11}^\bullet, M_{22}^\bullet, M_{12}^\bullet$) and the generalized displacements ($\psi_1^\bullet, \psi_2^\bullet, w^\bullet$)

NUMERICAL EXAMPLE

Here we present numerical results. The three-dimensional elasticity solutions of Pagano [1] and Pagano and Hatfield [2] for simply supported rectangular plates under sinusoidal loading are used to assess the present approach.
The following laminated plate problems are considered:

1. Square laminates consisting of 3-, 4-, 5-, 7-, and 9-layers.
2. A 3-ply laminate of rectangular geometry ($b/a = 3$).

All the laminates are with layers of equal thickness and subjected to sinusoidally distributed transverse loading $q_o \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{b}$.

Each layer is a unidireticonal fiber reinforced material with the following properties, which simulate a high modulus Graphite/Epoxy laminate

$$E_L = 172\,GPa\,,\ E_T = 6.9GPa\,, G_{LT} = 3.5GPa\,, G_{TT} = 1.4GPa\,, \nu_{LT} = \nu_{TT} = 0.25$$

Where *L* signifies the direction parallel to the fibers, *T* is the transverse direction and ν_{LT} is the major Poisson's ratio.

All the laminates considered are symmetric with respect the central plane, with fiber orientations alternating between 0^o and 90^o with respect to the x_1-axis, and the 0^o layers are the outer surfaces of the laminate.

The numerical results are summarized in the following sections. Also shown, for comparison purposes, are the results given by:

1. 3-D Elasticity (exact) [4],
2. Higher order theory, HSDT, [5], [8],
3. Finite element based on Layerwise theory, LWM, [6],
4. Three-dimensional finite element [7],
5. First order shear deformation theory, FSDT, [8].

which almost represent all the models used for the analysis of laminated plates. The analogous CPT results, quoted by [1] and [2], are given as well. Although equivalent single layer approach is adopted, a computer program, MathCad, was necessary to find the expressions of the eight unknowns, see Appendix.

Max Central Plane Deflection " $\bar{w}$ " at (a/2, b/2)

Table 1. contains the nondimenionalized deflections for 3-(square), 5-, 7-, and 9-layers laminate. The results were normalized using the following terms [2],

$\bar{w} = \pi^4 Q w / 12 q_o h S^4$ where $Q = 4G_{LT} + [E_L + E_T(1 + 2\upsilon_{TT})]/(1 - \upsilon_{LT}\upsilon_{TL})$ and $S = a/h$

While Table 2. contains results for 3-(rectangular), and 4-layers laminate. The results are normalized using the following formula [1]

$$\bar{w} = 100 E_{22} \frac{w}{\sigma h S^4} ; \sigma = q_o$$

The In-Plane Stress

The results of longitudinal stresses ($\bar{\sigma}_{11}$, $\bar{\sigma}_{22}$ and $\bar{\sigma}_{12}$) is shown in Tables 3., 4., and 5. respectively. For the sake of brevity, results of $\bar{\sigma}_{22}$ is shown only for 3-ply rectangular Laminate. All the results are normalized as follow; [1-2]

$$\bar{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} / q_o S^2$$

The Transverse Normal Stress

In Table 6., the present results of $\bar{\sigma}_{33}$(a/2, b/2, $\bar{X}_3$) for 3-ply square Laminate are compared with the elasticity (exact) results obtained by Senthil and Batra [4]. The results are normalized as follow[4]

$$\bar{\sigma}_{33} = \sigma_{33} / q_o$$

The Transverse Shear Stress

Tables7. and 8. give the nondimenionalized results of $\bar{\sigma}_{13}(0, b/2, 0)$ and $\bar{\sigma}_{23}(a/2, 0, 0)$ respectively. The following quantity is used for normalization; [1-2]

$$\bar{\sigma}_{\alpha 3} = \sigma_{\alpha 3} / \sigma S ; \ \sigma = q_o$$

The Longitudinal Displacement

The results for the in-plane displacement are obtained using [3]

$$u_1 = \psi_1 * X_3$$

Table 9. lists the elasticity results of $\bar{u}_1(a/4,\ b/2,\ \pm h/2)$ obtained by Senthil and Batra [4] as a comparison with the present results. The results are normalized by [4];

$$\bar{u}_1 = \frac{100\,E_T}{q_o S^3}\bar{X}_3\psi_1$$

CONCLUSION

A simplified approach, stress-based, was examined by the authors for laminated plate in cylindrical bending[3] has been extended for simply supported cross-ply symmetric laminates. Average stiffness moduli are considered to characterize the laminate properties, ESL. In order to verify the accuracy of the presented approach, a square plate with 3-, 7-, 5- and 9-layers and a rectangular plate with 3-layers orthotropic laminates of equal thickness has been examined. Aspect ratios (span-to-thickness)of 2, 4(thick), 10, 20(intermediate) and 50, 100(thin) have been considered. Numerical results have been compared with the elasticity (exact) solutions and other models (CPT, FSDT, HSDT, LWM, 3-D FE) as well and the following observations are made for

1. The present approach estimates the central deflection very well compared to the elasticity solutions, ES, almost for all the considered aspect ratios,
2. For the transverse shear stress, the present approach is in a good agreement with the elasticity solutions especially for intermediate aspect ratios and yields better results than both the FSDT and HSDT of Reddy [8],
3. The present approach estimates both the transverse normal stresses and the in-plane displacement, at the surface, very good as compared to the 3-D elasticity (exact) solutions by Senthil and Batra [4],
4. The present approach is consistency in the sense that it yields results with the same level of accuracy for all the multi-layered laminate considered,
5. The assumption of continuity of the longitudinal stress at the laminate interfaces which violates the continuity of the displacements causes significant differences with the elasticity solutions for the thick and intermediate laminate. However, the present approach compares fairly good for aspect ratio greater than 20,

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APPENDIX

Equations of The Generalized Displacements

$$\Psi 1 = \frac{6}{5}\cdot q\cdot \frac{b^{2}\cdot a\cdot\left(\pi^{4}\cdot b^{4}\cdot h^{4}\cdot B1\cdot Q66\cdot d1+\pi^{4}\cdot h^{4}\cdot B1\cdot b^{2}\cdot Q22\cdot a^{2}\cdot d1+12\cdot\pi^{2}\cdot b^{4}\cdot h^{2}\cdot B1\cdot a^{2}\cdot d2\cdot d1+\pi^{4}\cdot h^{4}\cdot B1\cdot a^{2}\right.}{\pi^{3}\cdot h^{3}\cdot\left(2\cdot\pi^{2}\cdot h^{2}\cdot a^{2}\cdot Q66\cdot Q12\cdot b^{4}\cdot d1-\pi^{2}\cdot h^{2}\cdot Q66\cdot Q11\cdot b^{6}\cdot d1-12\cdot Q11\cdot d2\cdot b^{6}\cdot a^{2}\cdot d1+\pi^{2}\right.}$$

$$\frac{\cdot Q66\cdot d2\cdot b^{2}+h^{4}\cdot B1\cdot\pi^{4}\cdot Q22\cdot a^{4}\cdot d2-\pi^{4}\cdot a^{2}\cdot h^{4}\cdot B2\cdot Q12\cdot b^{2}\cdot d1-\pi^{4}\cdot a^{2}\cdot h^{4}\cdot B2\cdot Q66\cdot b^{2}\cdot d1-\pi^{4}\cdot a^{4}\cdot h^{4}\cdot B2\cdot Q12\cdot d2}{\cdot b^{4}\cdot h^{2}\cdot Q12^{2}\cdot a^{2}\cdot d1-\pi^{2}\cdot b^{4}\cdot h^{2}\cdot Q22\cdot Q11\cdot a^{2}\cdot d1-48\cdot Q66\cdot a^{4}\cdot b^{4}\cdot d2\cdot d1-24\cdot a^{4}\cdot Q12\cdot b^{4}\cdot d2\cdot d1+2\cdot\pi^{2}\cdot a^{4}\cdot h^{2}\cdot Q12\cdot}$$

$$\frac{+12\cdot\pi^{2}\cdot a^{4}\cdot h^{2}\cdot B2\cdot d2\cdot b^{2}\cdot d1-\pi^{4}\cdot a^{4}\cdot h^{4}\cdot B2\cdot Q66\cdot d2-10\cdot a^{2}\cdot\pi^{2}\cdot h^{2}\cdot Q66\cdot b^{4}\cdot d1+10\cdot a^{4}\cdot h^{2}\cdot\pi^{2}\cdot Q12\cdot d2\cdot b^{2}-120\cdot a^{4}}{Q66\cdot d2\cdot b^{2}-\pi^{2}\cdot b^{2}\cdot h^{2}\cdot Q22\cdot Q66\cdot a^{4}\cdot d1-\pi^{2}\cdot h^{2}\cdot Q11\cdot Q66\cdot d2\cdot a^{2}\cdot b^{4}-\pi^{2}\cdot h^{2}\cdot Q11\cdot a^{4}\cdot b^{2}\cdot Q22\cdot d2+\pi^{2}\cdot a^{4}\cdot h^{2}\cdot Q12^{2}}$$

$$\frac{\cdot b^{4}\cdot d2\cdot d1-10\cdot a^{4}\cdot\pi^{2}\cdot b^{2}\cdot h^{2}\cdot Q22\cdot d1+10\cdot a^{4}\cdot h^{2}\cdot Q66\cdot\pi^{2}\cdot d2\cdot b^{2})}{\cdot d2\cdot b^{2}-12\cdot b^{2}\cdot Q22\cdot a^{6}\cdot d2\cdot d1-\pi^{2}\cdot h^{2}\cdot Q66\cdot Q22\cdot a^{6}\cdot d2)}$$

$$\Psi 2 = \frac{6}{5}\cdot q\cdot \frac{a^{2}\cdot b\cdot\left(\pi^{4}\cdot h^{4}\cdot B1\cdot b^{4}\cdot Q66\cdot d1+\pi^{4}\cdot h^{4}\cdot B1\cdot Q12\cdot b^{4}\cdot d1-12\cdot\pi^{2}\cdot h^{2}\cdot B1\cdot a^{2}\cdot d2\cdot b^{4}\cdot d1+\pi^{4}\cdot h^{4}\cdot B1\cdot b^{2}\cdot Q66\right.}{\pi^{3}\cdot h^{3}\cdot\left[(-2)\cdot\pi^{2}\cdot h^{2}\cdot a^{2}\cdot Q66\cdot Q12\cdot b^{4}\cdot d1+\pi^{2}\cdot h^{2}\cdot Q66\cdot Q11\cdot b^{6}\cdot d1+12\cdot Q11\cdot d2\cdot b^{6}\cdot a^{2}\cdot d1-\pi^{2}\cdot\right.}$$

$$\frac{\cdot a^{2}\cdot d2+\pi^{4}\cdot h^{4}\cdot B1\cdot a^{2}\cdot Q12\cdot d2\cdot b^{2}-\pi^{4}\cdot h^{4}\cdot B2\cdot Q11\cdot b^{4}\cdot d1-\pi^{4}\cdot a^{2}\cdot h^{4}\cdot B2\cdot b^{2}\cdot Q66\cdot d1-\pi^{4}\cdot a^{2}\cdot h^{4}\cdot B2\cdot Q11\cdot d2\cdot b^{2}-}{b^{4}\cdot h^{2}\cdot Q12^{2}\cdot a^{2}\cdot d1+\pi^{2}\cdot b^{4}\cdot h^{2}\cdot Q22\cdot Q11\cdot a^{2}\cdot d1+48\cdot Q66\cdot a^{4}\cdot d2\cdot b^{4}\cdot d1+24\cdot a^{4}\cdot Q12\cdot d2\cdot b^{4}\cdot d1-2\cdot\pi^{2}\cdot a^{4}\cdot h^{2}\cdot Q12\cdot}$$

$$\frac{12\cdot\pi^{2}\cdot a^{4}\cdot h^{2}\cdot B2\cdot d2\cdot b^{2}\cdot d1-\pi^{4}\cdot a^{4}\cdot h^{4}\cdot B2\cdot Q66\cdot d2-10\cdot a^{2}\cdot\pi^{2}\cdot h^{2}\cdot Q12\cdot b^{4}\cdot d1-10\cdot a^{2}\cdot h^{2}\cdot\pi^{2}\cdot b^{4}\cdot Q66\cdot d1+}{Q66\cdot d2\cdot b^{2}+\pi^{2}\cdot b^{2}\cdot h^{2}\cdot Q22\cdot Q66\cdot a^{4}\cdot d1+\pi^{2}\cdot h^{2}\cdot Q11\cdot Q66\cdot d2\cdot a^{2}\cdot b^{4}+\pi^{2}\cdot h^{2}\cdot Q11\cdot a^{4}\cdot b^{2}\cdot Q22\cdot d2-\pi^{2}\cdot a^{4}}$$

$$\frac{120\cdot a^{4}\cdot d2\cdot b^{4}\cdot d1+10\cdot a^{2}\cdot\pi^{2}\cdot h^{2}\cdot Q11\cdot d2\cdot b^{4}+10\cdot a^{4}\cdot\pi^{2}\cdot h^{2}\cdot b^{2}\cdot Q66\cdot d2)}{\cdot h^{2}\cdot Q12^{2}\cdot b^{2}\cdot d2+12\cdot b^{2}\cdot Q22\cdot a^{6}\cdot d2\cdot d1+\pi^{2}\cdot h^{2}\cdot Q66\cdot Q22\cdot a^{6}\cdot d2]}$$

$$w = \frac{1}{5}\cdot q\cdot a^{2}\cdot b^{2}\cdot\frac{\left[(-6)\cdot\pi^{4}\cdot h^{4}\cdot B1\cdot Q66\cdot b^{4}\cdot d1-72\cdot\pi^{2}\cdot h^{2}\cdot B1\cdot b^{4}\cdot a^{2}\cdot d2\cdot d1-6\cdot\pi^{4}\cdot h^{4}\cdot B1\cdot Q22\cdot a^{2}\cdot b^{2}\right.}{}$$

$$\frac{\cdot d1+6\cdot\pi^{4}\cdot h^{4}\cdot B1\cdot Q66\cdot a^{2}\cdot d2\cdot b^{2}+6\cdot\pi^{4}\cdot a^{2}\cdot h^{4}\cdot B1\cdot Q12\cdot d2\cdot b^{2}+6\cdot\pi^{4}\cdot b^{2}\cdot h^{4}\cdot B2\cdot Q12\cdot a^{2}\cdot d1+6\cdot\pi^{4}\cdot h^{4}\cdot B2\cdot Q66\cdot a^{2}\cdot}{\pi^{4}\cdot h^{3}\cdot\left(2\cdot\pi^{2}\cdot h^{2}\cdot Q66\cdot a^{2}\cdot Q12\cdot b^{4}\cdot d1-\pi^{2}\cdot h^{2}\cdot Q66\cdot Q11\cdot b^{6}\cdot d1-\pi^{2}\cdot h^{2}\cdot Q22\cdot Q11\cdot a^{2}\cdot b^{4}\cdot d1-12\cdot Q11\cdot b^{6}\cdot a^{2}\cdot d2\cdot d1\right.}$$

$$\frac{b^{2}\cdot d1-6\cdot\pi^{4}\cdot h^{4}\cdot B2\cdot Q11\cdot a^{2}\cdot d2\cdot b^{2}-6\cdot\pi^{4}\cdot h^{4}\cdot B2\cdot Q66\cdot a^{4}\cdot d2-72\cdot\pi^{2}\cdot h^{2}\cdot B2\cdot a^{4}\cdot d2\cdot b^{2}\cdot d1-10\cdot\pi^{4}\cdot h^{4}\cdot Q66\cdot a^{2}}{+\pi^{2}\cdot b^{4}\cdot h^{2}\cdot Q12^{2}\cdot a^{2}\cdot d1-48\cdot Q66\cdot a^{4}\cdot d2\cdot b^{4}\cdot d1-24\cdot a^{4}\cdot Q12\cdot d2\cdot b^{4}\cdot d1-\pi^{2}\cdot h^{2}\cdot Q22\cdot Q66\cdot a^{4}\cdot b^{2}\cdot d1-\pi^{2}\cdot h^{2}\cdot}$$

$$\frac{\cdot Q12\cdot b^{2}+5\cdot\pi^{4}\cdot h^{4}\cdot Q66\cdot Q11\cdot b^{4}+60\cdot\pi^{2}\cdot h^{2}\cdot Q11\cdot b^{4}\cdot a^{2}\cdot d2+60\cdot\pi^{2}\cdot h^{2}\cdot Q66\cdot b^{4}\cdot a^{2}\cdot d1+5\cdot\pi^{4}\cdot h^{4}\cdot Q22\cdot Q11\cdot a^{2}\cdot}{Q11\cdot Q66\cdot a^{2}\cdot d2\cdot b^{4}+2\cdot\pi^{2}\cdot a^{4}\cdot h^{2}\cdot Q12\cdot Q66\cdot d2\cdot b^{2}+a^{4}\cdot\pi^{2}\cdot h^{2}\cdot Q12^{2}\cdot b^{2}\cdot d2-\pi^{2}\cdot h^{2}\cdot Q11\cdot Q22\cdot a^{4}\cdot d2\cdot b^{2}-\pi^{2}\cdot h^{2}}$$

$$\frac{b^{2}-5\cdot\pi^{4}\cdot a^{2}\cdot b^{2}\cdot h^{4}\cdot Q12^{2}+720\cdot a^{4}\cdot b^{4}\cdot d2\cdot d1+60\cdot\pi^{2}\cdot h^{2}\cdot Q66\cdot a^{4}\cdot d2\cdot b^{2}+5\cdot\pi^{4}\cdot h^{4}\cdot Q22\cdot Q66\cdot a^{4}}{\cdot Q66\cdot Q22\cdot a^{6}\cdot d2-12\cdot Q22\cdot a^{6}\cdot d2\cdot b^{2}\cdot d1)}$$

$$\frac{+60\cdot\pi^{2}\cdot h^{2}\cdot Q22\cdot a^{4}\cdot b^{2}\cdot d1]}{}$$

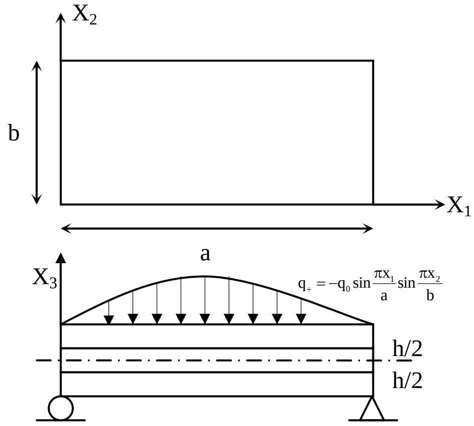

Fig. 1 Orthotropic laminated simply supported plate.

Table 1. Central deflections " $\bar{w}$ " at (a/2, b/2), normalized as Pagano and Hatfield[2]

3-ply *square* Laminate

S	ES[1]	Present	[5]	[6]
2	11.767	11.159	-------	14.731
4	4.491	3.57	------	4.934
5	-------	2.657	3.621	-----
10	1.709	1.435	1.788	1.758
20	1.189	1.128	------	1.196
50	1.031	1.041	------	1.026
100	1.008	1.029	------	1.002
	CPT	1		

[1]ES, Elasticity (Exact); [2]

5-ply Laminate

ES[1]	Present	[6]	[7]	[8]
12.278	11.197	13.862	-----	-----
4.291	3.571	4.331	-----	4.244
-----	-----	-----	-----	-----
1.57	1.433	1.57	-----	1.551
1.145	1.127	1.142	1.232	1.135
1.023	1.041	1.017	1.041	1.015
1.006	1.029	0.999	1.013	0.998
CPT	1			

7-ply Laminate

S	ES[1]	Present
2	12.342	11.194
4	4.153	3.569
10	1.529	1.432
20	1.133	1.126
50	1.021	1.041
100	1.005	1.029
	CPT	1

9-ply Laminate

ES[1]	Present	[8]
12.288	11.193	-----
4.079	3.568	4.088
1.512	1.432	1.505
1.129	1.126	1.123
1.021	1.041	1.016
1.005	1.029	1.000
CPT	1	

Table 2. Central deflections " $\bar{w}$ " at (a/2, b/2), normalized after Pagano [1]

3-ply *rectangular* Laminate

S	ES	Present	[8][2]	[8][3]	[5]	[6]
2	8.17	7.797	-----	-----	-----	-----
4	2.82	2.513	2.64	2.36	-----	-----
10	0.92	0.999	0.86	0.8	0.93	-----
20	0.61	0.78	0.59	0.58	0.61	0.67
50	0.52	0.719	-----	-----	0.52	0.53
100	0.51	0.71	0.51	0.51	0.51	0.51
	CPT	0.503				

4-ply Laminate

ES[1]	present	[8][2]	[8][3]	[7]
1.954	1.502	1.89	1.71	1.87
0.743	0.603	0.72	0.66	0.71
0.517	0.474	0.51	0.49	0.5
0.434	0.438	----	----	0.43
0.4385	0.433	0.43	0.43	0.42

[1]results quoted by [8] [2]HSDT [3]FSDT

Table 3. Normalized longitudinal stress; $\bar{\sigma}_{11}$ at $(a/2, b/2, h/2)$

3-ply *square* Laminate

S	present	ES[2]	ES[1]
2	0.331	1.388	------
4	0.338	0.72	------
5	0.343	-------	0.718
10	0.357	0.559	0.591
20	0.365	0.543	------
50	0.367	0.539	------
100	0.368	0.539	------
	CPT	0.539	

3-ply *rectangular* Laminate

present	ES	[8][2]	[8][3]	[6]
0.566	2.13	----	----	----
0.582	1.14	1.036	0.613	----
------	-----	------	------	----
0.593	0.726	0.693	0.622	----
0.595	0.65	0.641	0.623	0.651
0.595	0.628	----	----	0.64
0.595	0.624	0.624	0.623	0.638
	CPT	0.623		

[2]HSDT [3]FSDT

4-ply Laminate

S	ES[1]	present	[8][2]	[8][3]
4	0.72	0.28	0.67	0.41
10	0.56	0.28	0.55	0.5
20	0.54	0.28	0.54	0.53
100	0.54	0.28	0.54	0.54

[1]results quoted by [8] [2]HSDT [3]FSDT

5-ply Laminate

S	present	ES[2]	[6]
2	0.313	1.332 -0.903	-----
4	0.317	0.685 -0.651	-----
5	0319	---------	----
10	0.327	0.545	---
20	0.332	0.539	0.548
50	0.334	0.539	0.55
100	0.334	0.539	0.551
	CPT	0.539	

7-ply Laminate

S	present	ES[2]
2	0.306	1.284 -0.88
4	0.307	0.679 -0.645
10	0.314	0.548
20	0.318	0.539
50	0.319	0.539
100	0.319	0.539
	CPT	0.539

9-ply Laminate

S	present	ES[2]
2	0.301	1.26 -0.866
4	0.302	0.684 -0.649
10	0.307	0.551
20	0.31	0.541
50	0.311	0.539
100	0.311	0.539
	CPT	0.539

Table 4. Normalized longitudinal stress; $\bar{\sigma}_{22}$ at (a/2, b/2, h/6)

S	ES[1]	Present	[8][2]	[8][3]
2	0.23	0.066	-----	-----
4	0.119	0.036	0.103	0.093
10	0.044	0.019	0.04	0.038
20	0.0299	0.016	0.0289	0.0283
100	0.0253	0.015	0.0253	0.0253
	CPT	0.0252	[2]HSDT	[3]FSDT

Table 5. Normalized longitudinal stress; $\bar{\sigma}_{12}$ at (0, 0, h/2)

3-ply *square* Laminate

S	present	ES[2]	[4]
2	.019	-.0863 .0673	------
4	.021	-.0467 .0458	------
5	.021	------	.0403
10	.021	.0275	.0266
20	.021	.023	------
50	.021	.0216	-----
100	.021	.0214	-----
	CPT	.0213	

3-ply *rectangular* Laminate

S	present	ES[1]	[8][2]	[8][3]	[6]
2	.03	.0548	----	----	----
4	.02	.0281	.0263	.0205	----
10	.013	.0123	.0115	.0105	----
20	.012	.0093	.0091	.0088	.0099
50	.012	.0084	----	----	.0087
100	.012	.0083	.0083	.0083	.0085
	CPT	.0083			

[2]HODT [3]FSDT

4-ply Laminate

S	[8][1]	present	[8][2]	[8][3]
4	0.047	0.021	0.044	0.301
10	0.028	0.021	0.027	0.024
20	0.023	0.021	0.023	0.022
100	0.022	0.021	0.021	0.021

[1]results quoted by [8] [2]HSDT [3]FSDT

5-ply Laminate

S	present	ES[2]	[6]
2	0.019	0.0634	----
4	0.021	0.0384	----
10	0.021	0.0246	----
20	0.021	0.0222	0.0229
50	0.021	0.0214	0.0220
100	0.021	0.0213	0.0218
	CPT	0.0213	

7-ply Laminate

S	present	ES[2]
2	0.018	0.0579
4	0.021	0.0347
10	0.021	0.0238
20	0.021	0.0219
50	0.021	0.0214
100	0.021	0.0213
	CPT	0.0213

9-ply Laminate

present	ES[2]
0.018	0.0534
0.021	0.0328
0.021	0.0235
0.021	0.0218
0.021	0.0214
0.021	0.0213
CPT	0.0213

Table 6. Nondimenionalized transverse normal stress ; $\bar{\sigma}_{33}$ at $(a/2,\, b/2,\, \bar{X}_3)$

$\bar{X}_3$	S	Present	[4]
$\frac{h}{6}$	5	0.741	0.726
	10	0.741	0.74
0	5	0.5	0.496
	10	0.5	0.5

Table 7. Normalized transverse shear stress ; $\bar{\sigma}_{13}$ at $(0,\, b/2,\, 0)$

3-ply *square* Laminate

S	present	ES[2]	[4]
2	0.275	0.153	------
4	0.282	0.219	------
5	0.286	------	0.2653
10	0.297	0.301	0.3301
20	0.303	0.328	------
50	0.305	0.337	------
100	0.306	0.339	------
	CPT	0.339	

3-ply *rectangular* Laminate

S	present	ES[1]	[8][2]	[8][3]
2	0.452	0.257	----	----
4	0.463	0.351	0.273	0.1879
10	0.469	0.42	0.286	0.1894
20	0.47	0.434	0.288	0.1896
50	0.471	0.439	-----	-----
100	0.471	0.439	0.289	0.1897
	CPT	0.44	[2]HSDT	[3]FSDT

4-ply Laminate

S	ES[1]	Present	[8][2]	[8][3]	[7]
4	0.291	0.239	0.206	0.14	0.229
10	0.301	0.239	0.264	0.167	0.315
20	0.328	0.239	0.283	0.175	0.343
100	0.337	0.239	0.29	0.178	0.355

[1]results quoted by [8]   [2]HSDT   [3]FSDT

5-ply Laminate

S	Present	ES[2]	[7]
2	0.261	0.227	----
4	0.265	0.238	0.250
10	0.274	0.258	0.278
20	0.277	0.268	0.290
50	0.279	0.271	0.298
100	0.279	0.272	0.308
	CPT	0.272	

7-ply Laminate

S	present	ES[2]
2	0.254	0.178
4	0.257	0.219
10	0.264	0.255
20	0.266	0.267
50	0.267	0.271
100	0.267	0.272
	CPT	0.272

9-ply Laminate

S	present	ES[2]	[7]
2	0.251	0.204	---
4	0.253	0.223	0.229
10	0.258	0.247	0.256
20	0.26	0.255	0.265
50	0.261	0.258	0.269
100	0.261	0.259	0.270
	CPT	0.259	

Table 8. Normalized transverse shear stress ; $\bar{\sigma}_{23}$ at $(a/2, 0, 0)$

3-ply *square* Laminate

S	present	ES[2]	[4]
2	0.203	0.295	------
4	0.195	0.292	------
5	0.191	------	0.1911
10	0.18	0.196	0.1228
20	0.174	0.156	------
50	0.172	0.141	------
100	0.172	0.139	------
	CPT	0.138	

3-ply *rectangular* Laminate

S	present	ES[1]	[8][2]	[8][3]
2	0.075	0.0668	----	----
4	0.044	0.0334	0.0348	0.0308
5	0.037	------	----	----
10	0.025	0.0152	0.017	0.0159
20	0.022	0.0119	0.0139	0.0135
50	0.021	0.0110	-----	----
100	0.021	0.0108	0.0129	0.0127
	CPT	0.0108	[2]HSDT	[3]FSDT

4-ply Laminate (0/90/90/0); b=a

S	ES[1]	Present	[8][2]	[8][3]	[7]
4	0.292	0.239	0.239	0.1963	0.299
10	0.196	0.239	0.153	0.129	0.198
20	0.156	0.239	0.123	0.109	0.157
100	0.141	0.239	0.112	0.101	0.144

[1]results quoted by [8] [2]HSDT [3]FSDT

5-ply Laminate

S	present	ES[2]	[7]
2	0.217	0.186 0.268	---
4	0.212	0.229 0.233	0.257
10	0.204	0.223	0.247
20	0.2	0.212	0.233
50	0.199	0.206	0.231
100	0.199	0.205	0.229
	CPT	0.205	

7-ply Laminate

S	present	ES[2]
2	0.223	0.238
4	0.22	0.236
10	0.214	0.219
20	0.211	0.21
50	0.21	0.206
100	0.21	0.205
	CPT	0.205

9-ply Laminate

S	present	ES[2]	[7]
2	0.227	0.194 0.211	------
4	0.224	0.223 0.225	0.234
10	0.219	0.226	0.236
20	0.217	0.221	0.231
50	0.217	0.219	0.229
100	0.216	0.219	0.230
	CPT	0.205	

Table 9. Normalized in-plane displacement; $\bar{u}_1$ at $(a/4, b/2, \pm h/2)$

3-ply *square* Laminate

	S = 5		S = 10	
	Present	[4]	Present	[4]
h/2	0.6164	0.62	0.6532	-0.52
-h/2	0.6164	0.614	0.6532	0.522