A Closed Analytic Form for the Precision Matrix Of the Second Order Moving Average Processes

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Abstract

A difficult and challenging problem, not only to Bayesian but also to those interested in maximum likelihood estimation, is to express the precision matrix of the second order moving average processes in closed analytic from in terms of the parameters directly. The main objective of this article is to develop a convenient technique to obtain such closed analytic form. The proposed technique is based on approximating the covariance structure of the first two observations only. Then, a homogeneous difference equation of the second order is developed for the elements of the inverse of the transformation matrix and an exact solution for the difference equation is given in a closed and easily computed form.

Key words: Bayesian analysis, moving average processes, precision matrix.

1. Introduction

For well – understood reasons, most of publications to analyze autoregressive moving average, denoted by ARMA(p, q), processes using the Bayesian approach focus on the analysis of pure autoregressive, denoted by AR(p), processes and pay little attention to pure moving average, denoted by MA(q), processes. This void in the Bayesian literature due to the complexity of the likelihood function of MA (q) processes because there is not a closed mathematical form for the precision matrix in terms of the parameters directly. The main difficulty of MA (q) processes is that the previous errors ε_{t-j} are nonlinear functions in the model's coefficients. Therefore, the joint and marginal posterior distributions of the parameters are not standard; thus, the statistical inferences about the parameters should be done numerically. A simple mathematical form for the likelihood function is needed and this has not be done, so far, for MA (q), with $q \geq 2$, processes. Too many calculations are required in order to compute the joint and marginal distributions for any point in the parameter space. As the sample size increases, computation of the posterior distributions becomes increasingly laborious even for high- speed computers.

An efficient Bayesian analysis is not possible without finding a way to represent the likelihood function in such a way to produce analyticly tractable posterior distributions. Whittle (1951, chapter 4) presented an approximation to the likelihood function of ARMA (p, q) processes. Although Whittle's approximation reduces the number of calculations needed to characterize the posterior distributions, it requires the validity of the invertibility assumptions. Furthermore, the theoretical and numerical properties of the approximation have not been thoroughly studied. Wise and Siddique (1958) obtained the precision matrix of stationary AR (p) processes in a closed form. Box and Jenkins (1970) and Wilson (1973) used the idea of setting the initial values of the errors to zero in order to approximate the likelihood function. However, they did not provide a closed analytic form for the precision matrix. Newbold (1973) was concerned with the Bayesian estimation of the coefficients of the transfer –noise models. He used a nonlinear least squares approximation to show that the Bayesian inferences about the parameters can be done using Student's t distribution if Jeffreys' prior is used. Newbold (1974) derived an exact form for the likelihood function. However, he did not give a closed analytic form for the precision matrix. McLeod (1977) proposed replacing the determinant of the covariance matrix of ARMA processes by its asymptotic limit in order to approximate the likelihood function. However, his approach does not avoid the problem of computing the inverse of the covariance matrix. Phadke and Kedem(1978) presented three different techniques to obtain the exact likelihood function of MA(q) processes. However, none of these techniques can lead to a closed analytic form for the likelihood function in terms of the parameters directly. Their work has been extended to ARMA processes by Ansely (1979).

Another approach to approximate the likelihood function of MA (q) processes is used by Zellner and Reynolds (1978) and Hilmer and Tiao (1979). The idea of setting the initial values of the residuals to zero has been used in their work. Zellner and Reynolds (1978) have shown that the inferences about the coefficients of ARMA processes can approximately done using t distribution by replacing the exact coefficients values in the covariance matrix by initial consistent estimates. Their technique is equivalent to expand the errors sum of squares as a quadratic function in the coefficients around their nonlinear least squares estimates using Taylor's expansion.

By introducing a new spectral parameterization of time series data Shore (1980) has shown that Whittle's approximation of the likelihood function can be used in Bayesian analysis of ARMA processes. He derived a conjugate prior distribution for his approximation. He has shown also that the approximate precision matrix of MA(q) process is the covariance matrix of pure autoregressive process. Of special interest are the first and second moving average processes. With respect to the exact analysis of the first order moving average process, Shaarawy and Broemeling(1984) were able to derive the exact posterior distributions in closed analytic forms. Regarding the second and higher order of moving average processes, Broemeling and Shaarawy (1988) developed an approximate methodology to estimate the parameters using t distribution. Unlike the techniques of Newbold(1973) and Zellner and Reynolds (1978), Broemeling and Shaarawy (1988) proposed replacing the previous errors by their nonlinear least squares estimates. Their approach has been extended to bilinear models by Chen(1992). Some other investigations which attempt to approximate the likelihood function and posterior distributions may be found in Shaman (1975), Ljung and Box (1976) and Nicholls and Hall (1979). Shaarawy (1992) proved that it is possible to express the determinant of the covariance matrix of MA(2) process in a closed and analytic form. However, his approach does not avoid the problem of computing the precision matrix. For such process, it has not been found, so far, how to express the precision matrix in a closed analytic form in terms of the parameters directly. This causes a difficult and challenging problem, not only to Bayesian analysts but also to those interested in the maximum likelihood estimation.

In this article, a closed and analytic form for the precision matrix of MA(2) processes will be developed by setting the initial values of the errors to zero. This gives an approximate covariance structure for the first two observations (y_1, y_2), while most of the other approximations, outlined above, give an approximate covariance structure for all observations $(y_1, y_2, \ldots, y_n)$. The proposed approach is based on developing a homogeneous difference equation of the second order for the elements of the inverse of the transformation matrix and giving an exact solution for the difference equation in a closed and easily computed form.

2. The Second Order Moving Average Processes

Let {t} be a sequence of integers, θ_1 and θ_2 be real constants, $\{\varepsilon(t)\}$ is a sequence of $k\times 1$ independent and normally distributed unobservable random vectors with zero mean and a $k\times k$ unknown precision matrix, and y_t be a realization of the process {y(t)} at time t. Thus, the moving average model of the second order is defined for n successive time points by

$$y_t = \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2}, \quad t = 1, 2, \ldots, n \qquad (2.1)$$

In matrix notation

$$Y = A^*(\theta) = A^*(\theta_1, \theta_2)\,\varepsilon^*$$

Where

$$Y = (y_1, y_2, \cdots, y_n)', \varepsilon^* = (\varepsilon_{-1}, \varepsilon_0, \cdots, \varepsilon_n)',$$

and

$$A^*_{n\times n+2}(\theta_1, \theta_2) = \begin{bmatrix} -\theta_2 & -\theta_1 & 1 & 0 & 0 & 0 & \ldots\ldots\ldots 0 & 0 & 0 \\ 0 & -\theta_2 & -\theta_1 & 1 & 0 & 0 & \ldots\ldots\ldots 0 & 0 & 0 \\ 0 & 0 & -\theta_2 & -\theta_1 & 1 & 0 & \ldots\ldots\ldots 0 & 0 & 0 \\ & & & & & & & & \\ 0 & 0 & 0 & 0 \ldots\ldots\ldots\ldots & & & -\theta_2 & -\theta_1 & 1 \end{bmatrix}$$

Assume that

$$(\varepsilon_{-1} \quad \varepsilon_0)' = E(\varepsilon_{-1} \quad \varepsilon_0)' = (0 \quad 0)'$$

Thus, the model (2.1) can be written as

$$Y = A(\theta_1, \theta_2)\varepsilon \tag{2.2}$$

Where

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)',$$

And

$$A_{n\times n}(\theta_1,\theta_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots\dots\dots 0 & 0 & 0 \\ -\theta_1 & 1 & 0 & 0 & 0 & \dots\dots\dots 0 & 0 & 0 \\ -\theta_2 & -\theta_1 & 1 & 0 & 0 & \dots\dots\dots 0 & 0 & 0 \\ 0 & -\theta_2 & -\theta_1 & 1 & 0 & \dots\dots\dots 0 & 0 & 0 \\ & & & & & & & \\ & & & & & & & \\ 0 & 0 & 0 & 0 \dots\dots\dots & & -\theta_2 & -\theta_1 & 1 \end{bmatrix}$$

ε is assumed to be normally distributed with zero mean vector and covariance matrix $\sigma^2 I_n = \frac{1}{\tau} I_n, \tau = \frac{1}{\sigma^2}$. Thus Y is normally distributed with zero mean vector and covariance matrix $\sigma^2 A(\theta)A'(\theta)$. We can write the density of Y given the parameters θ_1, θ_2 and τ as

$$f(y \mid \theta_1, \theta_2, \tau) = (2\pi)^{-\frac{n}{2}} \left|\tau\left(A(\theta)A'(\theta)\right)^{-1}\right|^{\frac{1}{2}} \exp\frac{-\tau}{2}\left(y'\left(A(\theta)A'(\theta)\right)^{-1} y\right), y \in R^n, \theta \in R^2, \tau > 0$$

Clearly, $\left|A(\theta)A'(\theta)\right| = 1$.

Thus, the likelihood function of the parameters θ_1, θ_2, and τ can be written as

$$L(\theta, \tau \mid y) \propto (\tau)^{\frac{n}{2}} \exp\left[\frac{-\tau}{2}\left(y' A^{-1'}(\theta) A^{-1}(\theta) y\right)\right], y \in R^n, \theta \in R^2, \tau > 0 \tag{2.3}$$

The form of the likelihood function (2.3) is useless without finding a way to express the matrix $A^{-1}(\theta)$ in terms of the parameters θ_1 and θ_2 directly.

3. An Algorithm to Get $A^{-1}(\theta_1,\theta_2)$

Let D_k^n be a $n\times n$ matrix defined by

$$D_k^n=\left(d_{ij}^k\right)=\begin{cases}1 & ,i-j=k\\ 0 & ,i-j\neq k\end{cases}\quad , \; i,j=1,2,\dots,n\;;\;k=1,2,\dots,n-1$$

When k = 1,

$$D_1^n=\left(d_{ij}^1\right)=\begin{cases}1 & ,i-j=1\\ 0 & ,i-j\neq 1\end{cases}\qquad i,j=1,2,\dots,n$$

$$=\begin{bmatrix}0&0&0&&&0\\1&0&0&&&0\\0&1&0&&&0\\&&&&&\\0&0&0&&0&0\\0&0&0&0&1&0\end{bmatrix}$$

When k = 2,

$$D_2^n=\left(d_{ij}^2\right)=\begin{cases}1 & ,i-j=2\\ 0 & ,i-j\neq 2\end{cases}\qquad i,j=1,2,\dots,n$$

$$\begin{bmatrix}0&0&0&&&0\\0&0&0&&&0\\1&0&0&&&0\\0&1&0&&&0\\&&&&&\\0&0&0&&0&0\\0&0&0&1&0&0\end{bmatrix}$$

When k = n-1, $D^{n}_{n-1} = \left(d^{n-1}_{ij}\right) = \begin{cases} 1 & , i-j = n-1 \\ 0 & , i-j \neq n-1 \end{cases} \qquad i, j = 1, 2, \ldots, n$

$$\begin{bmatrix} 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & & & 0 \\ & & & & & \\ 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, we can write $A(\theta_1, \theta_2)$ as

$$A(\theta_1, \theta_2) = I_n - \theta_1 D^{n}_{1} - \theta_2 D^{n}_{2} = I_n - \sum_{k=1}^{2} \theta_k D^{n}_{k} = I_n - L, L = \sum_{k=1}^{2} \theta_k D^{n}_{k}$$

Theorem 1:

Let $A(\theta_1, \theta_2)$ and D^{n}_{k} be defined as above, then $A^{-1}(\theta_1, \theta_2)$ can be written as

$$A^{-1}(\theta_1, \theta_2) = I_n + \sum_{i=1}^{n-1} \alpha_i D^{n}_{i},$$

Where $(\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_{n-1}) = \alpha \in R^{n-1}$

Proof:

It is easy to see that

$$D^{n}_{i}\ D^{n}_{j} = \begin{cases} \{D^{n}_{i+j} & , i+j \leq n-1 \\ 0 & , otherwise \end{cases} \qquad i, j = 1, 2, \ldots, n$$

(3.1)

$$L^2 = \left(\theta_1 D_1^n + \theta_2 D_2^n\right)^2 = \theta_1^2 D_2^n + 2\theta_1\theta_2 D_3^n + \theta_2^2 D_4^n$$

$$L^3 = \left(\theta_1 D_1^n + \theta_2 D_2^n\right)^3 = \theta_1^3 D_3^n + 3\theta_1^2\theta_2 D_4^n + 3\theta_1\theta_2^2 D_5^n + \theta_2^3 D_6^n$$

$$L^4 = \left(\theta_1 D_1^n + \theta_2 D_2^n\right)^4 = \theta_1^4 D_4^n + 4\theta_1^3\theta_2 D_5^n + 6\theta_1^2\theta_2^2 D_6^n + 4\theta_1\theta_2^3 D_7^n + \theta_2^4 D_8^n$$

Similarly,

$$L^s = b_o D_s^n + b_1 D_{s+1}^n + b_2 D_{s+2}^n + ... + b_s D_{2s}^n, \qquad s \le n-1 \tag{3.2}$$

Where $b = \left(b_0 \quad b_1 \quad \dots \quad \alpha_{sn-1}\right)$ is a vector of constants

$$L^n = \left(\theta_1 D_1^n + \theta_2 D_2^n\right)^n = C_0^n \theta_1^n \left(D_1^n\right)^n + C_1^n \theta_1^{n-1} \left(D_1^n\right)^{n-1} D_2^n \theta_2 + ... + C_n^n \theta_2^n \left(D_2^n\right)^n = 0 \quad \text{from (3.1)}$$

$$L^{n+1} = LL^n = 0$$

Thus,

$$L^s = 0\,, \quad s \ge n$$

$$\left[I_n - L\right]\left[I_n + L + L^2 + ... + L^{n-1}\right] = I_n + L + L^2 + ... + L^{n-1} - L - L^2 - L^3 - ...... - L^n = I_n - L^n = I_n$$

Thus

$$A^{-1}(\theta) = I_n + L + L^2 + ... + L^{n-1}$$

From (3.2), there exists $\alpha^{(1)} \in R^{n-1}, \alpha^{(2)} \in R^{n-1}, ..., \alpha^{(n-1)} \in R^{n-1}$, such that

$$L^j = \sum \alpha_i^{(j)} D_i^n\,, \quad j = 1, 2, ..., n-1$$

Where

$$\alpha^{(j)} = \left(\alpha_1^{(j)} \quad \alpha_2^{(j)} \quad \dots \quad \alpha_{n-1}^{(j)}\right)$$

Thus,

$$A^{-1}(\theta) = I + \sum_{j=1}^{n-1}\sum_{i=1}^{n-1}\alpha_i^{(j)} D_i^n = I + \sum_{i=1}^{n-1} D_i^n \left(\sum_{j=1}^{n-1}\alpha_i^{(j)}\right)$$

$$= I + \sum_{i=1}^{n-1}\alpha_i D_i^n, \qquad \alpha_i = \sum_{j=1}^{n-1}\alpha_i^{(j)}$$

This completes the proof.

We can write $A^{-1}(\theta)$ in the following form

$$A^{-1}(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & & & 0 \\ \alpha_1 & 1 & 0 & 0 & & & 0 \\ \alpha_2 & \alpha_1 & 1 & 0 & & & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 & & & 0 \\ & & & & & & \\ \alpha_{n-1} & \alpha_{n-2} & & & \alpha_2 & \alpha_1 & 1 \end{bmatrix}$$

Let $a_1 = 1, a_2 = \alpha_1, a_3 = \alpha_2, \ldots, a_n = \alpha_{n-1}$, then we can rewrite $A^{-1}(\theta)$ as

$$A^{-1}(\theta) = \begin{bmatrix} a_1 & & & & & & \\ a_2 & a_1 & & & & & \\ a_3 & a_2 & a_1 & & & & \\ a_4 & a_3 & a_2 & a_1 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_n & a_{n-1} & \cdot & \cdot & \cdot & a_3 \quad a_2 & a_1 \end{bmatrix}$$

Theorem 2

Let $A(\theta_1, \theta_2)$ be defined as above, then the elements of $A^{-1}(\theta)$ can be written as

$$a_k = \begin{cases} \sum_{i=0}^{\frac{k-1}{2}} C_i^{k-i-1}\, \theta_2^i\, \theta_1^{k-2i-1}, & k \text{ is odd} \\ \sum_{i=0}^{\frac{k-2}{2}} C_i^{k-i-1}\, \theta_2^i\, \theta_1^{k-2i-1}, & k \text{ is even} \end{cases}$$

Proof

Multiplying the first row of the matrix $A^{-1}(\theta)$ by the first column of $A(\theta)$, we get $a_1 = 1$.

Multiplying the second row of the matrix $A^{-1}(\theta)$ by the first column of $A(\theta)$, we get $a_2 = \theta_1$.

A necessary condition for $A^{-1}(\theta)$ is

$$a'_{k+2}\, a_1^* = 0 \quad , k = 1, 2, \ldots, n-2 \tag{3.3}$$

Where a'_{k+2} is the (k+2)th row of the inverse $A^{-1}(\theta)$ and a_1^* is the first column of $A(\theta)$.

From (3.3), we get

$$a_{k+2} - \theta_1 a_{k+1} - \theta_2 a_k = 0, \qquad k = 1,\ 2,\ \ldots,\ n-2 \tag{3.4}$$

The equation (3.4) is a homogeneous difference equation of the second order. In order to solve this equation, we multiply both sides by ϕ^k and sum over all values of k. If we do that, we get the following

$$\sum_{k=1}^{\infty} \phi^k a_{k+2} = \theta_1 \sum_{k=1}^{\infty} \phi^k a_{k+1} + \theta_2 \sum_{k=1}^{\infty} \phi^k a_k$$

That is,

$$\frac{1}{\phi^2} \sum_{k=1}^{\infty} \phi^{k+2} a_{k+2} = \frac{\theta_1}{\phi} \sum_{k=1}^{\infty} \phi^{k+1} a_{k+1} + \theta_2 \sum_{k=1}^{\infty} \phi^k a_k \tag{3.5}$$

Define $p(\phi) = \sum_{k=1}^{\infty} \phi^k a_k$. From (3.5), we get

$$\frac{1}{\phi^2}\left[p(\phi) - a_1\phi - a_2\phi^2\right] = \frac{\theta_1}{\phi}\left[p(\phi) - a_1\phi\right] + \theta_2 p(\phi)$$

$$p(\phi)\left[1-\left(\phi\theta_1+\phi^2\theta_2\right)\right]=\phi a_1+\phi^2 a_2-\phi^2\theta_1 a_1 \quad , \tag{3.6}$$

Where $a_1 = 1$ and $a_2 = \theta_1$.

Thus, we can write (3.6) as

$$p(\phi)\left[1-\left(\phi\theta_1+\phi^2\theta_2\right)\right]=\phi$$

$$p(\phi)=\phi\left[1-\left(\phi\theta_1+\phi^2\theta_2\right)\right]^{-1} \ , \ \left|\phi\theta_1+\phi^2\theta_2\right|<1$$

That is,

$$p(\varphi)=\varphi\left[1+\left(\varphi\theta_1+\varphi^2\theta_2\right)+\left(\varphi\theta_1+\varphi^2\theta_2\right)^2+\dots\dots\right]$$

The coefficient of ϕ^k is a_k in the expansion of $p(\phi)$. It is clear that the coefficient of ϕ is $a_1 = 1$ and the coefficient of ϕ^2 is $a_2 = \theta_1$. Similarly,

$$a_3 = \theta_2 + \theta_1^2$$

$$a_4 = 2\theta_1\theta_2 + \theta_1^3$$

$$a_5 = 3\theta_1^2\theta_2 + \theta_2^2 + \theta_1^4$$

$$a_6 = 4\theta_1^3\theta_2 + 3\theta_1\theta_2^2 + \theta_1^5$$

$$a_7 = 5\theta_1^4\theta_2 + 6\theta_1^2\theta_2^2 + \theta_2^3 + \theta_1^6$$

In general, we have

$$a_k = \begin{cases} \sum_{i=0}^{\frac{k-1}{2}} C_i^{k-i-1}\,\theta_2^i\,\theta_1^{k-2i-1}, & k \text{ is odd} \\ \sum_{i=0}^{\frac{k-2}{2}} C_i^{k-i-1}\,\theta_2^i\,\theta_1^{k-2i-1}, & k \text{ is even} \end{cases} \tag{3.7}$$

To prove (3.7), we use the mathematical induction as follows:

i. Let k be an odd number

At k=1, we find that $a_1 = 1$.

At k=3, we find that $a_3 = \theta_2 + \theta_1^2$

Thus (3.7) is correct if k=1 or k=3

Assume that the relationship (3.7) is correct for k odd. Then

$$a_{k+2} = \theta_1 a_{k+1} + \theta_2 a_k$$

$$a_{k+2} = \sum_{i=0}^{\frac{(k+1)-2}{2}} C_i^{k-i}\ \theta_2^i\ \theta_1^{k-2i+1} + \sum_{i=0}^{\frac{k-1}{2}} C_i^{k-i-1}\ \theta_2^{i+1}\ \theta_1^{k-2i-1}$$

$$= \left[C_0^k\ \theta_1^{k+1} + C_1^{k-1}\ \theta_2\ \theta_1^{k-1} + \ldots + C_r^{k-r}\ \theta_2^r\ \theta_1^{k-2r+1} + \ldots + C_{\frac{k-1}{2}}^{k-\left(\frac{k-1}{2}\right)}\ \theta_2^{\frac{k-1}{2}}\ \theta_1^{k-2\left(\frac{k-1}{2}\right)+1} \right]$$

$$+ \left[C_0^{k-1}\ \theta_2 \theta_1^{k-1} + C_1^k\ \theta_2^2\ \theta_1^{k-3} + \ldots + C_{r-1}^{k-r}\ \theta_2^r\ \theta_1^{k-2r+1} + \ldots + C_{\frac{k-1}{2}}^{k-\left(\frac{k-1}{2}\right)-1}\ \theta_2^{\frac{k-1}{2}+1}\ \theta_1^{k-2\left(\frac{k-1}{2}\right)+1} \right]$$

That is,

$$a_{k+2} = \sum_{i=1}^{\frac{k-1}{2}} (C_i^{k-i} + C_{i-1}^{k-i})\ \theta_2^i\ \theta_1^{k-2i+1} + C_0^k\ \theta_1^{k+1} + C_{\frac{k-1}{2}}^{\frac{k-1}{2}}\ \theta_2^{\frac{k+1}{2}}\ \theta_1^2$$

$$= \sum_{i=1}^{\frac{k-1}{2}} C_i^{k-i+1}\ \theta_2^i\ \theta_1^{k-2i+1} + \theta_1^{k+1} + \theta_2^{\frac{k+1}{2}}\ \theta_1^2$$

$$= \sum_{i=0}^{\frac{k+1}{2}} C_i^{k-i+1}\ \theta_2^i\ \theta_1^{k-2i+1}$$

Thus, the formula (3.7) is correct for all odd values of k.

ii. When k is even, the same approach can be used to get the formula in (3.7).

This completes the proof.

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