Topp – Leone Pareto Type I Distribution

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Abstract

This paper is concerned with introducing Topp – Leone Pareto Type I distribution. The non-central moments, central moments, the skewness and the kurtosis measures are obtained. Reliability function, hazard function, reversed hazard function, quantile, mode and median are derived. Distribution of the smallest and the largest order statistics are obtained. The lower record and the upper record values are obtained. Parameters estimation using maximum likelihood method is obtained. To illustrate the proposed model, numerical example is given.

Key words: *Lower record and upper record values, Newton Raphson, Order Statistics, Maximum likelihood estimation, Topp – Leone Pareto Type I distribution.*

1. Introduction

Pareto distribution is used in describing social, scientific, and geophysical phenomena in the society. Pareto observed that 80% of the country's wealth was concentrated in the hands of only 20% of the population. The theory is now applied in many disciplines such as income. The Pareto family of distributions is often applied in economics, finance and actuarial science for example, income, loss or claim, severity estimation and fitting from data. Pareto distribution is useful for modeling and predicting tools with variety of socioeconomic contexts. Pareto Type-I distribution is first introduced by Pareto in (1965) originally published in 1896.

Pourdarvish *et al.* (2015) pointed out that the Topp-Leone distribution was first introduced by Topp and Leone (1955) as a probability distribution with bounded support which is useful for modelling life-time phenomena. They proposed a generalization of the Topp - Leone distribution referred to as the exponentiated Topp-Leone distribution. They studied many aspects of the new model like hazard rate function, the moments and the order statistics. They discussed maximum likelihood estimation of the model parameters. In application to a real data set, they showed that the exponentiated Topp-Leone model can be used quite effectively in analyzing data.

Al – Shomani *et al.* (2016) proposed a new family of distributions; the Topp-Leone family of distributions. They gave general expression for density and distribution functions of the new family. Expression for moments and hazard rate were given. They gave an example of the proposed family.

Bodhsuwan (2016) commenced the Topp – Leone Gumbel (TLGB) distribution. The moment generating function is also provided. Maximum likelihood and L – Moments methods are applied to estimate parameters of the TLGB distribution. Reliability properties, survival and hazard function of the proposed distribution are derived. He evaluated the effectiveness of the Gumble and TLGB distributions using real data sets.

Sangsanit and Bodhisuwan (2016) introduced a new framework for generating lifetime distributions, called the Topp-Leone generated (TLG) family of distributions. Some various properties of the TLG distribution are discussed, e.g., survival function, hazard function, moments and generating function. In addition, the TLG family improved fitted results and tail behavior of existing distributions. They presented the Topp-Leone generalized exponential (TLGE) distribution as an example of the TLG distribution. Some graphical representations related to the probability density function and hazard function of the TLGE distribution are provided. In application study, the goodness of fit test based on the TLGE, the generalized exponential (GE), and exponentiated generalized exponential (EGE) distributions are compared. The results emphasize that the TLGE distribution can be considered as a competitive distribution for the GE and EGE distributions.

Abbas *et al.* (2017) introduced a new three parameter life model called the Topp-Leone Inverse Weibull distribution. They provided comprehensive result of the mathematical characteristic, including moments, quantile function, random number generator, survival function, hazard rate function, and mode. Distributional properties of order statistics are analyzed. The parameters of the proposed model are estimated by the method of maximum likelihood. Simulation study is performed to investigate the performance of the maximum likelihood estimators. To assess the flexibility, empirical results of new model are obtained by modeling two real data sets.

Aryuyuen (2018) proposed a new framework for generating lifetime distributions, which is called the Topp-Leone Exponentiated Power Lindley (TL-EPL) distribution. Sub models of the TL-EPL distribution, such as the Topp-Leone Power Lindley, Topp-Leone Generalized Lindley, and Topp-Leone Lindley, are introduced. Some statistical characteristics of the distributions are investigated (i.e., mean, variance, and functions of survival, hazard, and quantile). The maximum likelihood estimation is used to estimate the parameters of each distribution. Some real data sets are fitted in order to illustrate the usefulness of the proposed distribution.

This paper is divided into seven sections. The first section contains the introduction. The second section is devoted to Topp – Leone generated family of distributions. Section three contains Topp – Leone Pareto Type I distribution $(TL-PI)$. Section four contains some properties of the $(TL-PI)$. Section five is devoted to the estimation of the parameters. Section sex contains numerical study. Section seven is the conclusion remarks.

2. Topp – Leone Generated Family of Distributions

If a random variable X is distributed as the Topp – Leone (TL) and bounded on $[0,1]$. Let X be a continuous random variable with $cdf\ G(x)$. The Topp – Leone generated (TLG) distribution has cdf written by

$$F_{TLG}(x) = (G(x))^{\alpha}(2-G(x))^{\alpha}, x > 0, \alpha > 0. \quad (1)$$

By differentiating the corresponding pdf is

$$f_{TLG}(x) = 2\alpha g(x)\big(1-G(x)\big)\big(G(x)\big)^{\alpha-1}\big(2-G(x)\big)^{\alpha-1}, x > 0, \alpha > 0 \quad (2)$$

The reliability function (rf) and hazard reliability function (hrf) of this family are given respectively by

$$R_{TLG}(x) = 1 - (G(x))^{\alpha}(2 - G(x))^{\alpha}, \quad x > 0, \alpha > 0 \tag{3}$$

and

$$hf_{TLG}(x) = \frac{2\alpha g(x)(1 - G(x))(G(x))^{\alpha-1}(2 - G(x))^{\alpha-1}}{1 - (G(x))^{\alpha}(2 - G(x))^{\alpha}}, \quad x > 0, \alpha > 0 \tag{4}$$

3. Topp – Leone Pareto Type I distribution $(TL - PI)$

The random variable X with Pareto Type I distribution has a distribution function given by

$$F(x; \theta, \sigma) = 1 - \left(\frac{x}{\sigma}\right)^{-\theta}, x > \sigma, \theta > 0 \tag{5}$$

and the pdf of Pareto Type I is given by

$$f(x; \theta, \sigma) = \frac{\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-\theta-1}, x > \sigma, \theta > 0 \tag{6}$$

where θ is the shape parameter and σ is the scale parameter

One can construct $(TL - PI)$ distribution by combining (5) and (6) into (1) and (2) hence the cdf and pdf of the $(TL - PI)$ are given respectively as follows

$$F_{TL-PI}(x) = \left(1 - \left(\frac{x}{\sigma}\right)^{-\theta}\right)^{\alpha}\left(2 - \left(1 - \left(\frac{x}{\sigma}\right)^{-\theta}\right)\right)^{\alpha} = \left(1 - \left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}, x > 0, \theta, \alpha, \sigma > x \tag{7}$$

and

$$f_{TL-PI}(x) = 2\alpha\frac{\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-\theta-1}\left[1 - \left(1 - \left(\frac{x}{\sigma}\right)^{-\theta}\right)\right]\left(1 - \left(\frac{x}{\sigma}\right)^{-\theta}\right)^{\alpha-1}\left[2 - \left(1 - \left(\frac{x}{\sigma}\right)^{-\theta}\right)\right]^{\alpha-1}$$

$$= \frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}, x > 0,$$
$$\theta, \alpha, \sigma > 0 \qquad (8)$$

4. Properties of $(TL - PI)$ Distribution

In this section, important and useful statistical characteristics of the proposed distribution are discussed.

4.1. Quantile and Median

The qth percentile of the distribution can be obtained using the following equation

$$q = F(x) \rightarrow q = \left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha} \rightarrow$$

$$x_q = \sigma\left(1 - q^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\theta}} \qquad (9)$$

The median of the $(TL - PI)$ distribution can be defined at $q = 0.5$.

$$x_{median} = \sigma\left(1 - (0.5)^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\theta}} \qquad (10)$$

and the inter – quantile range can be expressed as

$$IQR(x) = \sigma\left(1 - (0.75)^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\theta}} - \sigma\left(1 - (0.25)^{\frac{1}{\alpha}}\right)^{\frac{-1}{2\theta}} \qquad (11)$$

Also, the median can be obtained using the following equation:
Median=1/3 (Mode+2Mean)

4.2. Moments

4.2.1. Non – Central Moments

The rth non - central moments of $(TL - PI)$ distribution is computed as follows

$$E(X^r) = \acute{\mu}_r = \int_{\sigma}^{\infty} X^r \left[\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1 - \left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}\right] dX \qquad (12)$$

$$E(x^r) = 2\alpha\theta\sigma^r \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{r+2\theta+2\theta i}$$

at $r = 1$ the mean is as

$$E(X) = \acute{\mu}_1$$

$$= 2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i} \qquad (13)$$

at $r = 2$ the second moment is obtained

$$E(X^2) = \acute{\mu}_2$$

$$= 2\alpha\theta\sigma^2 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{2+2\theta+2\theta i} \qquad (14)$$

at $r = 3$ the third moment is obtained

$$E(X^3) = \acute{\mu}_3$$

$$= 2\alpha\theta\sigma^3 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{3+2\theta+2\theta i} \qquad (15)$$

at $r = 4$ the fourth moment is obtained

$$E(X^4) = \acute{\mu}_4$$

$$= 2\alpha\theta\sigma^4 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{4+2\theta+2\theta i} \qquad (16)$$

4.2.2. Central Moments

The central moments are derived as follows

The variance of the distribution is computed as

$$\mu_2 = \sigma^2 = \acute{\mu}_2 - \mu^2 = 2\alpha\theta\sigma^2 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{2+2\theta+2\theta i}$$

$$-\left[2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i}\right]^2 \qquad (17)$$

The third central moment equals to

$$\mu_3 = \acute{\mu}_3 - 3\acute{\mu}_2\acute{\mu}_1 + 2(\acute{\mu}_1)^3$$

$$\mu_3$$

$$= 2\alpha\theta\sigma^3 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{3+2\theta+2\theta i}$$

$$-3\left[2\alpha\theta\sigma^2 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{2+2\theta+2\theta i}\right]\left[2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i}\right]$$

$$+2\left[2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i}\right]^3 \qquad (18)$$

The fourth central moment equals to

$$\mu_4 = \acute{\mu}_4 - 4\acute{\mu}_3\acute{\mu}_1 + 6\acute{\mu}_2(\acute{\mu}_1)^2 - 3(\acute{\mu}_1)^4$$

$$\mu_4$$

$$= 2\alpha\theta\sigma^4 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{4+2\theta+2\theta i}$$

$$-4\left[2\alpha\theta\sigma^3 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{3+2\theta+2\theta i}\right]\left[2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i}\right]$$

$$+6\left[2\alpha\theta\sigma^2 \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{2+2\theta+2\theta i}\right]\left[2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i}\right]^2$$

$$-3\left[2\alpha\theta\sigma \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{1+2\theta+2\theta i}\right]^4 \qquad (19)$$

The moment generating function (MGF) of $(TL-PI)$ can be obtained as follows

$$M_x(t) = E(e^{tx}) = \int_{\sigma}^{\infty} e^{tx} \frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1} dx$$

$$= \sum_{j=0}^{\infty} \frac{t^j}{j!}\left[2\alpha\theta\sigma^j \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{j+2\theta+2\theta i}\right] \quad (20)$$

The non- central moments can be obtained using this (MGF)

The coefficient of skewness is derived as follows

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} \quad (21)$$

The coefficient of kurtosis is derived as follows

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} \quad (22)$$

4.3. Reliability Function

$$R_{TL-PI}(x) = 1 - F(x) = 1 - \left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}, x > 0, \theta, \sigma, \alpha > 0 \quad (23)$$

4.4. Hazard and Reversed Hazard Rate Functions

$$h_{TL-PI}(x) = \frac{f(x)}{R(x)} = \frac{\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}}{1-\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}}, x > 0, \theta, \sigma, \alpha > 0 \quad (24)$$

$$Rh_{TL-PI}(x) = \frac{f(x)}{F(x)} = \frac{\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}}{\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}}, x > 0, \theta, \sigma, \alpha > 0 \quad (25)$$

Plots of pdf and hrf of $(TL-PI)$ distribution are given, respectively in Figures 1 and 2. The plots, in Figures 1 and 2, indicate that the curves of the pdf and hrf are monotone decreasing at the values when the curves of the pdf and hrf are right skewed.

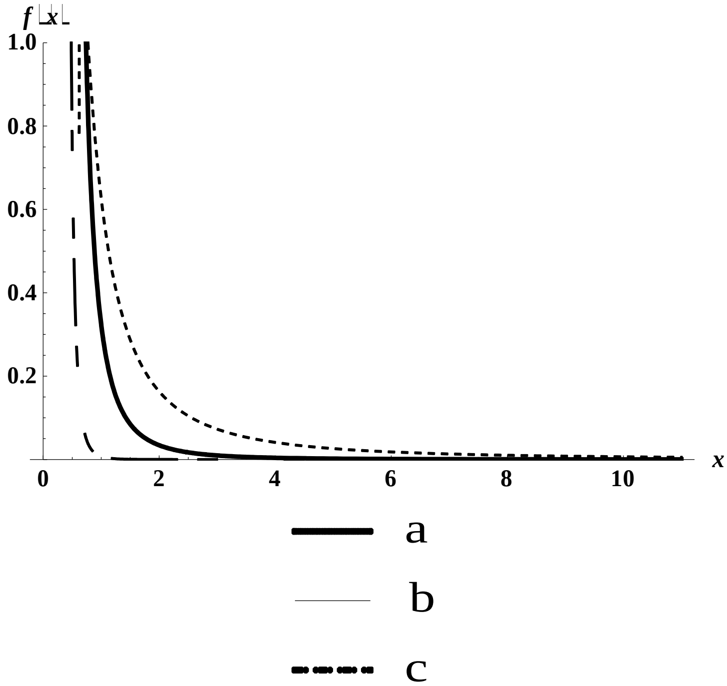

Figure (1): pdf for $(TL-PI)$ with

$\alpha_1 = 1.09, \theta_1 = 0.51, \sigma_1 = 0.495$ $for\ pdf(a)$

$\alpha_2 = 0.495, \theta_2 = 1.03, \sigma_1 = 0.53$ $for\ pdf\ (b)$

$\alpha_3 = 0.21, \theta_3 = 2.68, \sigma_1 = 0.39$ $for\ pdf\ (c)$

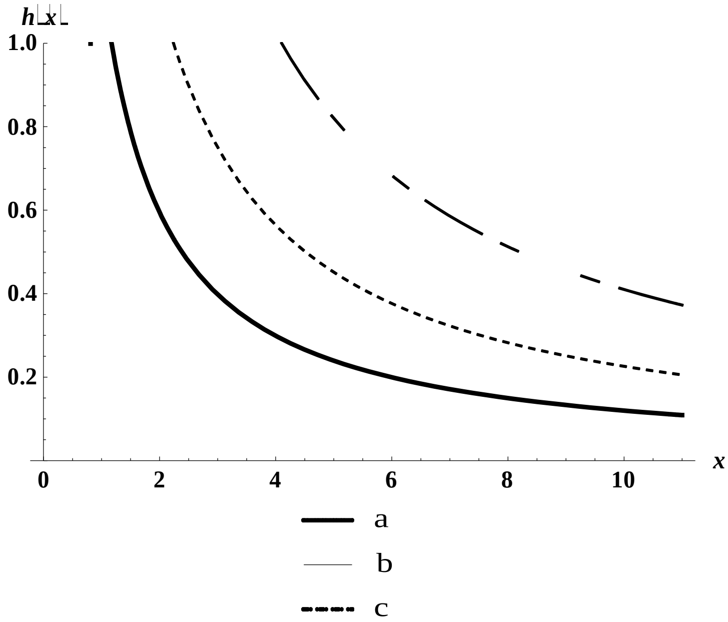

Figure (2): $h(x)$ for $(TL-PI)$ with

$$\alpha_1 = 1.04, \theta_1 = 1.13, \sigma_1 = 1.42 \; for \; pdf(a)$$

$$\alpha_2 = 1.04, \theta_2 = 0.6, \sigma_1 = 0.81 \; for \; pdf \; (b)$$

$$\alpha_3 = 1.26, \theta_3 = 2.05, \sigma_1 = 0.03 \; for \; pdf \; (c)$$

It is noticed that pdf and $h(x)$ for $(TL-PI)$ are approximately exponential.

4.5. The Mode of the $(TL-PI)$ Distribution

We consider the density function of $(TL-PI)$ distribution given in (8) and solve $\frac{df(x)}{dx} = 0$ for x to obtain the mode of $(TL-PI)$ distribution as follows

$$\frac{df(x)}{dx} = \frac{d}{dx}\left[\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}\right] = 0 \rightarrow x$$

$$Mode = \sigma\left(1+\frac{(\alpha-1)(2\theta)}{2\theta+1}\right)^{1/2\theta} \quad (26)$$

4.6. Order Statistics

Let $X_1, X_2, \dots, X_n$ be a simple random sample from $(TL-PI)X(\theta,\sigma,\alpha)$ with distribution and density functions given in (7) and (8). Let $X_{(1:n)} \leq X_{(2:n)}, \dots, X_{(n:n)}$ denote the order statistics obtained from this sample. The density function of $X_{(i:n)}, 1 \leq k \leq n$ is given as follows

$$f_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}[F(x)]^{i-1}[1 - F(x)]^{n-i} f(x). \quad (27)$$

The first order statistic is given by $X_1 = \min(X_1, X_2, \dots, X_n)$ and the last order statistic is given by $X_n = \max(X_1, X_2, \dots, X_n)$.

The distribution of first order statistic is given by

$$f_{1:n}(x) = \frac{n!}{(n-1)!}[1-F(x)]^{n-1} f(x).$$

$$f_{1:n}(x) = n\left[1-\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}\right]^{n-1}\left[\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}\right] \quad (28)$$

The distribution of the nth order statistic is given by

$$f_{n:n}(x) = \frac{n!}{(n-1)!}[F(x)]^{n-1} f(x)$$

the upper record value is

$$f_{U(n)}(x) = \frac{1}{\Gamma(n)}[-\log(1-F(x))]^{n-1} f(x).$$

$$= \frac{1}{\Gamma(n)}\left[-\log\left(1-\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}\right)\right]^{n-1} *$$

$$\left[\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}\right] \tag{29}$$

the lower record value is

$$f_{L(n)}(x) = \frac{1}{\Gamma(n)}[-\log(F(x))]^{n-1} f(x).$$

$$= \frac{1}{\Gamma(n)}\left[-\log\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha}\right]^{n-1}$$

$$\left[\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}\right] \tag{30}$$

5. Estimation of the Parameters:

The maximum likelihood method is used to estimate the parameters of the $(TL-PI)\ X(\theta,\sigma,\alpha)$ distribution. Let $x_1, x_2, \ldots, x_n$ be a random sample from $(TL-PI)X(\theta,\sigma,\alpha)$ distribution with density function as $f(\underline{x};\ \theta,\sigma,\alpha)$. The likelihood function of the $(TL-PI)\ X(\theta,\sigma,\alpha)$ distribution for the parameters θ,σ,α is given as follows:

$$L(\underline{x};\theta,\alpha,\sigma) = \prod_{i=1}^{n} f(\underline{x};\theta,\alpha,\sigma)$$

$$= \left(\frac{2\alpha\theta}{\sigma}\right)^{n} \prod_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^{-2\theta-1}$$

$$\prod_{i=1}^{n}\left[\left(1-\left(\frac{x_i}{\sigma}\right)^{-2\theta}\right)^{\alpha-1}\right] \tag{31}$$

The natural logarithm of the likelihood function is given by

$$\ell = \ln L(\underline{x};\theta,\alpha,\sigma) = n\ln\left(\frac{2\alpha\theta}{\sigma}\right) - (2\theta+1)\sum_{i=1}^{n}\ln\left(\frac{x_i}{\sigma}\right)$$

$$+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-\left(\frac{x_i}{\sigma}\right)^{-2\theta}\right) \tag{32}$$

Considering the three parameters θ, σ, α are unknown and partially differentiating the log likelihood function in (32) with respect to the parameters θ, α, σ respectively as follows

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - (\alpha - 1)\sum_{i=1}^{n} \frac{\ln\left(\frac{x}{\sigma}\right)}{\left[\left(\frac{x}{\sigma}\right)^{-\theta} - 1\right]\left(\frac{x}{\sigma}\right)^{\theta}}$$

$$-(\alpha - 1)\sum_{i=1}^{n} \frac{\ln\left(\frac{x}{\sigma}\right)}{\left[\left(\frac{x}{\sigma}\right)^{-\theta} + 1\right]\left(\frac{x}{\sigma}\right)^{\theta}} - 2\sum_{1}^{n} \ln\frac{x}{\sigma} \tag{33}$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n} \ln\left[\left(\frac{x}{\sigma}\right)^{-\theta} + 1\right]$$

$$+\frac{n}{\sigma}\sum_{i=1}^{n} \ln\left[1 - \left(\frac{x}{\sigma}\right)^{-\theta}\right] \tag{34}$$

$$\frac{\partial \ell}{\partial \sigma} = \frac{\theta(\alpha - 1)\sum_{i=1}^{n} \frac{x}{\left[\left(\frac{x}{\sigma}\right)^{-\theta} - 1\right]\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^2} - \frac{n}{\sigma}$$

$$+\frac{\theta(\alpha - 1)\sum_{i=1}^{n} \frac{x}{\left[\left(\frac{x}{\sigma}\right)^{-\theta} + 1\right]\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^2} \tag{35}$$

Equating the three nonlinear Equations (33) to (35) to zero and solving numerically one obtain the maximum likelihood estimated (MLEs) of the parameters θ, σ, α.

Asymptotic Variance – Covariance Matrix of Maximum Likelihood Estimators

The asymptotic variance-covariance matrix of the ML estimators for the three parameters is the inverse of the observed Fisher information matrix as follows

$$I^{-1} = \begin{bmatrix} var(\theta) & cov(\theta,\alpha) & cov(\theta,\sigma) \\ cov(\alpha,\theta) & var(\alpha) & cov(\alpha,\sigma) \\ cov(\sigma,\theta) & cov(\sigma,\alpha) & var(\sigma) \end{bmatrix}$$

$$\approx \frac{1}{|I|} \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \alpha} & \frac{\partial^2 \ell}{\partial \theta \partial \sigma} \\ \frac{\partial^2 \ell}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha \partial \sigma} \\ \frac{\partial^2 \ell}{\partial \sigma \partial \theta} & \frac{\partial^2 \ell}{\partial \sigma \partial \alpha} & -\frac{\partial^2 \ell}{\partial \sigma^2} \end{bmatrix}$$

with

$$\frac{\partial^2 \ell}{\partial \theta^2} =$$

$$-\frac{n}{\theta^2} - \sum_{i=1}^{n} \left[\frac{\ln\left(\frac{x}{\sigma}\right)^2}{\left[\left(\frac{x}{\sigma}\right)^{-\theta}+1\right]^2 \left(\frac{x}{\sigma}\right)^{2\theta}} - \frac{\ln\left(\frac{x}{\sigma}\right)^2}{\left[\left(\frac{x}{\sigma}\right)^{-\theta}+1\right]\left(\frac{x}{\sigma}\right)^{\theta}} \right] (\alpha - 1) -$$

$$\sum_{i=1}^{n} \left[\frac{\ln\left(\frac{x}{\sigma}\right)^2}{\left[\left(\frac{x}{\sigma}\right)^{-\theta}-1\right]^2 \left(\frac{x}{\sigma}\right)^{2\theta}} - \frac{\ln\left(\frac{x}{\sigma}\right)^2}{\left[\left(\frac{x}{\sigma}\right)^{-\theta}-1\right]\left(\frac{x}{\sigma}\right)^{\theta}} \right] (\alpha - 1)$$

$$\frac{\partial^2 \ell}{\partial \theta \, \partial \alpha} = -\sum_{i=1}^{n} \frac{\frac{\ln\left(\frac{x}{\sigma}\right)}{\left(\frac{x}{\sigma}\right)^{-\theta}-1}}{\left(\frac{x}{\sigma}\right)^{\theta}} - \sum_{i=1}^{n} \frac{\frac{\ln\left(\frac{x}{\sigma}\right)}{\left(\frac{x}{\sigma}\right)^{\theta}}}{\left(\frac{x}{\sigma}\right)^{-\theta}+1}$$

$$\frac{\partial^2 \ell}{\partial\theta\,\partial\sigma} =$$

$$(\alpha-1)\left[\frac{\sum_{i=1}^{n}\frac{1}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}-1\right)\left(\frac{x}{\sigma}\right)^{\theta}}}{\sigma}+\frac{\theta\sum_{i=1}^{n}\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}-1\right)^{2}\left(\frac{x}{\sigma}\right)^{2\theta+1}}}{\sigma^{2}}-\frac{\theta\sum_{i=1}^{n}\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^{2}}\right]+$$

$$(\alpha-1)\left[\frac{\sum_{i=1}^{n}\frac{1}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}+1\right)\left(\frac{x}{\sigma}\right)^{\theta}}}{\sigma}+\frac{\theta\sum_{i=1}^{n}\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}+1\right)^{2}\left(\frac{x}{\sigma}\right)^{2\theta+1}}}{\sigma^{2}}-\frac{\theta\sum_{i=1}^{n}\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^{2}}\right]-\frac{2}{\sigma}$$

$$\frac{\partial^2 \ell}{\partial\alpha\,\partial\theta} = -\sum_{i=1}^{n}\frac{\frac{\ln\left(\frac{x}{\sigma}\right)}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}-1\right)}}{\left(\frac{x}{\sigma}\right)^{\theta}}-\sum_{i=1}^{n}\frac{\frac{\ln\left(\frac{x}{\sigma}\right)}{\left(\frac{x}{\sigma}\right)^{\theta}}}{\left(\frac{x}{\sigma}\right)^{-\theta}+1}$$

$$\frac{\partial^2 \ell}{\partial\alpha^2} = -\frac{n}{\alpha^2}$$

$$\frac{\partial^2 \ell}{\partial\alpha\,\partial\sigma} = \frac{\theta\sum_{i=1}^{n}\frac{x}{\left(\left(\frac{x}{\sigma}\right)^{-\theta}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^{2}}+\frac{\theta\sum_{i=1}^{n}\frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^{2}}$$

$$\frac{\partial^2 \ell}{\partial \sigma\, \partial \theta} =$$

$$\frac{(\alpha-1)\sum_{i=1}^{n}\frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^2}+\frac{(\alpha-1)\sum_{i=1}^{n}\frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^2}+$$

$$\frac{\theta(\alpha-1)\sum_{i=1}^{n}\left[\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)^{2}\left(\frac{x}{\sigma}\right)^{2\theta+1}}-\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}\right]}{\sigma^2}+$$

$$\frac{\theta(\alpha-1)\sum_{i=1}^{n}\left[\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)^{2}\left(\frac{x}{\sigma}\right)^{2\theta+1}}-\frac{\ln\left(\frac{x}{\sigma}\right)x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}\right]}{\sigma^2}$$

$$\frac{\partial^2 \ell}{\partial \sigma\, \partial \alpha}=\frac{\theta\sum_{i=1}^{n}\frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^2}+\frac{\theta\sum_{i=1}^{n}\frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^2}$$

$$\frac{\partial^2 \ell}{\partial \sigma^2} = \frac{n}{\sigma^2} - \frac{\theta(\alpha-1)\left[\frac{\theta \sum_{i=1}^{n} \frac{x^2}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)^2 \left(\frac{x}{\sigma}\right)^{2\theta+2}}}{\sigma^2} - \frac{(\theta+1)\sum_{i=1}^{n} \frac{x^2}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+2}}}{\sigma^2}\right]}{\sigma^2} - \frac{2\theta(\alpha-1)\sum_{i=1}^{n} \frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}-1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^3} - \frac{2\theta(\alpha-1)\sum_{i=1}^{n} \frac{x}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+1}}}{\sigma^3} - \frac{\theta(\alpha-1)\left[\frac{\theta \sum_{i=1}^{n} \frac{x^2}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)^2 \left(\frac{x}{\sigma}\right)^{2\theta+2}}}{\sigma^2} - \frac{(\theta+1)\sum_{i=1}^{n} \frac{x^2}{\left(\frac{1}{\left(\frac{x}{\sigma}\right)^{\theta}}+1\right)\left(\frac{x}{\sigma}\right)^{\theta+2}}}{\sigma^2}\right]}{\sigma^2}$$

The asymptotic normality of ML estimation can be used to compute the asymptotic $100(1-w)\%$ confidence intervals for θ, σ, α as follows

$$\hat{\theta} \pm z_{(1-\frac{w}{2})}\sqrt{var(\hat{\theta})}\,,\ \hat{\alpha} \pm z_{(1-\frac{w}{2})}\sqrt{var(\hat{\alpha})} \text{ and } \hat{\sigma} \pm z_{(1-\frac{w}{2})}\sqrt{var(\hat{\sigma})}$$

The variance of the estimated parameters and the confidence intervals for them are obtained in Table (1).

6. Numerical Illustration

6.1. Monte Carlo Simulation

Some numerical results based on the MLE's of $(TL-PI)\ X(\theta,\alpha,\sigma)$ distribution is obtained according to the following steps:

1) Given initial values $\theta_0, \alpha_0, \sigma_0$ generate random samples of sizes $(n = 20,30,50,90)$ from the $(TL-PI)\ X(\theta,\alpha,\sigma)$ by

observing that if U is uniform $(0,1)$ then the inverse function is as follows

$$U = \left(1 - \left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha} \rightarrow x = \sigma\left(1 - u^{\frac{1}{\alpha}}\right)^{\frac{-1}{\theta}}$$

which has the Topp – Leone Pareto Type I distribution.

2) For each sample size n and the initial values for the parameters (θ, α, σ) the MLE's of the parameters θ, α, σ can be obtained by solving the nonlinear equations (33), (34) and (35) simultaneously using Mathcad iteration scheme
3) Repeat steps (1) - (2) N times where $N = 400$ for $n = 20,30,50,90$
4) The maximum likelihood estimates MLE's, mean square error (MSE),the relative mean square error $(RMSE)$, the variance and the confidence interval for the estimated parameters (θ, α, σ) are computed by averaging over the N repetitions

$$MSE\ (\theta) = \left(\text{bias}(\hat{\theta})\right)^2 + V(\hat{\theta})$$

$$MSE(\alpha) = \left(\text{bias}(\hat{\alpha})\right)^2 + V(\hat{\alpha})$$

$$MSE\ (\sigma) = \left(\text{bias}(\hat{\sigma})\right)^2 + V(\hat{\sigma})$$

$$RMSE(\theta) = \frac{\sqrt{MSE\ (\theta)}}{\theta}$$

$$RMSE(\alpha) = \frac{\sqrt{MSE\ (\alpha)}}{\alpha}$$

$$RMSE(\sigma) = \frac{\sqrt{MSE\ (\sigma)}}{\sigma}$$

The computation results are displayed in Table (1)

For the estimated parameters θ, α, σ we assume that the actual population values are $(\theta = 1.5, \alpha = .8, \sigma = 1.5)$.

Table (1) displays the estimation of the parameters, the estimated mean square error, the relative mean square, the variance and the confidence interval of the maximum likelihood estimates of θ, α, σ for different sample sizes. In this section MLE's of the scale and shape parameters, θ, α, σ of the $(TL-PI)$ distribution are obtained. MLE'S are found by solving the nonlinear equations (33), (34) and (35).

6.2. Results

Table (1): The estimates (MLE), mean square error (MSE), relative mean square error (RMSE), variance (Var.) and confidence interval for the parameters of the $(TL-PI)$ distribution at sample sizes $(n = 20{,}30{,}50{,}90)$

Sample size n	Parameter	MLE	Var	MSE	$RMSE$	Upper bound	Lower bound	length
20	θ	1.495	5.744×10^{-4}	5.973×10^{-4}	0.016	1.542	1.448	0.094
	α	0.949	6.141×10^{-4}	0.023	0.189	0.997	0.9	0.097
	σ	1.495	5.744×10^{-4}	5.973×10^{-4}	0.016	1.542	1.448	0.094
30	θ	1.493	2.597×10^{-4}	3.028×10^{-4}	0.012	1.525	1.462	0.063
	α	0.945	6.254×10^{-4}	0.022	0.183	0.994	0.896	0.098
	σ	1.493	2.597×10^{-4}	3.028×10^{-4}	0.012	1.525	1.462	0.063
50	θ	1.491	1.413×10^{-4}	2.117×10^{-4}	9.7×10^{-3}	1.515	1.468	0.047
	α	0.94	4.918×10^{-4}	0.02	0.177	0.983	0.897	0.087
	σ	1.492	1.413×10^{-4}	2.117×10^{-4}	9.7×10^{-3}	1.515	1.468	0.047
90	θ	1.491	5.862×10^{-5}	1.321×10^{-4}	7.661×10^{-3}	1.506	1.476	0.03
	α	0.936	3.373×10^{-4}	0.019	0.172	0.972	0.9	0.072
	σ	1.491	5.862×10^{-5}	1.321×10^{-4}	7.661×10^{-3}	1.506	1.476	0.03

From Table (1) it is noticed that the mean square error is decreased as the sample size increased.

6.3. An application using real data

In this section, one application of real data set is provided to compare the performance of the $(TL-PI)$ distribution with Pareto Type I distribution. The real data set represents the strength of 1.5 cm glass fibers measured at the National Physical laboratory, England (Shanker, *et al.* 2015) as follows in Table (2).

Table (2): the real data

0.55	0.93	1.25	1.36	1.50	1.55	1.62	1.68	1.77	1.89
0.74	1.04	1.27	1.39	1.51	1.58	1.63	1.69	1.78	2.00
0.77	1.11	1.28	1.42	1.52	1.59	1.64	1.70	1.81	2.01
0.81	1.13	1.29	1.48	1.53	1.60	1.66	1.73	1.82	2.24
0.84	1.24	1.30	1.49	1.54	1.61	1.67	1.76	1.84	

The descriptive statistics for these data are obtained in Table (3).

Table (3) Descriptive Statistics for the real data

N	min	max	range	mean	S.D.	mode	Q_1	Q_2	Q_3	Skewness	Kurtosis	Lower bound	Upper bound
49	0.55	2.24	1.69	1.47	0.36	0.55	1.28	1.54	1.69	-0.64	0.29	1.37	1.57

Since the value of the skewness measure is negative so the distribution of the data is skewed to the left. Also, since the value of the kurtosis measure is less than 3, so the distribution for the data is flat.

To compare the $(TL-PI)$ distribution with Pareto Type I distribution we consider the Akaike information criterion (AIC), Akaike information corrected criterion (AICC), Bayesian information criterion and one-sample Kolmogorov-Smirnov (KS) goodness of fit test for the data set. The better distribution is the one with the smallest values of the previous criteria.

These criteria are:

$AIC = 2k - 2\log L,$

$AICC = AIC + 2k(k+1)/(n-k-1),$

$BIC = 2L + k\log n.$

where

k: is the number of the parameters in each distribution

n: is the sample size

$\log L$: is the log of the likelihood function.

The estimated parameters of the $(TL-PI)$ and Pareto Type I distributions and the values of the criteria for goodness of fit are given in Table (4).

Table (4) Comparison between $(TL-PI)$ distribution and Pareto Type I distribution

Model	Estimates	$\log L$	AIC	AICC	BIC
Topp – Leone Pareto Type I	$\hat{\theta}=0.69$	-158.951	323.903	324.436	-312.832
	$\hat{\alpha}=1.524$				
	$\hat{\sigma}=1.946$				
Pareto Type I	$\hat{\theta}=1$	-34.446	72.892	73.153	-65.512
	$\hat{\sigma}=0.523$				

The results of Table (4) indicates that the proposed $(TL-PI)$ distribution fits well as it has the smallest values for $\log L$ and BIC as compared to Pareto Type I distribution. To test the goodness of fit for the $(TL-PI)$ distribution one – sample Kolmogorov – Smirnov test is used at level of significance $\alpha=0.01$. It is found that $p-value=0.029$ is greater than $\alpha=0.01$ so that these real data have the $(TL-PI)$ distribution. Also, it is found that $p-value=0.019$ is greater than $\alpha=0.01$ so that these real data have the Pareto Type I distribution.

7. Concluding remarks

In this paper, we propose the $(TL-PI)$ distribution. The method for generating the $(TL-PI)$ distribution is presented in Section 2. Some important properties of the $(TL-PI)$ distribution are discussed. This $(TL-PI)$ distribution has closed forms of cdf, pdf, hazard function and reversed hazard function. The moments can be derived using the moment generating function (MGF). Parameter estimation and its observed Fisher information matrix of the $(TL-PI)$ distribution are provided. In Section 6, the application of the $(TL-PI)$ distribution is demonstrated using a real data, and then compared with Pareto Type I distribution. According to the values of KS test, AIC, BIC, AICC and $log\,L$ the $(TL-PI)$ distribution can be considered a competitive distribution for the Pareto Type I distribution.

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Appendix

$$E(x^r) = \int_\sigma^\infty x^r \frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1} dx$$

Let $y = \frac{x}{\sigma} \rightarrow dy = \frac{dx}{\sigma}$

At $x = \sigma \rightarrow y = 1$

At $x = \infty \rightarrow y = \infty$

$$= \frac{2\alpha\theta}{\sigma}\int_1^\infty (\sigma y)^r y^{-2\theta-1}\left(1-y^{-2\theta}\right)^{\alpha-1}\sigma dy$$

$$= 2\alpha\theta\sigma^r \int_1^\infty y^{r-2\theta-1}\left(1-y^{-2\theta}\right)^{\alpha-1} dy$$

$$= 2\alpha\theta\sigma^r \int_1^\infty y^{r-2\theta-1}\sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\left(y^{-2\theta}\right)^i dy$$

$$= 2\alpha\theta\sigma^r \sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\int_1^\infty y^{r-2\theta-1-2\theta i} dy$$

$$= 2\alpha\theta\sigma^r \sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\frac{y^{r-2\theta-2\theta i}}{r-2\theta-2\theta i}\Big|_0^\infty$$

$$= 2\alpha\theta\sigma^r \sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\frac{y^{-(r+2\theta+2\theta i)}}{-(r+2\theta+2\theta i)}\Big|_0^\infty$$

$$E(x^r) = 2\alpha\theta\sigma^r \sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\frac{1}{r+2\theta+2\theta i}$$

$$M_x(t) = E(e^{tx}) = \int_\sigma^\infty e^{tx}\frac{2\alpha\theta}{\sigma}\left(\frac{x}{\sigma}\right)^{-2\theta-1}\left(1-\left(\frac{x}{\sigma}\right)^{-2\theta}\right)^{\alpha-1} dx$$

Since

$$e^{tx} = \sum_{j=0}^\infty \frac{(tx)^j}{j!}, E(e^{tx}) = E\left(\sum_{j=0}^\infty \frac{(tx)^j}{j!}\right) = \sum_{j=0}^\infty E\left(\frac{(tx)^j}{j!}\right) = \sum_{j=0}^\infty \frac{t^j}{j!} E(x^j)$$

$$= \sum_{j=0}^\infty \frac{t^j}{j!}\left[2\alpha\theta\sigma^j \sum_{i=0}^{\alpha-1}(-1)^i\binom{\alpha-1}{i}\frac{1}{j+2\theta+2\theta i}\right]$$