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Some Modified Kibria-Lukman Estimators for the Gamma Regression Model

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Abstract: This paper aims to propose the Gamma modified Kibria-Lukman estimator according to some selected formulas of the shrinkage parameter in order to overcome the effect of the multicollinearity problem in the Gamma regression model. The properties of the proposed estimator and the conditions of its superiority against the maximum likelihood estimator, Gamma ridge estimator, and Gamma Kibria-Lukman estimator based on the matrix of mean squared error criterion are presented. In addition, some selected formulas for the shrinkage parameter are used to improve the results of estimation. Moreover, a Monte Carlo simulation experiment and an application are implemented to assess the performance of the proposed estimator according to some selected formulas of the shrinkage parameter compared with other existing estimators by the scalar mean squared error criterion. The results confirm that the proposed estimator, the Gamma modified Kibria-Lukman estimator is preferred over other existing estimators in terms of scalar mean squared error.

Keywords: *Gamma regression, Multicollinearity, Ridge estimator, Shrinkage parameter, Kibria-Lukman estimator.*

1. Introduction

The Gamma regression model is an appropriate model for characterizing many life phenomena in which the response variable is positively skewed and follows a Gamma distribution. It is well known that when the explanatory variables are highly correlated, the problem of multicollinearity occurs, and hence, the estimated variance of the *maximum likelihood estimator* (MLE) becomes inflated and poor statistical inference will be result. Therefore, many methods for dealing with the effect of multicollinearity have been presented. The biased estimation methods are considered the most widely used methods for dealing with the problem of multicollinearity in several regression models. Actually, there are many biased estimators that have been introduced as alternatives to the MLE in order to achieve the best results, such as the ridge estimator proposed by Hoerl and Kennard (1970), the Liu estimator proposed by Liu (1993), the Liu-type estimator proposed by Liu (2003), and the ridge-type estimator which is called the Kibria-Lukman estimator proposed by Kibria and Lukman (2020) in the linear regression model. In addition, many biased estimators are adopted for generalized linear models, such as in the logistic regression model, the logistic ridge estimator by Schaefer *et al.* (1984), the logistic principal component estimator by Aguilera *et al.* (2006), the logistic Liu estimator by Månsson *et al.* (2012a), and the logistic Liu-type estimator by Inan and Erdogan (2013) were proposed, and in the Poisson regression model, the Poisson ridge estimator by Månsson and Shukur (2011), the Poisson Liu estimator by Månsson *et al.* (2012b), the Poisson Jackknifed Liu-type estimator by Alkhateeb and Algamal (2020), and the Poisson modified Kibria-Lukman estimator by Aladeitan *et al.* (2021) were proposed. Moreover, many biased estimators were developed for the Gamma regression model which is a member of the family of generalized linear models, such as the Gamma Liu estimator by Kurtoğlu and Özkale (2016), the Gamma Liu-type estimator by Algamal and Asar (2020), the *Gamma ridge estimator* (GRE) by Amin *et al.* (2020), the Gamma modified ridge-type estimator by Lukman *et al.* (2020), and the *Gamma Kibria-Lukman estimator* (GKLE) by Lukman *et al.* (2021).

The basis of the biased estimators is based on the shrinkage parameters that aim to reduce the inflated variance of the estimator. So, to achieve this purpose, several studies have been interested in providing various formulas for the shrinkage parameters used in biased estimators, such as the proposed formulas of the ridge shrinkage parameter by Khalaf and Shukur (2005) in the linear regression model, Kibria *et al.* (2012) in the logistic regression model, Kibria *et al.* (2015) in the Poisson regression model, and Amin *et al.* (2020) in the Gamma regression model, and the formulas of the Liu shrinkage parameter by Månsson *et al.* (2012a) in the logistic regression model and Qasim *et al.* (2018) in the Gamma regression model. About more details, one can see Hoerl and Kennard (1970), Liu (1993), Kibria (2003), Muniz and Kibria (2009), Khalaf and Iguernane (2014), and Algamal (2018).

Due to the importance of the shrinkage parameters in improving the results of estimation for the biased estimators, the objective of this paper is to propose the *Gamma modified Kibria-Lukman estimator* (GMKLE) according to some selected formulas of the shrinkage parameter to overcome the multicollinearity problem in the Gamma regression model.

This paper plans as follows: The Gamma regression model specification and estimation by the MLE, GRE, and GKLE are given in Section 2. The proposed estimator, GMKLE, its properties, and the conditions of its superiority against the existing estimators, MLE, GRE, and GKLE based on the *matrix of mean squared error* (MMSE) criterion are introduced in Section 3. Selection of the shrinkage parameter is explained in Section 4. The performance of the proposed estimator, GMKLE according to some selected formulas of the shrinkage parameter is evaluated by a Monte Carlo simulation experiment and compared to the performance of the estimators MLE, GRE, and GKLE based on the *scalar mean squared error* (MSE) criterion in Section 5. Also, an application of a real data set is analyzed in Section 6. Finally, the conclusions are presented in Section 7.

2. Gamma Regression Model: Specification and Estimation

Suppose that $Y_i, i = 1, 2, \dots, n$ is the positively skewed response variable that follows a Gamma distribution with the nonnegative shape parameter λ and scale parameter θ. Hence, the *probability density function* (pdf) of the response variable is given by

$$f(y_i) = \frac{1}{\Gamma(\lambda)\, \theta^{\lambda}} \; y_i^{\lambda-1} \; e^{-\frac{y_i}{\theta}}, \quad y_i > 0 \tag{1}$$

with $E(Y_i) = \lambda\, \theta = \mu_i$, and $Var(Y_i) = \lambda\, \theta^2 = \frac{\mu_i^2}{\lambda}$, where $\mu_i = exp\,(\mathbf{x}_i'\, \beta)$, $\mathbf{x}_i = (1, x_{i1},\ x_{i2}, \dots ,\ x_{iq})'$ is the ith row of the $n \times (q+1)$ data matrix X with q explanatory variables, and $\beta = (\beta_0, \beta_1, \dots, \beta_q)'$ is a $(q+1) \times 1$ vector of unknown coefficients.

By Hardin and Hilbe (2018), Equation (1) can be written also in the following form by re-parameterized λ and θ as $\lambda = \varphi^{-1}$, and $\theta = \varphi\, \mu$, where $\varphi > 0$ is the dispersion parameter.

Then,

$$f(y_i) = \frac{1}{\Gamma(\varphi^{-1})\ \ (\varphi\, \mu)^{\varphi^{-1}}} \; y_i^{\varphi^{-1}-1} \; e^{-\frac{y_i}{\varphi\, \mu}}, \ y_i > 0 \tag{2}$$

with $E(Y_i) = \mu_i$, and $Var(Y_i) = \varphi\, \mu_i^2$.

Since the pdf of the exponential family of distribution can be written as

$$f(y_i) = exp\left[\frac{y_i\,\vartheta_i - b(\vartheta_i)}{a(\varphi)} + c(y_i, \varphi)\right], \tag{3}$$

where ϑ_i is the link function, $b(\vartheta_i)$ is the cumulant function, $a(\varphi)$ is the function of dispersion parameter, and $c(y_i, \varphi)$ is the normalization term.

Then, from Equation (2), the log-likelihood function is given as follows:

$$\ell(y_i) = \sum_{i=1}^{n}[(-\varphi)^{-1}\left[\frac{y_i}{exp\,(\mathbf{x}_i'\,\beta)} + (\mathbf{x}_i'\,\beta)\right] + \varphi^{-1}(\varphi+1)ln(y_i) - \varphi^{-1}ln(\varphi) - ln\,(\Gamma(\varphi^{-1}))]. \tag{4}$$

Consider the maximum likelihood estimation method for estimating the parameters. By maximizing Equation (4) with respect to β, it is required to solve the following score vector

$$S(\beta) = \frac{\partial\,\ell(y_i)}{\partial\,\beta} = \varphi^{-1}\sum_{i=1}^{n}\left[y_i - exp\left(\mathbf{x}_i'\,\beta\right)\right]\mathbf{x}_i = 0. \tag{5}$$

Since Equation (5) is nonlinear in β, then, the algorithm of *iterative weighted least squares* (IWLS) or Fisher scoring method can be used. So, the parameters in the rth iteration are given as follows:

$$\beta^{(r+1)} = \beta^{(r)} + [I(\beta^{(r)})]^{-1}\,S\left(\beta^{(r)}\right), \tag{6}$$

where $I(\beta) = -E[\frac{\partial^2\ell(y_i)}{\partial\beta\,\partial\beta'}]$ is the Fisher information matrix which is evaluated, and $S\left(\beta^{(r)}\right)$ at $\beta^{(r)}$. When convergence, the MLE is given as follows:

$$\hat{\beta}_{\text{MLE}} = F^{-1}X'\widehat{W}\,\hat{\text{v}}, \tag{7}$$

where $F = X'\widehat{W}X$, $\widehat{W} = diag(\hat{\mu}_i^2)$, and $\hat{\text{v}}_i = \hat{\mu}_i + \frac{y_i - \hat{\mu}_i}{\hat{\mu}_i^2}$ is the ith element of the vector $\hat{\text{v}}$.

The MLE is asymptotically distributed for normal distribution with mean vector $E\left(\hat{\beta}_{\mathrm{MLE}}\right) = \beta$, and covariance matrix

$$Cov\left(\hat{\beta}_{\mathrm{MLE}}\right) = \left[-E\left[\frac{\partial^2 \ell(y_i)}{\partial\beta\, \partial\beta'}\right]\right]^{-1} \quad (8)$$

$$= \varphi\, (X'\widehat{W}X)^{-1}$$

$$= \varphi\, F^{-1},$$

where the dispersion parameter φ is estimated by

$$\hat{\varphi} = (n-q)^{-1} \sum_{i=1}^{n} \left(\frac{y_i - \hat{\mu}_i}{\hat{\mu}_i}\right)^2. \quad (9)$$

Then, the MMSE of $\hat{\beta}_{\mathrm{MLE}}$ is given by

$$\mathrm{MMSE}\left(\hat{\beta}_{\mathrm{MLE}}\right) = Cov\left(\hat{\beta}_{\mathrm{MLE}}\right) + B\left(\hat{\beta}_{\mathrm{MLE}}\right)B'\left(\hat{\beta}_{\mathrm{MLE}}\right) \quad (10)$$

$$= \varphi\, F^{-1},$$

where $B(.)$ is the bias vector.

The MSE of $\hat{\beta}_{\mathrm{MLE}}$ is as follows:

$$\mathrm{MSE}\left(\hat{\beta}_{\mathrm{MLE}}\right) = tr\left[\mathrm{MMSE}\left(\hat{\beta}_{\mathrm{MLE}}\right)\right] \quad (11)$$

$$= \varphi\;\; tr(F^{-1})$$

$$= \varphi \sum_{j=1}^{q} \frac{1}{\gamma_j},$$

where γ_j represents the jth eigenvalue of F.

2.1. The Gamma ridge estimator

Amin *et al.* (2020) proposed the ridge estimator for the Gamma regression model as an alternative estimator to the MLE which suffers from inflated variance of the estimated coefficients because of multicollinearity.

The GRE can be defined as follows:

$$\hat{\beta}_{\mathrm{GRE}} = (X'\widehat{W}X + k\,I)^{-1}\,X'\widehat{W}X\,\hat{\beta}_{\mathrm{MLE}} \quad (12)$$
$$= (F + k\,I)^{-1}\,F\,\hat{\beta}_{\mathrm{MLE}}$$
$$= A_k\,\hat{\beta}_{\mathrm{MLE}},$$

where $A_k = (F + k\,I)^{-1}\,F$, and $k > 0$ is the ridge shrinkage parameter.

The GRE has the mean vector, bias vector, and covariance matrix as follows:

$$E(\hat{\beta}_{\mathrm{GRE}}) = E[A_k\,\hat{\beta}_{\mathrm{MLE}}] \quad (13)$$
$$= A_k\,\beta,$$

$$B(\hat{\beta}_{\mathrm{GRE}}) = E[\hat{\beta}_{\mathrm{GRE}}] - \beta \quad (14)$$
$$= (A_k - I)\beta$$
$$= -k(F + k\,I)^{-1}\beta$$
$$= \xi_1,\ \text{(say)}$$

and

$$Cov(\hat{\beta}_{\mathrm{GRE}}) = Cov(A_k\,\hat{\beta}_{\mathrm{MLE}}) \quad (15)$$
$$= \varphi\,A_k\,F^{-1}A_k'$$
$$= \varphi\,(F + k\,I)^{-1}\,F\,(F + k\,I)^{-1}.$$

Also, the MMSE and MSE of $\hat{\beta}_{\mathrm{GRE}}$ are given by

$$\mathrm{MMSE}(\hat{\beta}_{\mathrm{GRE}}) = Cov(\hat{\beta}_{\mathrm{GRE}}) + B(\hat{\beta}_{\mathrm{GRE}})B'(\hat{\beta}_{\mathrm{GRE}}) \quad (16)$$
$$= \varphi\,A_k\,F^{-1}A_k' + \xi_1\xi_1'$$
$$= \varphi\,(F + k\,I)^{-1}\,F\,(F + k\,I)^{-1} + k^2\,(F + k\,I)^{-1}\beta\beta'(\,(F + k\,I)^{-1})',$$

and

$$\mathrm{MSE}(\hat{\beta}_{\mathrm{GRE}}) = tr[\mathrm{MMSE}(\hat{\beta}_{\mathrm{GRE}})] \tag{17}$$
$$= \varphi\, tr(A_k\ F^{-1}A_k') + \xi_1\xi_1'$$
$$= \varphi \sum_{j=1}^{q} \frac{\gamma_j}{(\gamma_j+k)^2} + k^2 \sum_{j=1}^{q} \frac{\alpha_j^2}{(\gamma_j+k)^2}\ ,$$

where α_j is the jth element of $\eta'\,\hat{\beta}_{\mathrm{MLE}}$ and η is an orthogonal matrix whose columns are the eigenvectors of F such that $\eta\,\Upsilon\,\eta' = F$, where $\Upsilon = \mathrm{diag}(\gamma_1,\ \gamma_2, \dots, \gamma_q)$.

2.2. The Gamma Kibria-Lukman estimator

For overcoming the effect of the high correlation between the explanatory variables, Kibria and Lukman (2020) proposed a new estimator called the KL estimator for the linear regression model and then was developed for the Gamma regression model by Lukman *et al.* (2021) and denoted as GKLE which is defined as follows:

$$\hat{\beta}_{\mathrm{GKLE}} = (X'\widehat{W}X +\ k\,I)^{-1}\,(X'\widehat{W}X - k\,I)\,\hat{\beta}_{\mathrm{MLE}} \tag{18}$$
$$= (F +\ k\,I)^{-1}\,(F - k\,I)\,\hat{\beta}_{\mathrm{MLE}}$$
$$= D_k\,\hat{\beta}_{\mathrm{MLE}},$$

where $D_k\ = (F +\ k\,I)^{-1}\,(F - k\,I)\,, k > 0.$

The mean and bias vectors of $\hat{\beta}_{\mathrm{GKLE}}$ are given by

$$E(\hat{\beta}_{\mathrm{GKLE}}) = E[D_k\,\hat{\beta}_{\mathrm{MLE}}] \tag{19}$$
$$= D_k\,\beta,$$

and

$$B(\hat{\beta}_{\mathrm{GKLE}}) = E[\hat{\beta}_{\mathrm{GKLE}}] - \beta \tag{20}$$
$$= (D_k - I)\beta$$

$$= -2\, k(F + k\, I)^{-1}\beta$$
$$= \xi_2, \text{(say)}$$

The covariance matrix of $\hat{\beta}_{\text{GKLE}}$ is

$$Cov(\hat{\beta}_{\text{GKLE}}) = Cov(D_k\, \hat{\beta}_{\text{MLE}}) \tag{21}$$
$$= \varphi\, D_k\, F^{-1} D_k'.$$

Thus, the MMSE and MSE of $\hat{\beta}_{\text{GKLE}}$ are as follows:

$$\text{MMSE}(\hat{\beta}_{\text{GKLE}}) = Cov(\hat{\beta}_{\text{GKLE}}) + B(\hat{\beta}_{\text{GKLE}})B'(\hat{\beta}_{\text{GKLE}}) \tag{22}$$
$$= \varphi\, D_k\, F^{-1} D_k' + 4\, k^2\, (F + k\, I)^{-1}\beta\beta'(\,(F + k\, I)^{-1})'$$
$$= \varphi\, D_k\, F^{-1} D_k' + \xi_2 \xi_2',$$

and

$$\text{MSE}(\hat{\beta}_{\text{GKLE}}) = tr[\text{MMSE}(\hat{\beta}_{\text{GKLE}})] \tag{23}$$
$$= \varphi\, tr(D_k\, F^{-1} D_k') + \xi_2 \xi_2'$$
$$= \varphi \sum_{j=1}^{q} \frac{(\gamma_j - k)^2}{\gamma_j\, (\gamma_j + k)^2} + 4\, k^2 \sum_{j=1}^{q} \frac{\alpha_j^2}{(\gamma_j + k)^2}\,.$$

3. The Gamma Modified Kibria-Lukman Estimator

In this section, following Aladeitan *et al.* (2021), the modified Kibria-Lukman estimator is proposed for the Gamma regression model by replacing $\hat{\beta}_{\text{MLE}}$ in (18) with the $\hat{\beta}_{\text{GRE}}$. The resulting estimator is denoted as GMKLE and defined as follows:

$$\hat{\beta}_{\text{GMKLE}} = (X'\widehat{W}X + k\, I)^{-1}\, (X'\widehat{W}X - k\, I)\, \hat{\beta}_{\text{GRE}} \tag{24}$$
$$= (F + k\, I)^{-1}\, (F - k\, I)(F + k\, I)^{-1}\, F\, \hat{\beta}_{\text{MLE}}$$
$$= D_k\, A_k\, \hat{\beta}_{\text{MLE}}\,.$$

3.1. Properties of the GMKLE

The GMKLE of $\hat{\beta}$ has the following properties:

$$E(\hat{\beta}_{\text{GMKLE}}) = E[D_k\, A_k\, \hat{\beta}_{\text{MLE}}] \tag{25}$$
$$= D_k\, A_k\, \beta,$$

$$B(\hat{\beta}_{\text{GMKLE}}) = E[\hat{\beta}_{\text{GMKLE}}] - \beta \tag{26}$$
$$= (D_k\, A_k\, - I)\beta$$
$$= (F +\, k\, I)^{-2}\, k(-3\, F - k\, I)\beta$$
$$= \xi_3, \text{(say)}$$

$$Cov(\hat{\beta}_{\text{GMKLE}}) = Cov(D_k\, A_k\, \hat{\beta}_{\text{MLE}}) \tag{27}$$
$$= \varphi\, D_k\, A_k\,\; F^{-1}\, A'_k D'_k,$$

$$\text{MMSE}(\hat{\beta}_{\text{GMKLE}}) = Cov(\hat{\beta}_{\text{GMKLE}}) + B(\hat{\beta}_{\text{GMKLE}})B'(\hat{\beta}_{\text{GMKLE}}) \tag{28}$$
$$= \varphi\, D_k\, A_k\,\; F^{-1}\, A'_k D'_k + k^2\, (F + k\, I)^{-2}(9F\, F' + k^2 I)\beta\beta'(\,(F + k\, I)^{-2})'$$
$$= \varphi\, D_k\, A_k\,\; F^{-1}\, A'_k D'_k + \xi_3 \xi'_3,$$

and

$$\text{MSE}(\hat{\beta}_{\text{GMKLE}}) = tr[\text{MMSE}(\hat{\beta}_{\text{GMKLE}})] \tag{29}$$
$$= \varphi\, tr(D_k\, A_k\,\; F^{-1}\, A'_k D'_k) + \xi_3 \xi'_3$$
$$= \varphi \sum_{j=1}^{q} \frac{\gamma_j\,(\gamma_j - k)^2}{(\gamma_j + k)^4} +\, k^2 \sum_{j=1}^{q} \frac{(3\gamma_j + k)^2\, \alpha_j^2}{(\gamma_j + k)^4}\,.$$

3.2. Superiority of the GMKLE

For comparison between the performance of the proposed estimator, GMKLE and the other existing estimators, MLE, GRE, and GKLE according to the MMSE criterion, the following lemmas are used:

Lemma 1. *Assume that R and S be two positive definite matrices, that is $R > 0$, and $S > 0$. Then, $R > S$ if and only if the maximum eigenvalue of $SR^{-1} < 1$.* [Rao *et al.* (2008)]

Lemma 2. *Let $\hat{\beta}_1$ and $\hat{\beta}_2$ are any two estimators of β, and $U = Cov(\hat{\beta}_1) - Cov(\hat{\beta}_2) > 0$. Then, $MMSE(\hat{\beta}_1) - MMSE(\hat{\beta}_2) = U + b_1 b_1' - b_2 b_2' > 0$ if and only if $b_2'[U + b_1 b_1']^{-1} b_2 < 1$, where b_1 and b_2 are the bias vectors of $\hat{\beta}_1$ and $\hat{\beta}_2$ respectively.* [Trenkler and Toutenburg (1990)]

3.2.1. Comparison of GMKLE with MLE

Based on (8) and (27),

$$Cov(\hat{\beta}_{\text{MLE}}) - Cov(\hat{\beta}_{\text{GMKLE}}) = \varphi F^{-1} - \varphi D_k A_k F^{-1} A_k' D_k' \quad (30)$$
$$= \varphi [F^{-1} - D_k A_k F^{-1} A_k' D_k'].$$

Since $F^{-1} - D_k A_k F^{-1} A_k' D_k'$ is positive definite. Then, by Lemma 2,

$$\text{MMSE}(\hat{\beta}_{\text{MLE}}) - \text{MMSE}(\hat{\beta}_{\text{GMKLE}}) = \varphi [F^{-1} - D_k A_k F^{-1} A_k' D_k'] - \xi_3 \xi_3' > 0,$$

if and only if $\xi_3' [\varphi (F^{-1} - D_k A_k F^{-1} A_k' D_k')]^{-1} \xi_3 < 1$. Thus, Theorem 1 can be stated as follows:

Theorem 1. *The estimator, GMKLE is superior to the MLE according to $MMSE$ if and only if*

$$\beta'(D_k A_k - I)' [\varphi (F^{-1} - D_k A_k F^{-1} A_k' D_k')]^{-1} (D_k A_k - I)\beta < 1.$$

3.2.2. Comparison of GMKLE with GRE

Based on (16) and (28),

$$\text{MMSE}(\hat{\beta}_{\text{GRE}}) - \text{MMSE}(\hat{\beta}_{\text{GMKLE}}) = \varphi A_k F^{-1} A_k' + \xi_1 \xi_1' - [\varphi D_k A_k F^{-1} A_k' D_k' + \xi_3 \xi_3']$$
$$= \varphi [A_k F^{-1} A_k' - D_k A_k F^{-1} A_k' D_k'] + \xi_1 \xi_1' - \xi_3 \xi_3'$$
$$= \varphi [R_1 - S_1] + \xi_1 \xi_1' - \xi_3 \xi_3', \quad (31)$$

where $R_1 = A_k\ F^{-1}A'_k$, and $S_1 = D_k\, A_k\ F^{-1}\ A'_k D'_k$ are positive definite. From Lemma 1, $R_1 - S_1 > 0$ if and only if $\gamma_{max}(S_1 R_1^{-1}) < 1$, where $\gamma_{max}(S_1 R_1^{-1})$ is the maximum eigenvalue of $S_1 R_1^{-1}$.

Then, by Lemma 2, $\text{MMSE}(\hat{\beta}_{\text{GRE}}) - \text{MMSE}(\hat{\beta}_{\text{GMKLE}}) > 0$ if and only if

$$\xi'_3\ [\varphi\ (R_1 - S_1) + \xi_1\xi'_1]^{-1}\ \xi_3 < 1.$$

Therefore, Theorem 2 can be stated as follows:

Theorem 2. *When $\gamma_{max}(S_1 R_1^{-1}) < 1$, the GMKLE is superior to the GRE according to MMSE if and only if*

$$\beta'(D_k\, A_k - I)'\ [\varphi\ (A_k\ F^{-1}A'_k - D_k\, A_k\ F^{-1}\ A'_k D'_k) + (A_k - I)\beta\beta'(A_k - I)']^{-1}(D_k\, A_k\ - I)\beta < 1.$$

3.2.3. Comparison of GMKLE with GKLE

Based on (22) and (28),

$$\begin{aligned}\text{MMSE}(\hat{\beta}_{\text{GKLE}}) - \text{MMSE}(\hat{\beta}_{\text{GMKLE}}) &= \varphi\ D_k\ F^{-1}D'_k + \xi_2\xi'_2 - [\varphi\ D_k\, A_k\ F^{-1}\ A'_k D'_k + \xi_3\xi'_3] \\ &= \varphi\ [D_k\ F^{-1}D'_k - D_k\, A_k\ F^{-1}\ A'_k D'_k] + \xi_2\xi'_2 - \xi_3\xi'_3 \\ &= \varphi\ [R_2 - S_1] + \xi_2\xi'_2 - \xi_3\xi'_3, \qquad (32)\end{aligned}$$

where $R_2 = D_k\ F^{-1}D'_k$ is positive definite. From Lemma 1, $R_2 - S_1 > 0$ if and only if $\gamma_{max}(S_1 R_2^{-1}) < 1$, where $\gamma_{max}(S_1 R_2^{-1})$ is the maximum eigenvalue of $S_1 R_2^{-1}$.

Then, by Lemma 2, $\text{MMSE}(\hat{\beta}_{\text{GKLE}}) - \text{MMSE}(\hat{\beta}_{\text{GMKLE}}) > 0$ if and only if

$$\xi'_3\ [\varphi\ (R_2 - S_1) + \xi_2\xi'_2]^{-1}\ \xi_3 < 1.$$

Thus, Theorem 3 can be established as follows:

Theorem 3. *When $\gamma_{max}(S_1 R_2^{-1}) < 1$, the GMKLE is superior to the GKLE according to MMSE if and only if*

$$\beta'(D_k\, A_k - I)'\ [\varphi\ (D_k\ F^{-1}D'_k - D_k\, A_k\ F^{-1}\ A'_k D'_k) + (D_k - I)\beta\beta'(D_k - I)']^{-1}(D_k\, A_k\ - I)\beta < 1.$$

4. Selection of the Shrinkage Parameter

A disciplined way of choosing the shrinkage parameter, k is required that minimizes the MSE. Many methods have been proposed for choosing the shrinkage parameter, k in various models since no specific method is available. Some popular methods for choosing k are considered as follows:
The classical estimator of k was proposed by Hoerl and Kennard (1970) as

$$k_1 = \frac{\hat{\varphi}}{\hat{\alpha}_{max}^2}, \tag{33}$$

where $\hat{\alpha}_{max}^2$ is the maximum value of α_j^2.

Also, Hoerl *et al.* (1975) proposed the following estimator

$$k_2 = \frac{q\,\hat{\varphi}}{\hat{\alpha}'\,\hat{\alpha}}. \tag{34}$$

In Khalaf and Iguernane (2014), two estimators were suggested as follows:

$$k_3 = \frac{\hat{\varphi}}{2}\left[\frac{1}{\hat{\alpha}_{max}^2} + \frac{q}{\hat{\alpha}'\,\hat{\alpha}}\right], \tag{35}$$

$$k_4 = \hat{\varphi}\sqrt{\frac{q}{\hat{\alpha}_{max}^2\,\hat{\alpha}'\,\hat{\alpha}}}. \tag{36}$$

5. Simulation Study

In the presence of multicollinearity in Gamma regression, the estimated MSE of the proposed estimator, GMKLE using $k_1 - k_4$ is computed as an evaluation measure to inspect their performance and compare the results with those of GRE, GKLE, and MLE.
Therefore, the response variable is generated from the Gamma distribution as $Y_i \sim G(\mu_i, \varphi)$ where $\mu_i = exp\,(\mathbf{x}_i'\,\beta)$ is the mean of the response variable with seven explanatory variables and the values of the

parameter vector, β are chosen so that $\beta_0 = 0$, $\sum_{j=1}^{q} \beta_j = 1$; $\beta_1 = \beta_2 = \cdots = \beta_q$, and the dispersion parameter, φ is selected as $0.25, 0.50$, and 0.75 values.

Following McDonald and Galarneau (1975) and Kibria (2003), the explanatory variables are generated as follows:

$$x_{ij} = (1-\rho^2)\, \mathcal{Z}_{ij} + \rho\, \mathcal{Z}_{iq}\,,\ i = 1,2, \dots, n, j = 1, 2, \dots, q, \qquad (37)$$

where ρ^2 is the degree of correlation between any two explanatory variables, and $\mathcal{Z}_{ij}$ are independent standard normal distribution pseudo random variables. For $q = 7$, the sample size $n = 20, 50$, and 100 and three levels of correlations corresponding to $0.90, 0.95$, and 0.99 are considered. For combinations of given values of n, φ and ρ, the Monte Carlo experiment is repeated 1000 times by the R 4.0.3 software and the estimated MSE is computed as

$$\text{MSE}(\hat{\beta}) = \frac{1}{1000} \sum_{r=1}^{1000} (\hat{\beta}_r - \beta)' \left(\hat{\beta}_r - \beta\right), \qquad (38)$$

where $\hat{\beta}_r$ is the rth estimated value of β.

Then, the estimated MSE values of MLE, GRE, GKLE, and GMKLE using $k_1 - k_4$ estimators and different combinations of given values of n, ϕ, and ρ are concluded in Tables 1-3.

Table 1. Estimated MSE of the estimators, MLE, GRE, GKLE, and GMKLE when $\varphi = 0.25$

n	ρ	MLE	GRE				GKLE				GMKLE			
			k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4
20	0.90	6.3435	5.4452	4.7699	5.0092	5.1136	4.8737	4.2961	4.4238	4.5080	4.6482	4.1861	4.3134	4.3836
	0.95	9.0366	6.9804	5.5335	6.0219	6.2510	5.7906	4.7890	4.9385	5.1387	5.3314	4.4776	4.6790	4.8104
	0.99	31.9785	19.8427	11.6748	14.2906	15.6190	13.4391	9.0868	9.2900	9.8931	10.8966	6.7176	7.5301	8.1743
50	0.90	4.5588	4.3786	4.2310	4.2896	4.3102	4.2443	4.0654	4.1255	4.1410	4.1814	4.0416	4.0919	4.1114
	0.95	5.1936	4.7598	4.4246	4.5461	4.5976	4.4764	4.1648	4.2431	4.2889	4.3627	4.1124	4.1874	4.2256
	0.99	10.5762	7.7429	5.7959	6.4252	6.7440	6.2214	5.1157	5.1990	5.3573	5.6283	4.5795	4.7975	4.9626
100	0.90	4.2736	4.1968	4.1350	4.1616	4.1690	4.1333	4.0425	4.0786	4.0893	4.0971	4.0255	4.0546	4.0632
	0.95	4.5805	4.3928	4.2447	4.3024	4.3233	4.2562	4.0860	4.1396	4.1632	4.1950	4.0648	4.1091	4.1275
	0.99	7.1865	5.8463	4.9170	5.2257	5.3751	5.0881	4.4806	4.5548	4.6443	4.8034	4.2723	4.3902	4.4729

Table 2. Estimated MSE of the estimators, MLE, GRE, GKLE, and GMKLE when $\varphi = 0.50$

n	ρ	MLE	GRE				GKLE				GMKLE			
			k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4
20	0.90	9.1113	7.1859	5.7544	6.2637	6.4836	5.9963	4.7827	5.0581	5.2370	5.5132	4.5192	4.8000	4.9528
	0.95	14.8531	10.5732	7.5384	8.5879	9.0615	8.1150	5.8536	6.2756	6.6108	7.1082	5.1700	5.6762	5.9774
	0.99	63.6359	39.2135	22.1788	27.8585	30.5992	26.1256	15.1072	16.5833	18.2224	20.5299	10.4539	12.8332	14.4235
50	0.90	5.2472	4.8078	4.4540	4.5860	4.6397	4.5055	4.1379	4.2406	4.2952	4.3775	4.0896	4.1710	4.2238
	0.95	6.6556	5.6276	4.8545	5.1239	5.2471	4.9946	4.3582	4.4861	4.5838	4.7467	4.2272	4.3627	4.4453
	0.99	18.6020	12.3528	7.9689	9.3830	10.1066	8.9617	6.3853	6.5718	6.9658	7.6142	5.2582	5.7200	6.0971
100	0.90	4.5415	4.3714	4.2199	4.2804	4.3020	4.2405	4.0487	4.1140	4.1410	4.1751	4.0252	4.0792	4.1006
	0.95	5.1752	4.7592	4.4156	4.5405	4.5945	4.4777	4.1399	4.2261	4.2775	4.3598	4.0935	4.1721	4.2137
	0.99	10.5664	7.7961	5.7993	6.4461	6.7780	6.2533	5.0094	5.1246	5.3107	5.6420	4.5327	4.7562	4.9338

Table 3. Estimated MSE of the estimators, MLE, GRE, GKLE, and GMKLE when $\varphi = 0.75$

n	ρ	MLE	GRE				GKLE				GMKLE			
			k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4
20	0.90	11.5915	8.7462	6.5696	7.3501	7.6874	6.9930	5.0721	5.5315	5.8218	6.2466	4.6634	5.1230	5.3708
	0.95	20.4830	14.2380	9.5733	11.2112	11.9473	10.5870	6.7968	7.6068	8.1892	9.0183	5.7899	6.6819	7.1960
	0.99	96.1377	60.7521	34.4510	43.4260	47.7106	41.1218	21.6187	25.1452	28.2101	32.4631	15.0911	19.5864	22.4266
50	0.90	5.9280	5.2384	4.6785	4.8838	4.9706	4.7734	4.2146	4.3619	4.4463	4.5763	4.1371	4.2689	4.3371
	0.95	8.1153	6.5460	5.3234	5.7477	5.9444	5.5773	4.5623	4.7653	4.9227	5.1803	4.3548	4.5668	4.6993
	0.99	26.7057	17.4530	10.5008	12.7716	13.9355	12.2877	7.6680	8.1969	8.9187	10.1012	6.0461	6.8967	7.5629
100	0.90	4.8046	4.5411	4.3075	4.3983	4.4333	4.3472	4.0692	4.1580	4.1992	4.2555	4.0366	4.1117	4.1448
	0.95	5.7319	5.1023	4.5878	4.7726	4.8550	4.6922	4.2069	4.3250	4.4012	4.5210	4.1309	4.2427	4.3058
	0.99	13.5951	9.5689	6.6767	7.6264	8.1059	7.3562	5.4869	5.6987	5.9901	6.4503	4.7750	5.1335	5.4092

It can be observed from Tables 1-3, the estimated values of MSE of all estimators MLE, GRE, GKLE, and GMKLE increase as ρ increases, whereas the estimated MSE of all estimators decreases as the sample size increases. In addition, when the dispersion parameter, φ increases, there is increasing in the MSE for all estimators in all cases. Furthermore, the MLE has the worst performance among other estimators the higher the degree of correlation since it has the largest estimated MSE in all cases, and the proposed estimators of GMKLE using $k_1 - k_4$ outperform other estimators in terms of MSE. Moreover, with respect to the estimators of the shrinkage parameter, it can be seen that the k_2 performs well with the lowest MSE of GMKLE. Although the GRE and GKLE attained good results compared to the MLE in terms of MSE, the GMKLE is superior to them in all cases.

6. Application

To evaluate the performance of the proposed estimator, GMKLE against the estimators MLE, GRE, and GKLE, a real data set of the number of persons employed in Pakistan is used. This data was taken from Pasha and Shah (2004) and comprises of 28 observations, where the response variable (y) is the number of persons employed (million) with five explanatory variables. These explanatory variables include the land cultivated (million hectares) (x_1), the inflation rate (%) (x_2), the number of establishments (x_3), population (million) (x_4), and the literacy rate (%) (x_5). For fitting the response variable to the Gamma distribution, the Chi-square test is considered. The result of Chi-square test confirms the suitability of the Gamma distribution to this data with a test statistic equals to 1.080387 and the $p-value$ is 0.955825. The dispersion parameter is estimated according to (9) equals to 0.000573. Further, the condition number (CN) of the data is used to check the existence of multicollinearity as

$$CN = \sqrt{\gamma_{max}/\gamma_{min}}\,, \tag{39}$$

where γ_{max} and γ_{min} are the maximum and minimum eigenvalues of the F matrix respectively. The value of CN is to be 41291.68 indicating the presence of severe multicollinearity among the explanatory variables. The correlation matrix among the five explanatory variables is as follows:

$$\begin{bmatrix} 1 & 0.66 & 0.94 & 0.98 & 0.96 \\ 0.66 & 1 & 0.66 & 0.73 & 0.68 \\ 0.94 & 0.66 & 1 & 0.96 & 0.87 \\ 0.98 & 0.73 & 0.96 & 1 & 0.95 \\ 0.96 & 0.68 & 0.87 & 0.95 & 1 \end{bmatrix}$$

It is clear that there are high correlations between x_1 and x_3, x_1 and x_4, x_1 and x_5, x_3 and x_4, x_3 and x_5, and x_4 and x_5.

As an evaluation measure, the MSE of the MLE, GRE, GKLE, and GMKLE is computed according to (11), (17), (23), and (29) respectively. In Table 4, the estimated coefficients, and MSE values of the estimators MLE, GRE, GKLE, and GMKLE are listed.

From Table 4, it is obvious seen that the proposed estimator, GMKLE shrinkages well the value of the estimated coefficients. Also, for the GRE, GKLE, and GMKLE, all considered estimators of k are superior to the MLE in terms of MSE and the k_2 has the lowest MSE. Additionally, the GMKLE using k_2 performs better than the other estimators in terms of MSE.

Table 4. The estimated coefficients and MSE of the estimators

Estimator		$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\hat{\beta}_5$	MSE
MLE		2.082394	0.016472	−0.057044	0.000013	0.009619	−0.002751	1.803159×10^{-6}
GRE	k_1	1.885615	0.027859	−0.055038	0.000010	0.009430	−0.003109	1.801214×10^{-6}
	k_2	1.418805	0.054871	−0.050257	0.000005	0.008980	−0.003958	1.794489×10^{-6}
	k_3	1.619086	0.043282	−0.052313	0.000007	0.009173	−0.003594	1.797840×10^{-6}
	k_4	1.705510	0.038281	−0.053197	0.000008	0.009257	−0.003437	1.799047×10^{-6}
GKLE	k_1	1.688837	0.039246	−0.053031	0.000008	0.009241	−0.003467	1.799274×10^{-6}
	k_2	0.755216	0.093270	−0.043470	−0.000003	0.008341	−0.005165	1.785750×10^{-6}
	k_3	1.155778	0.070092	−0.047582	0.000002	0.008728	−0.004437	1.792563×10^{-6}
	k_4	1.328626	0.060090	−0.049351	0.000004	0.008894	−0.004123	1.794960×10^{-6}
GMKLE	k_1	1.529383	0.048473	−0.051402	0.000006	0.009087	−0.003757	1.797342×10^{-6}
	k_2	0.516092	0.107106	−0.040968	−0.000006	0.008110	−0.005599	1.777527×10^{-6}
	k_3	0.899381	0.084928	−0.044940	−0.000001	0.008480	−0.004903	1.787343×10^{-6}
	k_4	1.088659	0.073976	−0.046887	0.000001	0.008663	−0.004559	1.790908×10^{-6}

7. Conclusions

In this paper, the GMKLE was proposed in the Gamma regression model to tackle the effect of multicollinearity using some estimators of the shrinkage parameter. The Monte Carlo simulation experiment and application were conducted to evaluate the performance of the proposed estimator, GMKLE with considered estimators of shrinkage parameter against the estimators MLE, GRE, and GKLE based on the MSE criterion. The results of simulation and application showed that the proposed estimator, GMKLE outperformed the other estimators in terms of MSE. In addition, for the GRE, GKLE, and GMKLE, all selected estimators of the shrinkage parameter performed better than the MLE in terms of MSE and the GMKLE with k_2 had the lowest MSE. Hence, the GMKLE is a good choice to apply in the Gamma regression model when the problem of multicollinearity is present.

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