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## RESTRICTIVE PADE' APPROXIMATION FOR SOLVING SINGULARLY PERTURBED INITIAL BOUNDARY VALUE PROBLEM FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

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### Abstract

This paper is concerned with the numerical treatment of the first-order hyperbolic partial differential equation by small parameter with the time derivative term. This problem is reduced to stiff system of ODE<sub>s</sub> in time. The resulting system is solved by the restrictive Pade' approximation. The stability condition and the error upper bound are introduced. The numerical results are given and the considered method gives better results compared with the classical advantages of the considered method compared with the classical Pade' approximation.

### KEY WORDS

Pade' approximation, Restrictive Pade' approximation, finite difference and hyperbolic partial differential equations.

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the restrictive Pade` approximation as done in [1] and [2] to approximate the exponential matrix.

### 2. Restrictive Pade` Approximation (RPA)

The restrictive Pade` approximation can be written as done in [1] in the form

$$RPA[M + \alpha / N]_{f(x)}(x) = \frac{\sum_{i=0}^M a_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i x^{M+i}}{1 + \sum_{i=1}^N b_i x^i} \tag{8}$$

where  $\alpha$  is a positive integer dose not exceed the degree of the denominator  $N$ , i.e.  $\alpha = 1(1)N$ , such that

$$f(x) - RPA[M + \alpha / N]_{f(x)}(x) = o(x^{M+N+1}). \tag{9}$$

Let  $f(x)$  have a Maclaurin series  $f(x) = \sum_{i=0}^{\infty} c_i x^i$ , then from equations (8) and (9)

we have

$$\left( \sum_{i=0}^{\infty} c_i x^i \right) \left( 1 + \sum_{i=1}^N b_i x^i \right) - \left( \sum_{i=0}^M a_i x^i \right) - \left( \sum_{i=1}^{\alpha} \varepsilon_i x^{i+M} \right) = o(x^{M+N+1}). \tag{10}$$

The vanishing of the first  $(M+N+1)$  powers of  $x$  on the left hand side of (10) implies a system of  $(M+N+1)$  equations.

$$\left. \begin{aligned} a_r &= c_r + \sum_{i=1}^r c_{r-i} b_i, & r &= 0(1)M, \\ & & (b_i &= 0 \text{ if } i > M) \\ c_{M+N-s} + \sum_{i=1}^N c_{M+N-i-s} b_i &= \varepsilon_{N-s}, & s &= 0(1)N-1, \\ & & (c_i &= 0 \text{ if } i < 0) \end{aligned} \right\} \tag{11}$$

Hence we can determine the coefficient,  $a_i$  and  $b_i$  as a function of  $a_i, i=1(1)\alpha$ , where the parameters  $\varepsilon_i$  are to be determined, such that

$$f(x_i) = RPA[M + \alpha / N]_{f(x)}(x_i), \quad i = 1(1)\alpha. \tag{12}$$

It means that the considered approximation is exact at  $(\alpha+1)$  points.

Consider the function  $f(x) = \left( \frac{1 + 0.5x + 0.25x^2}{1 + 5x} \right)^{0.5}$ ,

its Pade` approximation and restrictive Pade` approximation takes the forms:

$$PA [2 / 1]_{f(x)}(x) = \frac{1 + 1.9311x - 0.563724x^2}{1 + 4.1811x},$$

$$RPA[2/1]_{f(x)}(x) = \frac{1 + 1.73134x - 0.114257x^2}{1 + 3.98134x} \text{ where } \alpha = 1 \text{ and } x_{\alpha} = 0.6$$

### 3. Restrictive Pade` Approximation (RPA) for Solving Singularly Perturbed Hyperbolic Partial Differential Equation

The restrictive Pade` approximation of order [0/1] and [1/1] of the exponential matrix  $\exp(k A)$  as done in [3] can take the forms:

i-)  $RPA[0/1]_{\exp(kA)} = (I + (\varepsilon - k)A)^{-1}$  (13)

using equation (13) to approximate the  $\exp(k A)$  in equation (6), which can the form:

$$\underline{U}^{j+1} = -A^{-1} \underline{F} + (I + (\varepsilon - k)A)^{-1} [\underline{U}^j + A^{-1} \underline{F}], \quad \Delta t = k, \quad (14)$$

or

$$(I + (\varepsilon - k)A) \underline{U}^{j+1} = \underline{U}^j - (\varepsilon - k) \underline{F}. \quad (15)$$

which can take the equivalent scalar form:

$$\left( \frac{(\varepsilon - k)a}{2\delta h} \right) u_{i-1,j+1} + u_{i,j+1} - \left( \frac{(\varepsilon - k)a}{2\delta h} \right) u_{i+1,j+1} = u_{i,j} - \left( \frac{\varepsilon - k}{\delta} \right) f(x_i, t). \quad (16)$$

Similarly the restrictive Pade` approximation of order [1/1] of the exponential matrix  $\exp(k A)$  as done in [3] can take the form:

ii-)  $RPA[1/1]_{\exp(kA)} = (I + (\varepsilon - \frac{1}{2}k)A)^{-1} (I + (\varepsilon + \frac{1}{2}k)A)$  (17)

using equation (13) to approximate the  $\exp(k A)$  in equation (6), which can the form:

$$\underline{U}^{j+1} = -A^{-1} \underline{F} + (I + (\varepsilon - \frac{1}{2}k)A)^{-1} (I + (\varepsilon + \frac{1}{2}k)A) [\underline{U}^j + A^{-1} \underline{F}] \quad (18)$$

or

$$(I + (\varepsilon - \frac{1}{2}k)A) \underline{U}^{j+1} = (I + (\varepsilon + \frac{1}{2}k)A) \underline{U}^j + k \underline{F}. \quad (19)$$

It can take the equivalent scalar form:

$$\begin{aligned} & \left( \frac{(\varepsilon - 0.5k)a}{2\delta h} \right) u_{i-1,j+1} + u_{i,j+1} - \left( \frac{(\varepsilon - 0.5k)a}{2\delta h} \right) u_{i+1,j+1} \\ & = \left( \frac{(\varepsilon + 0.5k)a}{2\delta h} \right) u_{i-1,j} + u_{i,j} - \left( \frac{(\varepsilon + 0.5k)a}{2\delta h} \right) u_{i+1,j} + \left( \frac{k}{\delta} \right) f(x_i, t). \end{aligned} \quad (20)$$

Putting  $\varepsilon = 0$  in equations (16) and (20), we have the schemes arising of applying classical Pade` approximations [0/1] and [1/1] respectively to the exponential matrix  $\exp(kA)$  in equation (6). To determine the restrictive parameters  $\varepsilon$ , we must have the exact solution or a highly accurate numerical solution at the first level.

#### 4. The Stability Analysis

By using Von-Neumann stability analysis method we have, the amplification factors  $G_1$  and  $G_2$  for the difference equations (16) and (20) respectively are:

$$G_1 = \frac{1}{1 - I \left( \frac{(\varepsilon - k)a}{\delta h} \sin \theta \right)}, \quad G_2 = \frac{1 - I \left( \frac{(\varepsilon + 0.5k)a}{\delta h} \sin \theta \right)}{1 - I \left( \frac{(\varepsilon - 0.5k)a}{\delta h} \sin \theta \right)}, \quad I = \sqrt{-1}$$

i.e.  $|G_1| < 1, \forall \varepsilon$  and  $|G_2| < 1, \forall \varepsilon < 0$  consequently the considered methods (16) and (20) are unconditionally stable.

#### 5. The Local Truncation Error Upper Bound

##### i-) For the RPA[0/1]

Using Taylor expansion, we can obtain the local truncation error of the difference equation (16) as done in [5],[6],[7],[8] and [9] as:

$$T_{i,j} = \frac{(\varepsilon - k)a}{\delta h} \sum_{n=0}^{\infty} \left( \frac{h^{2n+1}}{(2n+1)!} \frac{\partial^{2n+1} u}{\partial x^{2n+1}} \right)_{(i,j)} + \sum_{n=1}^{\infty} \left( \frac{k^n}{n!} \frac{\partial^n u}{\partial t^n} \right)_{(i,j)} - \frac{(\varepsilon - k)a}{\delta h} \sum_{m=0}^{\infty} \frac{h^{2m+1}}{(2m+1)!} \sum_{n=1}^{\infty} \left( \frac{k^n}{n!} \frac{\partial^{2m+n+1} u}{\partial x^{2m+1} \partial t^n} \right)_{(i,j)}$$

Then, if there exists a positive real numbers  $M_1, M_2, M_3$  for all sufficiently large positive integer  $n$  such that

$$\left| \frac{\partial^n u}{\partial x^n} \right| \leq M_1, \quad \left| \frac{\partial^n u}{\partial t^n} \right| \leq M_2, \quad \left| \frac{\partial^{m+n} u}{\partial x^m \partial t^n} \right| \leq M_3 \quad \forall n.$$

Let  $M = \max \{M_1, M_2, M_3\}$

then, the local truncation error  $T_{i,j}$  of the difference equation (16) will have an upper bound as:

$$|T_{i,j}| \leq M \left| e^k \left( 1 + \frac{(\varepsilon - k)a}{\delta h} \sinh h \right) - 1 \right|.$$

##### i- For the RPA[1/1]

Also, in the same way using Taylor expansion, we can obtain the local truncation error of the difference equation (20) as:

$$T_{i,j} = (r_1 - r_2) \sum_{n=0}^{\infty} \left( \frac{h^{2n+1}}{(2n+1)!} \frac{\partial^{2n+1} u}{\partial x^{2n+1}} \right)_{(i,j)} + \sum_{n=1}^{\infty} \left( \frac{k^n}{n!} \frac{\partial^n u}{\partial t^n} \right)_{(i,j)} - r_1 \sum_{m=0}^{\infty} \frac{h^{2m+1}}{(2m+1)!} \sum_{n=1}^{\infty} \left( \frac{k^n}{n!} \frac{\partial^{2m+n+1} u}{\partial x^{2m+1} \partial t^n} \right)_{(i,j)},$$

where  $r_1 = \frac{(\varepsilon - 0.5k)a}{\delta h}$ ,  $r_2 = \frac{(\varepsilon + 0.5k)a}{\delta h}$ , then if there exists a positive real numbers  $M_1, M_2, M_3$  for all sufficiently large positive integer  $n$  such that

$$\left| \frac{\partial^n u}{\partial x^n} \right| \leq M_1, \quad \left| \frac{\partial^n u}{\partial t^n} \right| \leq M_2, \quad \left| \frac{\partial^{m+n} u}{\partial x^m \partial t^n} \right| \leq M_3 \quad \forall n.$$

and  $M = \max \{M_1, M_2, M_3\}$ ,

then, the local truncation error  $T_{i,j}$  of the difference equation (20) will have an upper bound as:

$$|T_{i,j}| \leq M |(r_1 e^k - r_2) \sinh h + e^k - 1|,$$

### 6. Numerical Results

We present a numerical example to compare the considered methods (16) and (20) with the corresponding classical methods which arising of applying Pade' approximations [0/1] and [1/1] respectively to the exponential matrix  $\exp(kA)$  in equation (6).

Consider the singularly perturbed hyperbolic partial differential equation

$$\delta \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 2x - 1,$$

with the initial and boundary conditions

$$u(x,0) = \exp(-t/\delta), \quad u(x,0) = \exp(x) + x^2 - x \text{ and } u(x,1) = \exp(x - 1/\delta) + x^2 - x$$

its exact solution is:

$$u(x,t) = \exp(x - t/\delta) + x^2 - x.$$

We consider two cases:

**i-) Case I :** We apply our methods such that the exact solution is given at the first level to determine the restrictive parameters  $\varepsilon_i$ , and hence we use it for another levels for calculation. Tables (1) gives the absolute errors along  $x=0.1, 0.5, 0.9$  where  $h=0.1$  and  $\delta = 0.01, k=0.001$ . Tables (2) gives the absolute errors along  $x=0.1, 0.5, 0.9$  where  $h=0.1$  and  $\delta = 0.001, k=0.00001$ .

**ii-) Case II :** In general the exact solution at the first level is unknown, and we use the classical method in the case of PA[1/1], to evaluate the solutions at the first time level by large number of very small space and time steps lengths  $h=0.01$  and  $k=1 \times 10^{-10}$ , after 100 time step,  $k=1 \times 10^{-8}$  and we can choice space step  $h=0.1$ , hence we determine the restrictive parameters  $\varepsilon$  i.e. we can use large space and time steps lengths  $h$  and  $k$  to evaluate the solution at another levels. Tables (3) give the absolute errors along  $x=0.1, 0.5, 0.9$  where  $h=0.1$  and  $\delta = 0.0001, k=1 \times 10^{-8}$ .

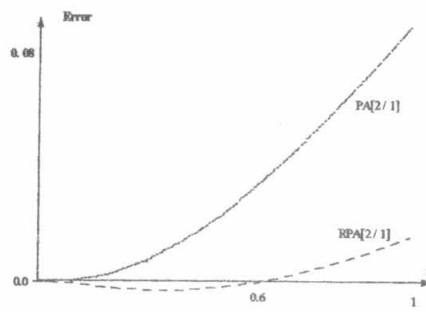


Fig (1)  
Comparison of the errors between PA[2 / 1] and RPA[2 / 1]

Table (1)  
Comparison of the absolute errors (A.E.) between the considered methods RPA[1/1], RPA[0/1] and The classical methods PA[1/1], PA[0/1], for  $h=0.1$ ,  $k=1 \times 10^{-3}$  and  $\delta=1 \times 10^{-2}$  for case I.

x	No. of Steps	A. E. of The considered methods		A. E. of The classical methods	
		RPA[1/1]	RPA[0/1]	PA[1/1]	PA[0/1]
0.1	500	$4.30 \times 10^{-16}$	$2.44 \times 10^{-15}$	$2.73 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.5		$4.44 \times 10^{-16}$	$2.38 \times 10^{-15}$	$3.73 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.9		$2.77 \times 10^{-16}$	$2.38 \times 10^{-15}$	$5.32 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.1	1000	$1.92 \times 10^{-15}$	$2.44 \times 10^{-15}$	$4.55 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.5		$1.83 \times 10^{-15}$	$2.38 \times 10^{-15}$	$4.04 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.9		$1.87 \times 10^{-15}$	$2.38 \times 10^{-15}$	$2.93 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.1	1500	$3.63 \times 10^{-15}$	$2.44 \times 10^{-15}$	$3.98 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.5		$3.66 \times 10^{-15}$	$2.38 \times 10^{-15}$	$3.70 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.9		$3.44 \times 10^{-15}$	$2.38 \times 10^{-15}$	$4.40 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.1	2000	$4.19 \times 10^{-15}$	$2.44 \times 10^{-15}$	$2.79 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.5		$4.27 \times 10^{-15}$	$2.38 \times 10^{-15}$	$4.07 \times 10^{-3}$	$7.38 \times 10^{-2}$
0.9		$4.48 \times 10^{-15}$	$2.38 \times 10^{-15}$	$4.72 \times 10^{-3}$	$7.38 \times 10^{-2}$

Table (2)  
Comparison of the absolute errors (A.E.) between the considered methods RPA[1/1], RPA[0/1] and The classical methods PA[1/1], PA[0/1], for  $h=0.1$ ,  $k=1 \times 10^{-5}$  and  $\delta=1 \times 10^{-3}$  for case I.

x	No. of Steps	A. E. of The considered methods		A. E. of The classical methods	
		RPA[1/1]	RPA[0/1]	PA[1/1]	PA[0/1]
0.1	500	$8.68 \times 10^{-15}$	$1.35 \times 10^{-14}$	$1.78 \times 10^{-3}$	$4.03 \times 10^{-3}$
0.5		$9.82 \times 10^{-15}$	$1.37 \times 10^{-14}$	$2.81 \times 10^{-3}$	$5.60 \times 10^{-3}$
0.9		$1.12 \times 10^{-14}$	$1.51 \times 10^{-14}$	$3.91 \times 10^{-3}$	$7.25 \times 10^{-3}$
0.1	1000	$1.21 \times 10^{-14}$	$1.07 \times 10^{-15}$	$4.03 \times 10^{-3}$	$6.99 \times 10^{-3}$
0.5		$1.01 \times 10^{-14}$	$8.52 \times 10^{-15}$	$2.73 \times 10^{-3}$	$5.63 \times 10^{-3}$
0.9		$8.99 \times 10^{-15}$	$7.97 \times 10^{-15}$	$1.82 \times 10^{-3}$	$4.37 \times 10^{-3}$
0.1	1500	$8.54 \times 10^{-15}$	$2.94 \times 10^{-15}$	$1.69 \times 10^{-3}$	$4.62 \times 10^{-3}$
0.5		$1.06 \times 10^{-14}$	$4.19 \times 10^{-15}$	$2.96 \times 10^{-3}$	$5.67 \times 10^{-3}$
0.9		$1.03 \times 10^{-14}$	$4.12 \times 10^{-15}$	$1.88 \times 10^{-3}$	$6.66 \times 10^{-3}$
0.1	2000	$1.11 \times 10^{-15}$	$3.60 \times 10^{-16}$	$3.81 \times 10^{-3}$	$6.39 \times 10^{-3}$
0.5		$9.90 \times 10^{-15}$	$1.60 \times 10^{-15}$	$2.84 \times 10^{-3}$	$5.65 \times 10^{-3}$
0.9		$9.02 \times 10^{-15}$	$2.51 \times 10^{-15}$	$1.88 \times 10^{-3}$	$4.92 \times 10^{-3}$

Table (3)  
Comparison of the absolute errors (A.E.) between the considered methods RPA[1/1], RPA[0/1] and The classical methods PA[1/1], PA[0/1], for  $h=0.1$ ,  $k=1 \times 10^{-8}$  and  $\delta=1 \times 10^{-4}$

x	No. of Steps	A. E. of The considered methods				A. E. of The classical methods	
		Case I		Case II		PA[1/1]	PA[0/1]
		RPA[1/1]	RPA[0/1]	RPA[1/1]	RPA[0/1]		
0.1	500	$2.7 \times 10^{-14}$	$4.2 \times 10^{-14}$	$7.7 \times 10^{-7}$	$7.7 \times 10^{-7}$	$7.7 \times 10^{-5}$	$7.5 \times 10^{-5}$
0.5		$3.5 \times 10^{-14}$	$4.6 \times 10^{-14}$	$1.3 \times 10^{-6}$	$1.3 \times 10^{-6}$	$1.3 \times 10^{-4}$	$1.2 \times 10^{-4}$
0.9		$2.7 \times 10^{-14}$	$1.1 \times 10^{-13}$	$2.2 \times 10^{-6}$	$2.2 \times 10^{-6}$	$2.2 \times 10^{-4}$	$2.1 \times 10^{-4}$
0.1	1000	$9.4 \times 10^{-14}$	$6.5 \times 10^{-14}$	$1.2 \times 10^{-6}$	$1.2 \times 10^{-6}$	$1.2 \times 10^{-4}$	$1.2 \times 10^{-4}$
0.5		$6.7 \times 10^{-14}$	$1.2 \times 10^{-13}$	$2.4 \times 10^{-6}$	$2.4 \times 10^{-6}$	$2.4 \times 10^{-4}$	$2.4 \times 10^{-4}$
0.9		$2.4 \times 10^{-14}$	$3.2 \times 10^{-13}$	$4.7 \times 10^{-6}$	$4.7 \times 10^{-6}$	$4.7 \times 10^{-4}$	$4.6 \times 10^{-4}$
0.1	1500	$2.3 \times 10^{-14}$	$7.4 \times 10^{-14}$	$1.5 \times 10^{-6}$	$1.5 \times 10^{-6}$	$1.5 \times 10^{-4}$	$1.5 \times 10^{-4}$
0.5		$8.3 \times 10^{-14}$	$2.0 \times 10^{-13}$	$3.5 \times 10^{-6}$	$3.5 \times 10^{-6}$	$3.5 \times 10^{-4}$	$3.4 \times 10^{-4}$
0.9		$1.3 \times 10^{-13}$	$5.9 \times 10^{-13}$	$7.4 \times 10^{-6}$	$7.4 \times 10^{-6}$	$7.4 \times 10^{-4}$	$7.2 \times 10^{-4}$
0.1	2000	$6.8 \times 10^{-14}$	$6.9 \times 10^{-14}$	$1.6 \times 10^{-6}$	$1.6 \times 10^{-6}$	$1.6 \times 10^{-4}$	$1.6 \times 10^{-4}$
0.5		$8.1 \times 10^{-14}$	$2.8 \times 10^{-13}$	$4.5 \times 10^{-6}$	$4.5 \times 10^{-6}$	$4.5 \times 10^{-4}$	$4.3 \times 10^{-4}$
0.9		$3.0 \times 10^{-13}$	$9.2 \times 10^{-13}$	$1.0 \times 10^{-5}$	$1.0 \times 10^{-5}$	$1.0 \times 10^{-3}$	$1.0 \times 10^{-3}$

**Conclusion**

The numerical results presented in case I in each of tables (1), (2) and (3) shows that the absolute errors obtained by the considered methods is almost of order  $10^{-10}$  of that absolute errors obtained by the classical methods. Also, the numerical results presented in case II in table (3) shows that the best absolute errors estimation for the classical methods is not better than  $10^{-5}$ , while for the



considered methods the maximum absolute errors estimation dose not exceed that  $10^{-5}$ .

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