Military Technical College, Kobry El-Kobbah, Cairo, Egypt

9th International Conference On Aerospace Sciences & Aviation Technology

LIQUID SLOSHING DYNAMICS WITH AEROSPACE APPLICATIONS

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ABSTRACT

Liquid containers constitute major components in a number of dynamical systems such as aerospace vehicles, road tankers, liquefied natural gas carriers, and elevated water towers. The dynamic behavior of these systems is greatly affected by the dynamics of the free liquid surface. The basic problem of liquid sloshing involves the estimation of hydrodynamic pressure distribution, forces, moments and natural frequencies of the free liquid surface. These parameters have direct effect on the dynamic stability and performance of moving containers. The aerospace technology has promoted the research activities in many problems pertaining to liquid sloshing and the special NASA research monograph edited by Abramson [1] documents these problems. Recently, Ibrahim, et al, [2] presented an extensive review of recent advances in liquid sloshing dynamics and the present paper is an abridged form. A liquid free surface in partially filled containers can experience a wide spectrum of motions such as planar, non-planar, rotational, quasi-periodic, chaotic, and disintegration. Since the early 1960's, the problem of liquid sloshing dynamics has been of major concern to aerospace engineers studying the influence of liquid propellant sloshing on the flight performance of jet vehicles. Since then, new areas of research activities have emerged. The modern theory of nonlinear dynamics has indeed promoted further studies and uncovered complex nonlinear phenomena. These include rotary sloshing, Faraday waves, nonlinear liquid sloshing interaction with elastic structures, internal resonance effects, stochastic sloshing dynamics, hydrodynamic sloshing impact dynamics, g-jitter under microgravity field, cross-waves, and spatial resonance. The dynamic stability of liquid gas tankers and ship cargo tankers, and liquid hydrodynamic impact loading are problems of current interest to the designers of such systems.

FREE AND FORCED FREE-SURFACE MOTIONS

Fluid Field Equations

The general equations of motion for a fluid in closed containers can be simplified by assuming that the container is rigid and impermeable, and the fluid is inviscid, incompressible, and initially irrotational. Capillary or surface tension effects will be

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ignored in a gravitational field. The tank may be displaced along some trajectory in space. It is convenient to refer the fluid motion to a moving coordinate system as the variables are measured by a measuring device that is moving relative to the inertial frame. It is useful to write the fluid equations of motion with reference to the stationary and moving coordinates as shown in Fig. 1. The tank is allowed to move in planar curvilinear motion without rotation in the *X-Z* plane. Let *O'X'Y'Z'* be the stationary Cartesian coordinate inertial reference frame. For irrotational fluid motion there exists a velocity potential function, Φ, whose negative gradient gives the fluid velocity,

$$q = -\nabla\Phi \tag{1}$$

where $\nabla = \frac{\partial}{\partial r}\boldsymbol{i}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{i}_\theta + \frac{\partial}{\partial z}\boldsymbol{i}_z$ for cylindrical coordinates or $\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$ for Cartesian coordinates, and $\boldsymbol{q}$ is the fluid vector velocity. Let *Oxyz* be another coordinate frame fixed to the tank such that the *Oxy*-plane coincides with the undisturbed free surface. Let $\boldsymbol{V_0}$ be the velocity of the origin *O* relative to the fixed origin *O'*. The fluid particle velocity $\boldsymbol{q}_{rel}$ relative to the moving coordinate is

$$q_{rel} = q - V_0 = -\nabla\Phi - V_0 \tag{2}$$

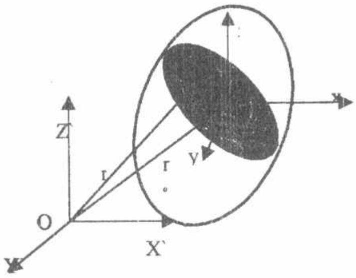

Fig. 1. Schematic diagram of a liquid tank and the coordinate frames

The velocity potential function Φ should satisfy Laplace's (continuity) equation,

$$\nabla^2\Phi = 0 \tag{3}$$

The velocity of the tank

$$V_0 = \dot{X}_0\boldsymbol{i} + \dot{Z}_0\boldsymbol{k} \tag{4}$$

can also be expressed in terms of cylindrical coordinates as

$$V_0 = \left(\dot{X}_0 \cos\theta\right)\boldsymbol{i}_r - \left(\dot{X}_0 \sin\theta\right)\boldsymbol{i}_\theta + \dot{Z}_0\,\boldsymbol{i}_z \tag{5}$$

In this case, one can obtain the Kelvin's equation for an unsteady flow

$$\frac{P}{\rho}+\frac{1}{2}(\nabla\Phi.\nabla\Phi)+\left(g+\ddot{Z}_0\right)z+\ddot{X}_0 r\cos\theta-\frac{\partial\Phi}{\partial t}=C(t) \tag{6}$$

Equation (6) is the fluid field equation referred to the moving coordinate system with acceleration components $\ddot{X}_0$ and $\ddot{Z}_0$. The complete solution of equation (3) must satisfy the relevant boundary conditions of the problem. These conditions are:

i. At the wetted rigid wall and bottom, the velocity component normal to the boundary must vanish.
ii. At the free surface, the pressure is zero, which gives the dynamic free surface condition, and is obtained from equation (6) after setting $P=0$.
iii. The vertical velocity of a fluid particle located on the free surface should equal the vertical velocity of the free surface itself. This condition is known as the kinematic free surface condition.

The solution of such boundary value problem for a rectangular tank is

$$\Phi(x,y,z,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\alpha_{mn}(t)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cosh[k_{mn}(z+h)] \tag{7a}$$

where a and b are the tank width and breadth, respectively, and $k_{mn}=\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$, m and n are positive integers.

For an upright circular container the velocity potential function takes the form:

$$\Phi(r,\theta,z,t)=\sum_{m=0}^{\infty}.\sum_{n=1}^{\infty}\alpha_{mn}(t)J_m(\lambda_{mn}r)\cos m\theta\frac{\cosh[\lambda_{mn}(z+h)]}{\cosh\lambda_{mn}h} \tag{7b}$$

where $J_m(.)$ is the Bessel function of the first kind of order m, $\lambda_{mn}=\xi_{mn}/R$ are the roots of $\partial J_m(\lambda_{mn}r)/\partial r\,|_{r=R}=0$. A series expansion for the free surface elevation η can be written for the rectangular tank,

$$\eta(x,y,t)=\sum_{m=0}^{\infty}.\sum_{n=1}^{\infty}A_{mn}(t)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \tag{8a}$$

and for the circular tank,

$$\eta(r,\theta,t)=\sum_{m=0}^{\infty}.\sum_{n=1}^{\infty}A_{mn}(t)J_m(\lambda_{mn}r)\cos m\theta \tag{8b}$$

The generalized coordinates, α_{mn} and A_{mn}, are time dependent, obtained by satisfying the free surface boundary conditions. Note that the time dependent of the wave height results in variation of the position of the vehicle center of mass. Introducing expressions (7) and (8) in the nonlinear fluid free surface boundary conditions results in a major difficulty due to the high degree nonlinearity of different modes. In order to get quantitative information, it appears essential to introduce approximation in the free surface equations. There are some methods used for treating the liquid surface amplitude in rectangular and circular cylindrical containers and three of the main theories are briefly summarized:

- Moiseev's Theory [3] constructs normal mode functions and characteristic numbers by integral equations in terms of Green's function of the second kind.
- Penny and Price [4] carried out a successive approximation approach where the potential function was expressed as a Fourier series in space with coefficients that are functions of time. These coefficients were again approximated by Fourier time series using the method of perturbation. The resulting solution was given as a double Fourier series in space and time.
- Hutton's theory [5] expanded the dynamic and kinematic free surface equations in Taylor series about a stationary surface position.

Note that the analysis can significantly be simplified if the fluid field equations are linearized for small displacements. In this case, one can determine all dynamical parameters such as natural frequencies, mode shapes, hydrodynamic pressure, and sloshing forces and moments. Hydrodynamic pressure can be estimated in terms of the potential function using equation (6). The normal mode frequencies can be determined from the linearized free surface boundary condition

$$\frac{\partial \tilde{\Phi}}{\partial t} + g\eta = 0 \tag{9}$$

Substituting (7) and (8) into (9) gives the natural frequencies in a rectangular tank

$$\omega_{mn}^2 = g k_{mn} \tanh(k_{mn} h) \tag{10a}$$

and in a cylindrical container

$$\omega_{mn}^2 = \frac{g\xi_{mn}}{R} \tanh(\xi_{mn} h / R) \tag{10b}$$

For a circular cylindrical tank, under forced sinusoidal excitation $x(t) = X_0 \cos \Omega t$ along $\theta = 0$, the total force exerted by the fluid on the tank walls is estimated by integrating the hydrodynamic pressure over the wetted area. The force along x-axis is

$$F_x(t) = M X_0 \Omega^2 \sin \Omega t \left\{ 1 + \sum \frac{R/h}{\xi_{mn}} \frac{\Omega^2}{(\omega_{mn}^2 - \Omega^2)} \frac{2 \tanh(\xi_{mn} h / R)}{(\xi_{mn}^2 - 1)} \right\} \tag{11}$$

where M is the total mass of the liquid. The liquid sloshing force given by expression (11) is accurate for excitation frequencies not close to the liquid natural frequency. For excitations near resonance, nonlinear analysis should be performed to calculate the hydrodynamic forces.

I.2 Free Sloshing

The modal analysis of free liquid oscillations in a partially filled container determines the free surface natural frequencies and the corresponding free surface mode shapes. These parameters are essential in the design process of liquid tanks and in implementing active control systems in space vehicles. It has been noted that when the tank is very deep, the bottom shape has no influence on the free liquid surface motion and the bottom can be treated as flat based on equal liquid volumes. On the other hand, for extremely shallow liquid depths, the bottom shape governs the fluid motion and the problem is reduced to two-dimensional.

Because of the complexity of liquid sloshing in sector tanks with perforated walls, the natural frequencies were measured experimentally. Note that if the tank is under lateral excitation, the resonance frequency is significantly affected by the excitation amplitude due to liquid intermixing from one sector to another. Experimental investigations on liquid tanks with perforated baffles showed a decrease of resonance frequencies as the size of perforated holes increases. It was also shown that the values of the resonance frequency could be maintained equal to or greater than that of solid wall sector up to a Reynolds number of 50,000 for the 23% open area sector wall, and to Reynolds number of 20,000 for the 30% open area sector wall. Above these numbers, the resonance frequencies drop to experimental values for an uncompartmented cylindrical tank.

It was observed that the liquid resonance frequencies are dependent on the ring baffle area and its location below the free surface. For ring baffles having a width-to-tank-radius ratio of 0.157, the liquid resonance frequency exhibits a maximum value when the baffle is located at the free liquid surface. It then decreases to a minimum value near a baffle depth of one tenth of the tank radius. For perforated baffles, the frequency was found to increase as the percentage of the perforated area is increased. The influence of damping on the natural frequency was found that for higher viscosities the resonance frequency is slightly higher than the predicted value for an ideal liquid.

Forced Sloshing

The dynamic behavior of a free liquid surface depends on the type of excitation and its frequency content. In the design process, it is important to keep the liquid natural frequencies away from all normal and nonlinear resonance conditions. The excitation can be impulsive, sinusoidal, periodic and random. Its orientation with respect to the tank can be lateral, parametric, pitching/ yaw or roll and a combination.

The dynamic sloshing loads are of great importance on the stability and trajectories of liquid propellant rockets. The influence of liquid sloshing loads on the stability of aerospace vehicles was studied by Bauer (Chapter 7 in reference [1]).

Abramson [1] documented analytical and measured values of the pressure distribution, net horizontal force and moment for steady state horizontal and pitch excitation of an upright circular cylindrical tank. Under lateral harmonic excitation, the free liquid surface may exhibit two types of nonlinearities. The first is large amplitude response, and the second involves different forms of liquid behavior produced by coupling or instabilities of various sloshing modes. The most important of these is the rotary sloshing or swirl motion. This type of motion usually occurs very near the lowest liquid natural frequency [5,6]. Hutton [5] reported three types of fluid motion in circular cylindrical tanks: stable planar, stable non-planar, and unstable motion near resonance. Stable planar motion is a steady-state liquid motion with a constant peak wave height with a stationary single nodal diameter perpendicular to the direction of excitation. Stable non-planar motion is a steady-state liquid motion with a constant peak wave height with a single nodal diameter that rotates at a constant rate around the tank vertical axis. This motion occurred primarily above the natural frequency of the free surface. Unstable motion never attains a steady-state harmonic response.

Koval'chuk and Podchasov [7] considered the regular traveling wave modes of a liquid under quasi-stationary vibrations of its container. The excitation frequency was allowed to slowly increase with time, passing through the resonance zone. They expressed the free surface wave height in the form

$$\eta(r,\theta,t) = a(t)\sin[\theta + \alpha(t)]R_1(r) \qquad (12)$$

where $a(t)$ and $\alpha(t)$ are the amplitude and phase, both are unknown functions of time. $R_1(r)$ is the eigenfuction of the homogeneous boundary-value problem

$$\frac{d^2R_1(r)}{dr^2} + \frac{1}{r}\frac{dR_1(r)}{dr} + \left(\lambda_1^2 - \frac{1}{r^2}\right)R_1(r) = 0 \qquad (13a)$$

$$\left.\frac{dR_1(r)}{dr}\right|_{r=r_0} = 0, \quad R_1(r)\big|_{r=0} = \infty \qquad (13b)$$

where r_0 is the tank radius.

Koval'chuk and Kubenko [8] studied the free surface motion under translational harmonic excitation, $s(t) = \delta \cos \Omega t$, $(\delta, \Omega = const.)$, along the lateral horizontal axis. They obtained the following coupled differential equations of the amplitude and phase

$$\ddot{a} + \left(\omega^2{}_1 - \dot{\alpha}^2\right)a = G_1(a,\dot{a},\alpha) + f(t)\cos\alpha$$
$$\alpha\ddot{\alpha} + 2\dot{a}\dot{\alpha} = G_2(a,\dot{a},\alpha) - f(t)\sin\alpha \qquad (14)$$

where

$$G_1 = b_1a\left(\dot{a}^2 - \omega^2{}_1a^2\right) + (b_1 - 2b_2)a^3\dot{\alpha}^2, \quad G_2 = 2b_2a^2\dot{a}\dot{\alpha}, \quad \omega_1^2 = g\frac{\lambda_1}{r_0}\tanh\frac{\lambda_1 h}{r_0}, \quad f(t) = \gamma\ddot{s}$$

By making use of a special coordinate, they transformed the differential equations of motion into the standard system of first order differential equations. It was shown that the traveling waves propagating in the circular direction are possible for the detuning parameter $\Delta = \omega_1^2 - \Omega^2$, which is below some critical value. As the detuning parameter is increased, the motion of the liquid surface becomes "chaotic." Further increase of the parameter Δ leads to the standing wave vibration. In order to investigate the case of passing through the resonance zone, they assumed the frequency of the container to be a slowly varying function of time, $\Omega = \Omega_0 + \varepsilon t$, where ε is a relatively small parameter, and $\Omega_0 = 0.9\omega_1$. Their numerical results showed that the maximum amplitude decreases to a minimum value $a_{min}(\varepsilon)$, which is reached at some time instance, $T(\varepsilon)$.

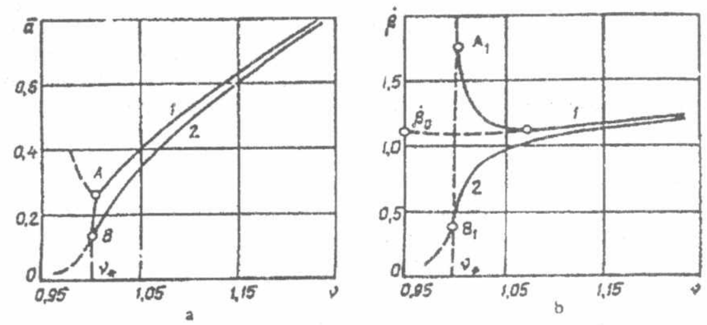

Fig. 2. Amplitude (a) and phase (b) frequency response curves near resonance [7]

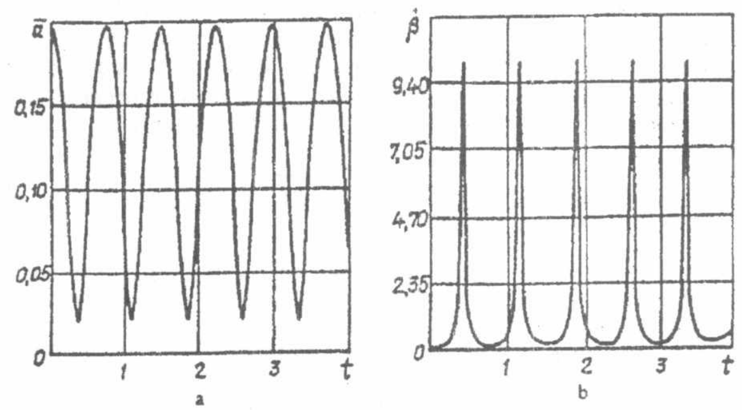

Fig. 3. Amplitude (a) and phase (b) time history records near resonance during rotational sloshing [7]

Fig. 2 shows the dependence of the circular wave maximum amplitude on the frequency parameter $\nu = (\Omega_0 + \varepsilon t)/\omega_1$. It is seen that the amplitude increases monotonically with the external excitation frequency Ω and reaches its minimum value at the time of exact resonance $\nu = 1$. With increasing , (starting from $\Omega = \omega_1$), the maximum phase velocity

$\dot{\beta} = \dot{\alpha}/\omega_1$ first decreases to a minimum $\dot{\beta}_0$ (which occurs at $(v_c = 0.95$ and 1.07) after which it begins to increase slightly. Note that curves 1 correspond to the maximum steady-state amplitudes and phase velocities, while curves 2 to their minimum values.
Free vibration ($E = 0$) with initial conditions $a(0) = 0.2r_0$, $\alpha(0) = 0$, and $\dot{\alpha}(0) = 0$, the amplitude and phase velocity time evolutions are periodic as shown in Fig. 3. It is seen that both the amplitude and phase velocity are exchanging energy. The phase velocity assumes only positive values indicating that the liquid "rotates" in one direction compatible with the initial condition. The displacement of the wave crest in the circular direction corresponds to a "pulse" mode. When the wave amplitudes are small, the position of the crest along the circular coordinate θ changes abruptly and the wave motion is "retarded". These features also occur in the case of quasi-periodic motion of the liquid container.

Parametric Sloshing

By setting $\ddot{X}_0 = 0$ and $\ddot{Z}_0$ as a sinusoidal function, in equation (6), and after linearization , their analysis led to a system of Mathieu equations in the form

$$\frac{d^2 A_{mn}}{dt^2} + 2\zeta_{mn}\omega_{mn}\frac{dA_{mn}}{dt} + \omega_{mn}^2\left(1 - 2Z_0 \cos\Omega t\right)A_{mn} = 0 \qquad (15)$$

where A_{mn} is nondimensional wave height amplitude, ω_{mn} and ζ_{mn} are the corresponding natural frequency and damping ratio of mode mn, respectively, Ω is the excitation frequency, Z_0 is the excitation amplitude parameter. Depending on the excitation amplitude, frequency, and damping ratio, the solutions of equation (15) can be stable or unstable. The regions of instability are given by the inequality

$$1 - \sqrt{Z_0^2 - 4\zeta_{mn}^2\omega_{mn}^2} < \left(\frac{\Omega}{2\omega_{mn}}\right)^2 < 1 + \sqrt{Z_0^2 - 4\zeta_{mn}^2\omega_{mn}^2} \qquad (16)$$

It can be shown that if the plane free surface is unstable, the resulting motion could have frequency $k/2$ times the excitation frequency, where k is an integer. Recent studies reported complex free liquid surface motions, which occur in the presence of nonlinear resonance conditions such as *internal resonance* and parametric resonance conditions. Internal resonance implies the presence of a linear algebraic relationship among the natural frequencies of the interacted modes.
Under parametric harmonic excitation, chaotic sloshing was experimentally observed by Ciliberto and Gollub [9]. They conducted a series of experimental investigations on a fluid layer in a circular tank with depth-to-radius ratio of 0.16. For such fluid depth, the sloshing natural frequency is strongly dependent on the fluid depth. They measured the regions of parametric instability of two sloshing modes. Above the stability boundaries, the fluid surface oscillates at half the driving frequency in a single stable mode. However, another region of mode competition emerged in which the fluid surface can be described as a superposition of the two modes with amplitudes having a slowly varying

envelope. These slow variations can be either periodic or chaotic. At the intersection of the two stability boundaries, both modes oscillate simultaneously.
The stochastic stability of a liquid surface under random parametric excitation can be studied in terms of one of the stochastic modes of convergence. These modes include convergence in probability, convergence in the mean square and almost sure convergence [10]. The linear stability analysis is based on the stochastic differential equation of the sloshing mode *mn*, i.e.,

$$\ddot{A}_{mn} + 2\zeta_{mn}\dot{A}_{mn} + \left[1 + \ddot{\xi}(\tau)\right] A_{mn} = 0 \tag{17}$$

where A_{mn} is a dimensionless free liquid surface amplitude of mode mn, a prime denotes differentiation with respect to the nondimensional time parameter $\tau = \omega_{mn} t$, ω_{mn} is the natural frequency of the sloshing mode mn, ζ_{mn} is the corresponding damping ratio, and $\ddot{\xi}(\tau)$ is a dimensionless vertical random acceleration of spectral density $2D$. Mitchell [11] determined the mean square stability condition of the response of equation (17), which is given by the inequality

$$D / 2\zeta_{mn} < 1 \tag{18a}$$

On the other hand, the sample stability condition is

$$D / 2\zeta_{mn} < 2 \tag{18b}$$

The nonlinear motion of the free liquid surface under random parametric excitation involves the estimation of stochastic stability and response statistics of the free surface [12-14]. The free liquid surface height of a sloshing mode mn in a cylindrical container was found to be governed by the nonlinear differential equation

$$\ddot{A}_{mn} + 2\zeta_{mn}\dot{A}_{mn} + \left[1 + \ddot{\xi}(\tau)\right] A_{mn}(1 - K_1 A_{mn} - K_2 A_{mn}^2) + K_3 \dot{A}_{mn}^2 + K_4 A_{mn}\ddot{A}_{mn} + K_5 A_{mn}\dot{A}_{mn}^2 + K_6 A_{mn}^2 \ddot{A}_{mn} = 0 \tag{19}$$

The last four terms in equation (19) represent quadratic (for symmetric modes) and cubic (for asymmetric modes) inertia nonlinearities. Equation (19) represents the nonlinear modeling of any mode mn and does not include nonlinear coupling with other sloshing modes. Ibrahim and Heinrich [14] experimentally observed that the free liquid surface might follow one of the possible regimes:

i. *Zero free liquid surface motion*, which is characterized by a delta Dirac function of the response probability density function.
ii. *On-off intermittent motion* of the free liquid surface.
iii. *Partially developed random sloshing*. This regime is characterized by undeveloped sloshing where significant liquid-free surface motion occurs for a certain time-period and then ceases for another period. At higher excitation levels, the time-period of liquid motion exceeds the period of zero motion.

iv. *Fully developed sloshing* is characterized by continuous random liquid motion for all excitation levels exceeding the previous regime. When the first symmetric sloshing mode is excited, higher sloshing modes are excited as well.

Note that the excitation spectral level was limited to a lower level within a narrow-band in order to avoid parametric excitation with other modes. Mixed mode interaction under random excitation has not been treated in the literature. The measured probability density function of the liquid response was found to be non-Gaussian for regions of large subharmonic motion with non-zero mean.

EQUIVALENT MECHANICAL MODELS

The liquid dynamic pressure in moving rigid containers has two distinct components. One component is directly proportional to the acceleration of the tank and is caused by part of the fluid moving in unison with the tank. The second component is known as "convective" pressure and experiences sloshing at the free surface. This component can be modeled by a set of mass-spring-dashpot systems or by a set of pendulums as shown in Fig. 4. A realistic representation of the liquid dynamics inside closed containers can be approximated by an equivalent mechanical system. The technique of equivalent mechanical models is a very useful mathematical tool for solving the complete dynamic problem of a system containing liquid.

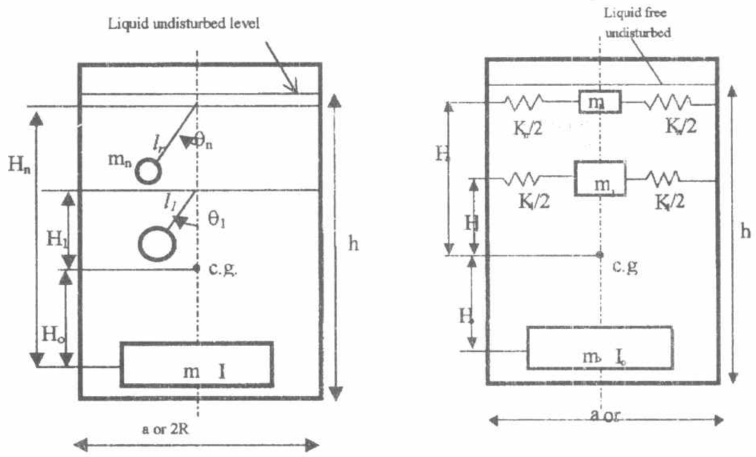

Fig. 4. Pendulum and mass-spring equivalent models

The model parameters are determined only from consideration of fluid motion in a stationary tank. The principals for constructing a mechanical model are based on the following conditions:

1. The equivalent masses and moments of inertia must be preserved.
2. The center of gravity must remain the same for small oscillations.

3. The system must possess the same modes of oscillations and produce the same damping forces.
4. The force and moment components under certain excitation of the model must be equivalent to that produced by the actual system.

For the first liquid sloshing mode, an equivalent pendulum may be used. Three dynamic regimes are possible, as shown in **Figure 4**:

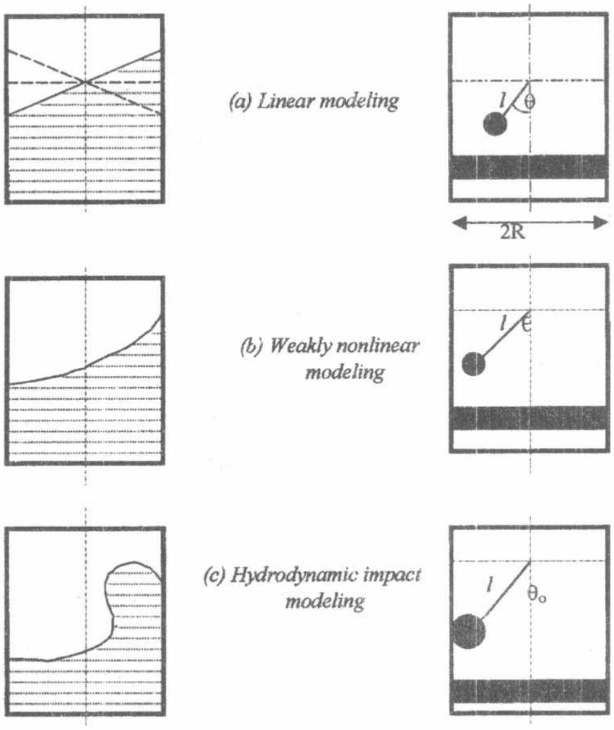

Figure 5. Regimes of free liquid surface motion and their equivalent mechanical modeling

(1) Small oscillations in which the fluid free surface remains planar without rotation of its nodal diameter (see Fig. 5(a)). This regime can be described by a linear equation for the first asymmetric mode, equivalent to a pendulum describing small oscillations with $\sin\theta \approx \theta$.

(2) Relatively large amplitude oscillations in which the free liquid surface experiences non-planar motion (see Fig. 5(b)). This regime is described by a differential equation with weak non-linearity and can be analyzed using the standard perturbation

techniques. The equivalent mechanical model is a simple pendulum describing relatively large motion such that $\sin\theta \approx \theta - \theta^3/3!$.

(3) Strongly nonlinear motion where the non-linearity is mainly due to rapid velocity changes associated with hydrodynamic pressure impacts of the liquid motion close to the free surface (see Fig. 5(c)). The velocity changes of the free liquid surface are usually treated as being instantaneous (velocity jumps) giving various strong non-linear features to the system behavior. The equivalent mechanical model of this regime is a pendulum describing impacts with the tank walls.

Expressions for sloshing forces and moments are available for a number of simple tank shapes such as rectangular, cylindrical, and ellipsoidal (see Chapter 6 in [1]). Since the fluid is not rigid the above relation overestimates the fluid mass moment of inertia. The actual moment of inertia can be obtained by determining the ratio of the actual liquid to rigid liquid moment of inertia for a cylindrical tank having an identical fluid height and fluid mass.

SLOSHING INTERACTION WITH ELASTIC STRUCTURES

The problem of liquid sloshing interaction with structural dynamics falls into one of the following categories:

1. Interaction between the free liquid surface motion and the breathing elastic modes of liquid container structure.
2. Interaction between liquid sloshing modes and the motion of the supporting elastic structure.
3. Liquid-structure interaction of immersed structures subjected to water waves. This class will not be addressed in this article.

Sloshing interaction with elastic container

In aerospace applications, the elastic container of liquid propellant can experience bending and breathing motions, which can couple with the propellant free surface motion. The combined liquid-structure system is very difficult to model and any analysis was based on some assumed simplifications. However, the interaction of the liquid sloshing dynamics with elastic deformations of the tank must be considered in studying the overall vehicle dynamics. Note the available multibody dynamic codes do not handle such interaction. Kana (Chapter 9 in [1]) presented an excellent overview of the interaction with elastic tank bending and breathing vibrations.

The influence of the elastic bottom on the liquid response is more significant as the tank diameter increases and as the bottom thickness decreases. The normal mode frequencies of the liquid-membrane bottom system resulted in an infinite order determinant for the coupled natural frequencies. The general trend was found such that the bottom elasticity tends to lower the liquid surface natural frequencies below their values for the case of rigid bottom. Within the framework of the linear theory of small oscillations, the liquid -elastic container coupling was studied to determine the natural frequencies and mode shapes.

The interaction of liquid motions with elastic tank breathing was considered by several investigators. Different forms for the frequency equation of the combined system were derived. The linear formulation was adequate as long as the free liquid surface did not exceed a certain range above which non-linearity became important. For a thin elastic tank wall, the fundamental axi-symmetric coupled frequency was much smaller than the liquid frequency with rigid walls. As the tank wall thickness increases the coupling effect diminishes and the liquid frequency approaches the case of a rigid tank.

Longitudinal excitation of elastic shells partially filled with liquid was considered in many studies [15,16]. For moderate and high frequency excitation, the dynamic response can follow one of the following scenarios:

i. Direct linear harmonic response in axi-symmetric modes occurs primarily as large pressure amplifications and accompanied by very small wall deflections.

ii. Response in non-axi-symmetric modes appears in the form of only small pressure amplitudes but with rather large harmonic wall motions.

iii. The dominant form of response occurs as parametric modes in which large symmetric harmonic pressure oscillations in the fluid are accompanied by large-amplitude ½-subharmonic shell wall motions in non-axi-symmetric modes. This nonlinear response results from instabilities of the linear response and is usually observed in shells with little circumferential stiffening.

The shell-liquid dynamic behavior is complex in nature and the reported results indicated that most parameters considered have a significant influence on initial-state axi-symmetric response, dynamic instability, and subsequent nonlinear responses. Boyarshina [17] examined the nonlinear interaction between liquid sloshing modes and a circular shell vibration in the neighborhood of three internal resonance cases. These are (i) the natural frequency of the shell is close to one of the sloshing natural frequencies, (ii) the natural frequency of the shell is twice the natural frequency of the sloshing modes, and (ii) a 3:1 internal resonance.

Sloshing interaction with an elastic support structure

This type of interaction takes place between the free liquid surface motion and the supported elastic structure dynamics based on the assumption that the liquid container is rigid. Under the base motion of the supporting structure, the fluid container experiences motion in a certain trajectory governed by the excitation and the liquid response. The free liquid surface motion will result in hydrodynamic forces that are fed back to the supporting structure. The nonlinear interaction in elevated water towers subjected to vertical sinusoidal ground motion was examined in the neighborhood of internal resonance [18-21]. In these studies, the free liquid surface sloshing modes and the elastic support structure were coupled through inertia nonlinearity, which results in internal resonance conditions among the interacting modes (i.e., $\sum k_j \omega_j = 0$, where k_j are integers and ω_j are the natural frequencies of the coupled modes). This type of coupling is referred to as *autoparametric interaction* when an externally excited mode can act as a parametric excitation to other modes. The problem of internal resonances in nonlinearly coupled oscillators is of interest in connection with redistribution of energy among the various natural modes. This energy sharing is usually brought about by resonant interactions among the natural modes of the system. The coupling among these modes plays a crucial role in such interactions. In a straightforward perturbation theory, internal resonances lead to the problem of small divisors.

Under the principal internal resonance condition (i.e., when one of the normal mode frequencies is twice one of the other mode frequency), the system possesses a steady-state response. Ibrahim and Barr [19] found that under the summed or difference internal resonance conditions (i.e., one of the normal mode frequencies equals the sum or difference of another two mode frequencies) the system does not achieve a constant steady-state response.

Nonstationary responses of cases including violent system motion, which can lead to collapse of the system, were reported in the neighborhood of multiple internal resonances [20]. The multiple internal resonances may occur when two or more sloshing modes are interacting with the vertical and horizontal motions of the structure. In the neighborhood of the summed internal resonance and one-to-one internal resonance, the structure and free liquid surface simultaneously oscillate with a continuous increase in their amplitude. This growth could lead to structural failure if the shaker excitation is not stopped. In the presence of one-to-two and one-to-one internal resonance conditions, experimental observations showed a steady-state response over a frequency range defined by the regions of instability. The regions of instability were indicated by the occurrence of collapse in response amplitudes. Another type of instability, manifested by a jump in amplitudes, was caused by a weak energy flow between the fluid modes and structure modes for a few cycles. Within a short period of time, the system achieves a steady state response.

Ibrahim and Li [22] studied liquid-structure interaction under horizontal periodic motion. Ikeda and Nakagawa [23,24] and Ikeda [25] considered the nonlinear interaction of liquid sloshing in rectangular and cylindrical tanks with an elastic structure whose motion is orthogonal to the tank vertical walls. They showed that the response frequency curves experience change from soft to hard response characteristics as the water depth decreases. Under vertical sinusoidal excitation of an elastic structure carrying a rigid rectangular tank, Ikeda [25] determined the response of the coupled system when the structure natural frequency is about twice the liquid sloshing frequency. As the excitation frequency approached the structure natural frequency the free liquid surface was excited through the autoparametric resonance and energy was transferred from the structure to the free liquid surface.

Soundararajan and Ibrahim [26] examined more realistic cases such as simultaneous random horizontal and vertical ground excitations in the presence of 1:3 internal resonance. They used a Gaussian and non-Gaussian closure schemes to determine the system response statistics. They found that both Gaussian and non-Gaussian solutions deviate appreciably from the linear solution as the system approaches internal resonance but they converge when the system is detuned away from the exact internal resonance. The autoparametric interaction was identified by an irregular energy exchange between the two modes.

SLOSHING HYDRODYNAMIC IMPACT

An impulsive acceleration to a liquid container can result in impact hydrodynamic pressure of the free liquid surface on the tank walls. It can also occur during maneuvering or docking of spacecraft in an essentially low gravity field. Methods for estimating liquid impact and the associated pressure are not well developed and are only identified by experimental studies. It was found that when hydraulic jumps or traveling waves are present, extremely high impact pressures can occur on the tank walls. The influence of fluid sloshing impact forms a serious problem in underground radioactive waste storage tanks when subjected to earthquakes. Some tanks were damaged at the roofs due to sloshing impact caused by strong earthquakes. The hydrodynamic pressure distribution of such impact loads is an important factor in studying the integrity of the tanks and resolving related safety problems.

Milgram 27] studied the sloshing impact pressure in roofed liquid tanks. Milgram carried out a series of experiments to distinguish some nonlinear sloshing phenomena in the reactor vessel of a pool type, which may cause damage to the vessel or inner structures. Test results for three types of models using a long period large amplitude shaking table provided information on how the scales and the configurations of the method affect the sloshing wave crest impact pressure. Minowa, et al. [28] conducted a series of shaking table tests of a rectangular tank to measure roof impact pressures, natural frequencies and modes of bulging vibrations. Their measured results showed that the roof impact pressures possess great potential damage to tank as the pressure reached as high as 30 *psi*, under 400 gal El-Centro seismic excitation. The liquid viscosity effects on sloshing response were found to be significant.

Sloshing impact loading cannot be viewed as a single loading event since it can be repeated due to the inertia and restoring forces. If the system is linear with constant coefficients and is subjected to impact loading it will experience non-linear behavior. Liquid pressure impacts are a source of strong nonlinearity in a liquid tank system. This non-linearity becomes clear when trying to exclude the impact finite relations by using the Dirac delta-functions in the equations of motion. This can be easily done for the case of linear equations of motion between the impacts. In this case, the complete equations of motion involve delta-function terms. A thorough description of various methods for the analysis of vibro-impact systems may be summarized by the following schemes:

i. Step-by-step integration method, which is also known as the point-wise mapping method.
ii. Approximate methods established for the theory of non-linear oscillations [29-31]. These methods include perturbation techniques, asymptotic approximation methods, and averaging methods.
iii. Non-smooth coordinate transformation originally proposed by Zhuravlev [32]. This transformation assumes rigid barriers and converts the vibro-impact system into an oscillator without barriers such that the equations of motion do not contain any impact terms. This technique has recently been used by Pilipchuk and Ibrahim [33] to analyze the dynamic response behavior of a system involving hydrodynamic sloshing impact.
iv. Saw-tooth-time-transformation (STTT) method developed for non-rigid barriers. This technique is based on a special transformation of time and gives explicit form of analytical solutions for the power non-linearities. The physical and mathematical principles of the STTT have been formulated by Pilipchuk [34]. This technique has also been used to analyze the response of systems involving liquid sloshing impact by Pilipchuk and Ibrahim [35].

Let m be the equivalent sloshing mass of the first asymmetric mode of the liquid. The free fluid surface is modeled as a pendulum of length ℓ. The pendulum can reach the walls of the tank if its angle with the vertical axis is $\theta = \pm\theta_0$. Considering the pendulum and the tank walls as rigid bodies, one must introduce the constraint $|\theta| \leq \pm\theta_0$. From the point of view of analytical techniques in nonlinear mechanics such constraints essentially complicate the analysis because one must match solutions at points of interaction $\{t : \theta(t) = \pm\theta_0\}$, which is a *priori* unknown. Hence, it is useful to avoid operations with constraints. One can phenomenologically describe the interaction between the pendulum and the tank walls with a special potential field of interaction, which is very weak in the region $|\theta| < \pm\theta_0$, but becomes fast growing in the neighborhood of the points $|\theta| = \pm\theta_0$. For example, the desirable properties of the potential field can be provided by means of the following potential energy function, [35],

$$\Pi_{impact} = \frac{b}{2n}\left(\frac{\theta}{\theta_0}\right)^{2n} \tag{20}$$

where n>>1, is a positive integer, and b is a positive constant parameter.

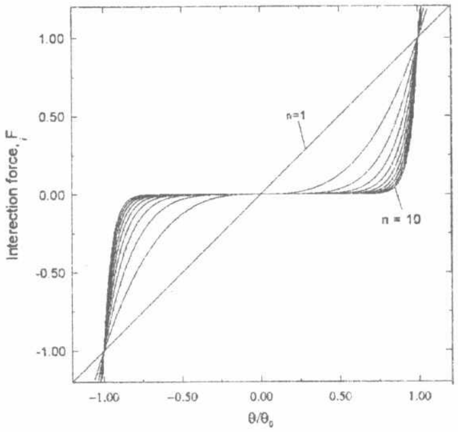

Fig. 6 Variation of liquid impact force between the tank walls for different values of the exponent n

The force of interaction is

$$F_{impact} = \frac{d\Pi_{impact}(\theta)}{d\theta} = b\left(\frac{\theta}{\theta_0}\right)^{2n-1} \tag{21}$$

One has a limit of absolutely rigid bodies interaction, if $n \to \infty$. For this case, the potential energy (20) takes the square well form. If the exponent *2n-1* is large and finite, then the interaction field is not absolutely localized at the points $\theta = \pm\theta_0$. This means that the tank walls and the pendulum mass are not rigid, but admit a small deformation about the points of contact $\theta = \pm\theta_0$. Accordingly, a finite value of n seems more realistic than the rigid body limit; see Fig. 6, yet the approach considered includes the rigid body limit as a particular case.

Suppose that the energy dissipation of the pendulum results from the pendulum interaction with the container walls. This means that the dissipation is spatially localized around the points $\theta = \pm\theta_0$. The localized dissipative force will be approximated by the expression

$$F_d = d\left(\frac{\theta}{\theta_0}\right)^{2p}\dot{\theta} \tag{22}$$

where *d* is a constant coefficient, $p >> 1$ is a positive integer (generally $p \neq n$), and a dot denotes differentiation with respect to time *t*. Note that the constants b and d are determined experimentally. Pilipchuk and Ibrahim [35] introduced this modeling into the equations of motion of a nonlinear system simulating liquid sloshing impact in tanks supported by an elastic structure. They employed the saw-tooth time transformation to describe the in-phase and out-of phase nonlinear periodic regimes. Based on explicit forms of analytical solutions, all basic characteristics of nonlinear free and forced response regimes were estimated. It was found that a high frequency out-of-phase nonlinear mode takes place with relatively small tank amplitude and is more stable than the in-phase oscillation mode under small perturbations. The in-phase mode has relatively large tank amplitudes and does not preserve its symmetry under periodic parametric excitation.

When the first normal mode was parametrically excited the system exhibits hard nonlinear behavior and the impact loading reduced the response amplitude. On the other hand, when the second mode was parametrically excited, the impact loading results in complex response behavior characterized by multiple steady-state solutions where the response switches from soft to hard nonlinear characteristics. Under combined parametric resonance the system behaves like a soft system in the absence of impact and as a hard system in the presence of impact. Under simultaneous parametric and internal resonance conditions the system response was studied using the multiple scales method by [31] and by applying the Lie group transformations [35]. Both studies lead to the same system response characteristics. For example under first- and mixed-mode parametric excitation, the normal modes interact through internal resonance. Depending on the initial conditions and internal detuning parameter, the response can be quasi-periodic or chaotic with irregular jumps between two unstable equilibria. In the presence of impact forces, the system preserves fixed response amplitude response within a small range of internal detuning parameter. Beyond that range, the response exhibits quasi-periodic motion mainly governed by the initial conditions, internal detuning parameter, damping ratios and excitation level. Under second mode parametric excitation the second mode reaches fixed response amplitude, depending on initial conditions, with no energy sharing with the first mode. However, the phase angles were found to vary with time. Under combination parametric resonance, and in the absence of impact forces, the response was found to be sensitive to initial conditions.

SLOSHING UNDER A LOW GRAVITATIONAL FIELD

Regular gravitational potential has a stabilizing effect in that it brings the liquid volume toward the bottom of its container. When this body force diminishes, the liquid volume can assume any location inside its container in an unpredictable manner. Liquid sloshing dynamics under a microgravity field experiences different problems from those encountered under regular gravitational field. These problems include the reorientation of the liquid in its container and the difficulty of moving and handling it, since the body forces are almost negligible. Under microgravity, surface tension forces become predominant. The Bond number, given by the ratio of the gravitational to capillary forces, plays a major role in the free liquid surface characteristics. For very small values of the Bond number <<1, capillary forces predominate and the free liquid surface will not be flat in its container, but will rise around its vertical walls. Reynolds and Saterlee (Chapter 11 in [1]) addressed different problems of liquid behavior at a low and zero *g*. These include the mechanics and thermodynamics of capillary systems, heat transfer in cryogenic tanks and mechanisms of energy transport, capillary hydrostatics, low-g sloshing and some related problems, and fluid handling at low g. The early work of free liquid surface behavior under low gravity field considered different problems of the surface vapor interface. The following issues are considered the main problems encountered in microgravity liquid sloshing.

i. Natural Frequencies and Damping: The experimental results of Salzman and Masica [36], using the 5-seconds free-fall facility, revealed that the value of the centerline liquid depth depends on the magnitude of the Bond number and the liquid volume. For large liquid depth $h/R > 2$ and zero static contact-angle, they obtained empirically the following relationship for the liquid first mode natural frequency ω_1

$$\omega_1^2 = (2.6 + 1.84B_0)\sigma / (\rho R^3) \tag{23}$$

where the constant 2.6 represents the capillary contribution to the lateral natural frequency. Equation (23) reveals that capillary effects begin to appear for Bond numbers less than 20. The dependence of the nondimensional natural frequency $\Omega_1^2 = \omega_1^2 R^3 / \beta$, where $\beta = \sigma / \rho$, on the Bond number parameter $(B_0 + 1.4)$ is plotted on a log-log graph as shown in Fig. 7. For shallow liquid depth $h/R < 2$ and zero Bond number, the following relation was obtained

$$\omega_1^2 = \frac{2.6\beta}{R^3} tanh \frac{2h}{R} \tag{24}$$

The fundamental sloshing mode shape exhibited a similar dependence on Bond number. In the fundamental mode, the vertex of the liquid surface remains at the centerline of the cylinder and maximum displacement occurred at the cylinder wall.
The damping coefficient c was obtained by Clark and Stephens (1967) in the form

$$\frac{c}{(c)_{B_0=0}} = \frac{1}{35.7} K_d (2.6 + 1.8 B_0)^{1/4}, \text{ with } (c)_{B_0=0} = \frac{28.1}{2\pi} \sqrt{\frac{\mu}{\rho R^2}} \left(2.6 \frac{\sigma}{\rho R^3} \right)^{1/4} \quad (25)$$

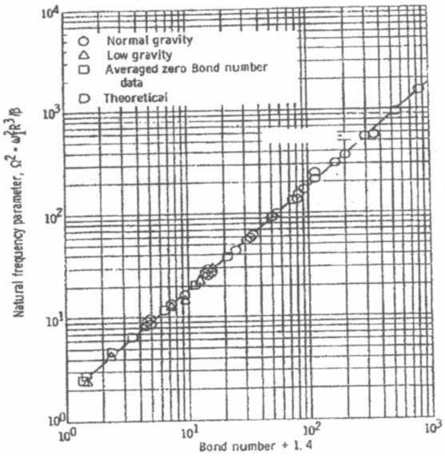

Fig. 7. Dependence of liquid natural frequency on the Bond number for liquid depth ratio >2 [36]

where $(c)_{B_0=0}$ is the damping coefficient at zero Bond number, K_d is an explicit function of Bond number, and μ is the liquid viscosity. The measured results showed that the normalized damping coefficient $c/(c)_{B_0=0}$ remained constant for all Bond numbers below 100. The decrease in the natural frequency compensates for the increase in the damping coefficient. It was concluded that for identical radii and liquids, the damping coefficient c is relatively independent of acceleration in the 0 to 100 Bond number region.

ii. Liquid Handling: The problem arises during liquid drainage where the free liquid surface experiences distortion. A major problem in the design of orbital propellant transfer is vapor pull-through caused by large-amplitude deformation of the liquid surface during outflow. Pull-through results in a portion of the propellant in the tank being unstable, either for transfer to another tank for engine supply. It is a serious

problem in systems incorporating turbo-pumps, because of the potential danger of pump cavitations and destruction if gas is introduced into the system.
Ward, et al. [38] outlined three possible interfacial configurations of a vapor-fluid system in an isothermal cylindrical container subjected to a reduced gravitational intensity. The first one is the "single interface" which is normally observed under the regular gravitational field. The curvature of the interface would be altered at the reduced gravitational field. If the gravitational intensity were reduced sufficiently, this configuration could become metastable, and as a result the system makes a transition out of this configuration based on the Bond number criterion.. The second takes place in the form of "two-interface" configuration in which a portion of the liquid phase would be above the vapor phase and a portion below. The third possibility is for the system to adopt the "bubble-configuration" in which a vapor volume is present and is surrounded by the liquid phase.

iii. Free liquid Surface Shape at a Low g Field: The most important force is the capillary force usually measured in terms of Bond number. Under low gravitational field the Bond number is much greater than 1. Furthermore, the shape of the free surface of the liquid will no longer be flat but will have some curvature. For axi-symmetric meniscus, the curvature κ was given by the following expression

$$\kappa = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{rf_r}{\sqrt{1+f_r^2+\frac{1}{r^2}f_\theta^2}}\right)+\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{f_\theta}{\sqrt{1+f_r^2+\frac{1}{r^2}f_\theta^2}}\right) \tag{27}$$

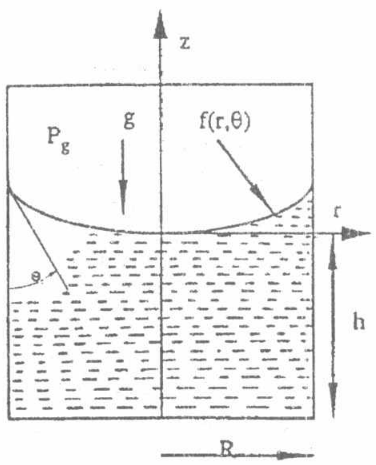

Fig. 8. Configuration of undisturbed free liquid surface showing the contact-line region

where a subscript denotes differentiation with respect to the subscripted variable, and $f(r,\theta)$ is the undisturbed free surface height of the meniscus, see Fig. 8. The role of curvature κ is very important in establishing the free surface boundary condition. The associated normal pressure across the free surface experiences discontinuity by an amount proportional to the product of the interfacial tension and the mean surface curvature, assuming inviscid fluid, i.e.,

$$P_g - P = \sigma\kappa \tag{28}$$

where P_g is the gas pressure outside the free liquid surface and P is the pressure just inside of the interface.

Under low gravity, the meniscus will have large curvature and some conditions must be applied to the slope of the interface at the tank wall. The angle between the tangent to the fluid interface at the contact-line and water-gas interface is known as the angle of contact ϑ_c. Three possibilities of the slope at the tank wall are: i) the slope remains constant for which dynamic contact-angle hysteresis is absent (free-edge condition), ii) the edge of the interface remains fixed (stuck-edge condition), or iii) some intermediate condition prevails. The third scenario was considered by Reynolds and Satterlee [1]. It was assumed that the contact-angle measured in the liquid of undisturbed surface is zero, which is typical of several liquid tank systems. However, it is possible that the angle at which the moving wave meets the wall is not the same as the static contact-angle. This phenomenon is called contact-angle hysteresis. It was assumed that the influence of hysteresis can be represented by the relationship

$$\frac{\partial \eta}{\partial r}\Big|_{r=R} = C_1 \eta \tag{29a}$$

If C_1 is a constant then relation (29a) does not explain the damping or energy dissipation caused by the hysteresis. Rather than introducing an arbitrary functional relationship, hysteresis will be neglected in the analysis, and the contact-line is assumed to slide easily along the tank walls (the free-edge condition). In other words $\frac{\partial \eta}{\partial r}\big|_{r=R} = 0$, and in this case the contact-angle ϑ_c is defined by the expression

$$\cot \vartheta_c = \frac{\partial \zeta}{\partial r} / \sqrt{1 + \left(\frac{\partial \zeta}{r \partial \vartheta}\right)}\Big|_{r=R} \tag{29b}$$

This means that the static equilibrium free surface shape should be defined a priori. For very low values of Bond number, the equilibrium interface can be assumed spherical and can be expressed by the shape function $f(r) = R - \sqrt{R^2 - r^2}$ for $B_0 << 1$. As B_o increases, the interface becomes flatter and a modified shape was suggested by Satterlee and Chin [39] for values of Bond number in the range $10 < B_o < 100$.

iv. g-Jitter Effects: During space missions, microgravity experiments revealed significant levels of residual accelerations referred to as g-jitter. The residual acceleration field can be decomposed into a quasi-steady or systematic component and a fluctuating component known as g-jitter. Typical values of the quasi-steady component $\|g_s\|$ are around $10^{-6} g_E$ (g_E is the gravitational acceleration on the earth's surface). The fluctuating contribution $g(t)$ is random in nature and has characteristic frequency of 1 Hz or higher. Zhang, et al. [40] described the free surface motion by a linear differential equation with a random coefficient and examined the stochastic stability boundary of the equilibrium state. Analytical results for the stability of the response second moment were presented in the limits of low-frequency oscillations and near the region of subharmonic parametric resonance.

v. Thermocapillary Convection: Thermocapillary convection plays an important role in microgravity fluid dynamics. Thermocapillary flows are driven by temperature-induced surface tension gradients at the interface between two immiscible fluids. For most liquids, the surface tension decreases with increasing temperature. Thus, when the interface experiences a positive temperature gradient the bulk fluids on each side of the interface must balance an effective negative shear stress. Through this mechanism, the thermal fields in the fluids are coupled to the velocity fields [41]. Surface tension driven convection and its instabilities have been a subject of great interest in resent years. For example, Marangoni flow (or thermo-capillary convection of the Benard type) occurs in a fluid layer with at least one free surface. It starts once there is a temperature gradient established along the free surface such that there is no first transitions. Along the free surface, surface tension can act to drive convection if the surface tension varies in magnitude along the surface due to its dependence upon a spatially varying temperature associated with a thermal disturbance.

CLOSING REMARKS

There is no doubt that a significant progress in the area of liquid sloshing dynamics has been dône since the publication of the monumental research monograph of Abramson [1] as reflected by the recent review article [2]. Thanks for the modern theory of dynamics as it has promoted our understanding of the free liquid surface motion under different types of parametric and nonlinear resonance conditions. The observed rotational motion near normal resonance and reported in earlier investigations has been recently described analytically and numerically in the literature. The influence of complex free surface motion on the system dynamics has not been examined. Note that these fluid motions have been examined in the absence of their interaction with the system dynamics.

Finite element and boundary element algorithms have been developed to determine the fluid motion in rigid containers and to simulate the hydroelastic coupling of elastic tank with the free liquid surface dynamics. However, these algorithms could not handle the problem of liquid-structure interaction in the neighborhood of internal resonance. The most significant contributions have dealt with the influence of sloshing loads on the stability and behavior of moving liquid tankers, ships, liquid storage subjected to earthquakes, and near zero-gravity liquid behavior. Most of the published literature has been devoted to deterministic aspects; however, few attempts have considered some stochastic cases. The difficulty arises when the excitation spectral density exceeds a certain level above which other sloshing modes exist and the liquid may experience different response regimes such as large amplitude motion and surface disintegration. This problem may be understood by studying the stochastic bifurcation of the liquid surface motion. Stochastic analysis needs to be applied to studies of regular and micro gravitational fields in order to establish the possibility of stabilizing the free surface through a multiplicative noise. Under microgravity, there are several problems of great importance and have been recently promoted by *NASA Fluid Physics Program*. These problems include thermo-capillary flows, interfacial phenomena, g-jitter induced and stochastic flows, and dynamics and stability of liquid bridges. The differential inclusion techniques should be considered in studying interfacial phenomenon.

Acknowledgment:

- This work was conducted under research grant from the ***National Science Foundation***, Grant number CMS-9634223 and by the ***Institute for Manufacturing Research at Wayne State University***.

- This paper is dedicated to the occasion of the 77th birth date of ***professor M. I. Rashed.***

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