A STUDY ON BIVARIATE BURR TYBE III DISTRIBUTION

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Abstract

Burr Type III distribution have been mainly used in statistical modeling of events in a variety of applied mathematical contexts such as fracture roughness, life testing, meteorology, modeling crop prices, forestry, reliability analysis. Our aim of this work is to construct a bivariate Burr Type III distribution and some of its structural properties such as bivariate probability density function and it's marginal, joint cumulative distribution and it's marginal, reliability and hazard rate function are studied. The maximum likelihood estimators of the parameters are derived. The Bayes estimators of the parameters based on the squared error loss function and Bayesian prediction of the future observations are presented. The performance of the proposed bivariate distribution is examined using a simulation study. Finally, one data set under the proposed distributions to illustrate their flexibility for real-life applications is analyzed.

Keywords: *Inverted Weibull; Gamma Distributions; Bivariate Distributions; Compounding, Maximum Likelihood Estimation; Bayes Estimation; Bayesian Prediction.*

1. Introduction

In this paper, a *bivariate Burr Type III* (BBIII) distribution is considered based on AL-Hussaini and Ateya (2005). Among the features bivariate distribution are important both on theoretical and applied grounds. Their uses in bivariate analysis that have been applied to a variety of disciplines are numerous.

In the statistical literature, several methodologies of constructing bivariate and multivariate distributions based on marginal and conditional distributions were proposed by Kotz *et al*. (2000), Arnold *et al*. (1999; 2001),Balakrishnan and Lai (2009), Mahmoud *et al.*(2021) and among others.

Domma (2009) extended some results related to the dependence structure of the bivariate Burr Type III distribution; proposed by Rodriguez (1980) using copula representations of bivariate distributions. Headrick *et al.* (2010) presented method for simulating univariate and multivariate Burr Type III and Type XII distributions with specified correlation matrices. The methodology is based on the derivation of the parametric forms of a *probability density function* (pdf) and *cumulative distribution function* (cdf) for this family of distributions. Ismail and Khalid (2015) used some copulas as *Ali-Mikhail-Haq* (AMH), Clayton and Gumbel on uncensored data to joint specific Burr Type III and XII distributions using theorem and algorithm of construction the copula Capitani *et al.* (2016) showed a bivariate Burr III copula to the trivariate case. This copula seems to be very general and analytically manageable and it provides an alternative to the commonly employed elliptical copulas (such as the Gaussian or the Student' sons since they have, roughly the same number of parameters. They showed that the trivariate Burr-III copula is, in general, able to capture the dependence structure implicit in observed trivariate data. Ogana *et al*. (2018) considered Frank and Plackett copulas, the two copulas evaluated on seven distributions models using data from temperate and tropical forests, one of these distributions is Burr Type III distribution. Azizi and Sayyareh (2019) constructed bivariate Burr Type III by using Marshall-Olkin (1967) technique; they presented some properties and estimated the parameters using *maximum likelihood* (ML) method.

One of the objectives of this paper is to construct BBIII distribution; based on the method suggested by AL-Hussaini and Ateya (2005). It could be useful in studying reliability maintainability of complicated systems. AL-Hussaini and Ateya (2005) considered a class of multivariate distributions. The same technique can be used to introduce and construct bivariate Burr Type III distribution, and some properties are studied.

In Section 2, a construction of BBIII distribution based on AL-Hussaini and Ateya (2005) technique is introduced, also some properties of the distribution are obtained. Maximum likelihood estimation and prediction are considered in Section 3. In Section 4, a numerical illustration for ML estimation is introduced. In Section 5, Bayesian estimation and two-sample prediction is presented. In Section 6,
a numerical illustration for Bayesian estimation and prediction is given.

2. Bivariate Burr Type III Distribution

This section aims to construct bivariate Burr Type III distribution by applying same technique introduced by AL-Hussaini and Ateya (2005), but in this study special case, bivariate is considered .Also, in this section, some properties and description of this distribution are discussed.

Assume that t_1, t_2 have conditional upon a common scale parameter θ, independent *inverted Weibull* (IW) distributions with pdf given by

$$f(t_i|\theta) = \alpha_i \theta t_i^{-(\alpha_i+1)} e^{-\theta t_i^{-\alpha_i}}, \qquad t_i, \alpha_i, \theta > 0. \tag{1}$$

It was assumed that θ is a positive random variable following the gamma (*a*, *b*) distribution with pdf $g(\theta)$ given by

$$g(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}\exp(-b\theta), \qquad \theta > 0, \qquad (a, b > 0). \tag{2}$$

The joint pdf $f(\underline{t})$ of BBIII can be obtained as follows:

$$f(\underline{t}) = \int_0^\infty \left[\prod_{i=1}^{2} f(t_i|\theta)\right] g(\theta)d\theta. \tag{3}$$

The joint pdf can be written as follows

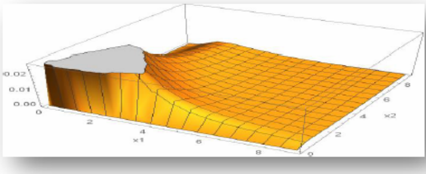
$$f(t_1, t_2) = a(a+1)\left[\prod_{i=1}^{2} c_i \dot{\lambda}_{\eta_i}(t_i)\right]\left[1 + \sum_{i=1}^{2} c_i\lambda_{\eta_i}(t_i)\right]^{-(a+2)}, \quad c_i = \delta_i/b.$$

In this case, $\lambda_\eta(t) = t^{-\alpha}$. Then the joint pdf can be obtained as follows:

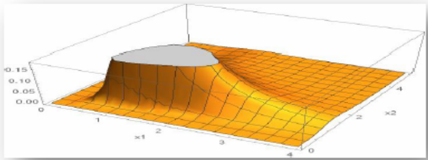
$$f(t_1, t_2) = a(a+1)\left[\prod_{i=1}^{2} c_i\,\alpha_i t_i^{-(\alpha_i+1)}\right]\left[1 + \sum_{i=1}^{2} c_i t_i^{-\alpha_i}\right]^{-(a+2)}$$

$$= a(a+1)\frac{\alpha_1\alpha_2\delta_1\delta_2}{b^2} t_1^{-(\alpha_1+1)} t_2^{-(\alpha_2+1)}\left[1 + \frac{\delta_1}{b}t_1^{-\alpha_1} + \frac{\delta_2}{b}t_2^{-\alpha_2}\right]^{-(a+2)}, \tag{4}$$

The contour plots of the joint pdf of the BBIII distribution for different parameter values are presented in Figure 1.



(1.a)



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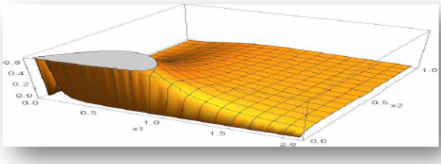

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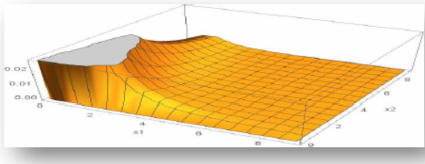
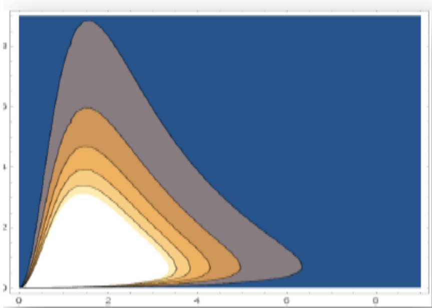
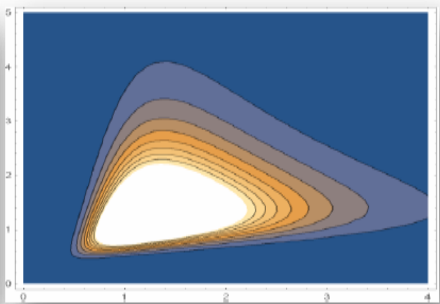


(1.d)

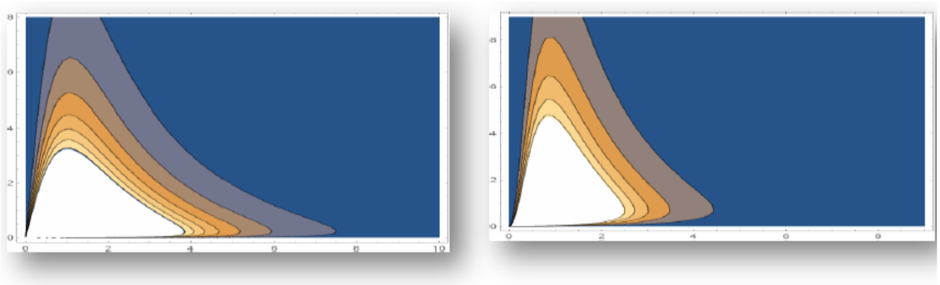
Figure 1: The surface plots of the joint pdf of the BBIII distribution for different parameter values:
(1.a) $(k = 1.5, a = 0.5, \alpha_1 = 2, \alpha_2 = 1, \delta_1 = 1, \delta_2 = 0.5)$,
(1.b) $(k = 3, a = 3, \alpha_1 = 3, \alpha_2 = 3, \delta_1 = 3, \delta_2 = 3)$,
(1.c) $(k = 0.5, a = 0.5, \alpha_1 = 2, \alpha_2 = 1, \delta_1 = 1, \delta_2 = 0.5)$ and
(1.d) $(k = 1.5, a = 1.5, \alpha_1 = 2, \alpha_2 = 1, \delta_1 = 1, \delta_2 = 1.5)$.



(1.e)



(1.f)



(1.g) (1.h)

Figure 1: The contour plots of the joint pdf of the BBIII distribution for different parameter values:
(1.e) $(k = 1.5, a = 0.5, \alpha_1 = 2, \alpha_2 = 1, \delta_1 = 1, \delta_2 = 0.5)$,
(1.f) $(k = 3, a = 3, \alpha_1 = 3, \alpha_2 = 3, \delta_1 = 3, \delta_2 = 3)$,
(1.g) $(k = 0.5, a = 0.5, \alpha_1 = 2, \alpha_2 = 1, \delta_1 = 1, \delta_2 = 0.5)$ and
(1.h) $(k = 1.5, a = 1.5, \alpha_1 = 2, \alpha_2 = 1, \delta_1 = 1, \delta_2 = 1.5)$.

And the joint cdf of the BBIII is given by

$$F(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} f(t_1, t_2) dt_1 dt_2 .$$

$$= \int_0^{t_1} \int_0^{t_2} a(a+1) \frac{\alpha_1 \alpha_2 \delta_1 \delta_2}{b^2} t_1^{-(\alpha_1+1)} t_2^{-(\alpha_2+1)} \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-(a+2)} dt_2 dt_1$$

$$= a(a+1) \int_0^{t_1} \frac{\alpha_1 \delta_1}{b} t_1^{-(\alpha_1+1)} \int_0^{t_2} \frac{\alpha_2 \delta_2}{b} t_2^{-(\alpha_2+1)} \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-(a+2)} dt_2 dt_1$$

$$= a(a+1) \int_0^{t_1} \frac{\alpha_1 \delta_1}{b} t_1^{-(\alpha_1+1)} \left[\frac{\left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-(a+1)}}{-(a+1)}\right]_0^{t_2} dt_1$$

$$= a \int_0^{t_1} \frac{\alpha_1 \delta_1}{b} t_1^{-(\alpha_1+1)} \left\{\left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1}\right]^{-(a+1)} - \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-(a+1)}\right\} dt_1$$

$$= a \left[\frac{\left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1}\right]^{-a}}{-a}\right]_0^{t_1} - \left[\frac{\left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-a}}{-a}\right]_0^{t_1}$$

$$= 1 - \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1}\right]^{-a} - \left[1 + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-a} + \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-a}. \qquad (5)$$

The marginal cdf of the BBIII distribution can be written as

$$F(t_i) = 1 - \left[1 + \frac{\delta_i}{b} t_i^{-\alpha_i}\right]^{-a}.$$

And the joint *reliability function* (rf) of the BBIII distribution is given as follows

$$R(t_1, t_2) = p(T_1 > t_1, T_2 > t_2) = 1 - F(t_1) - F(t_2) + F(t_1, t_2)$$

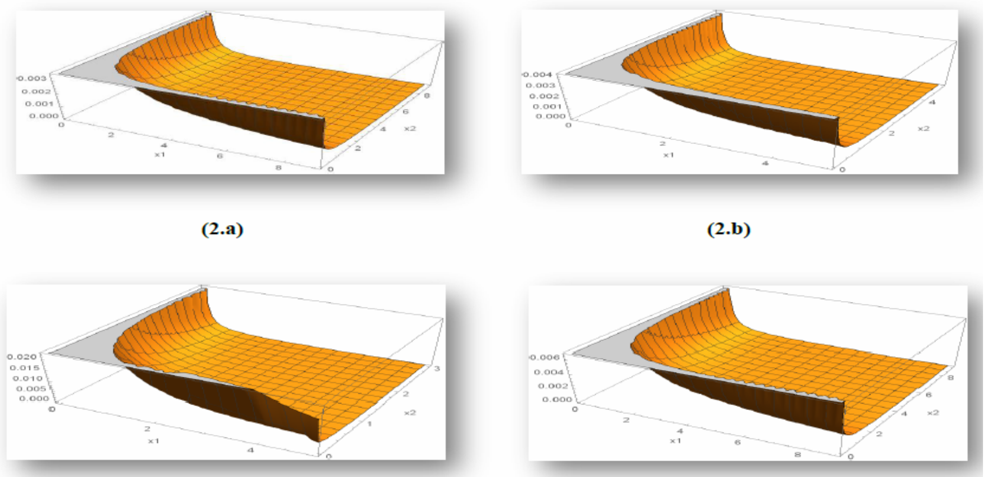
$$= \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-a}.$$

The joint *hazard rate function* (hrf) can be defined as

$$h(t_1, t_2) = \frac{f(t_1, t_2)}{R(t_1, t_2)}$$

$$= a(a+1)\frac{\alpha_1 \alpha_2 \delta_1 \delta_2}{b^2} t_1^{-(\alpha_1+1)} t_2^{-(\alpha_2+1)} \left[1 + \frac{\delta_1}{b} t_1^{-\alpha_1} + \frac{\delta_2}{b} t_2^{-\alpha_2}\right]^{-2}, \qquad (8)$$

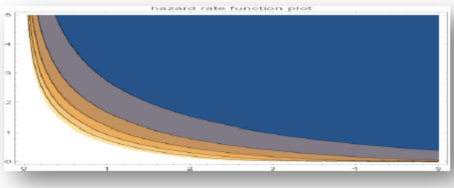
If $(T_1, T_2) \sim$BBIII, then for all values for $t_1 > 0$ and $t_2 > 0$ both components of $h(t_1, t_2)$ are decreasing function of t_1 andt_2.The contour plots of the joint hrf of the BBIII distribution for different parameter values are presented in Figure 2.



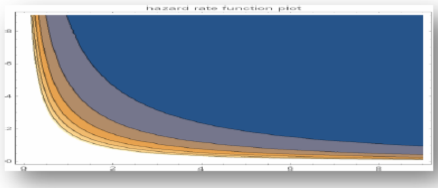
(2.a) (2.b)

(2.c) (2.d)

Figure 2: The surface plots of the joint hrf of the BBIII distribution for different parameter values:
(2.a) $(k = 0.3, a = 0.3, \alpha_1 = 0.2, \alpha_2 = 0.5, \delta_1 = 0.4, \delta_2 = 0.8)$,
(2.b) $(k = 0.4, a = 0.6, \alpha_1 = 0.1, \alpha_2 = 0.4, \delta_1 = 0.9, \delta_2 = 0.3)$,
(2.c) $(k = 0.1, a = 0.2, \alpha_1 = 0.5, \alpha_2 = 0.9, \delta_1 = 0.3, \delta_2 = 0.4)$ and
(2.d) $(k = 0.5, a = 0.3, \alpha_1 = 0.2, \alpha_2 = 0.5, \delta_1 = 0.4, \delta_2 = 0.8)$.



(2.e)



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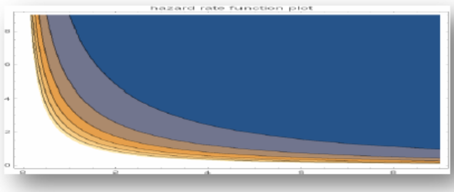
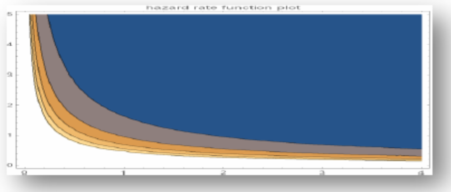


(2.g)



(2.h)

Figure 2: The contour plots of the joint hrf of the BBIII distribution for different parameter values:
(2.e) $(k = 0.3, a = 0.3, \alpha_1 = 0.2, \alpha_2 = 0.5, \delta_1 = 0.4, \delta_2 = 0.8)$,
(2.f) $(k = 0.4, a = 0.6, \alpha_1 = 0.1, \alpha_2 = 0.4, \delta_1 = 0.9, \delta_2 = 0.3)$,
(2.g) $(k = 0.1, a = 0.2, \alpha_1 = 0.5, \alpha_2 = 0.9, \delta_1 = 0.3, \delta_2 = 0.4)$ and
(2.h) $(k = 0.5, a = 0.3, \alpha_1 = 0.2, \alpha_2 = 0.5, \delta_1 = 0.4, \delta_2 = 0.8)$.

The mixed moments of the BBIII distribution

In this case, $\lambda_\eta(t) = t^{-\alpha}, \lambda_\eta^{'}(t) = -\alpha t^{-(\alpha+1)}$ and from (1), if $[(1-z)/cz]^p = \lambda_\eta(t) = t^{-\alpha}$, then $\lambda_\eta^{-1}[(1-z)/cz]^p = [(1-z)/cz]^{-p/\alpha}$.

A BBIII density function is given by (4) can be used to catch the mixed moments of order $p = \sum_{i=1}^{n} p_i$ for BBIII distribution can be shown as

$$E(t_1^{p_1},t_2^{p_2})=\frac{\Gamma(a+2)}{\Gamma(a)}\int_0^1\int_0^1\left(\prod_{i=1}^{2}\left(\lambda_{\eta_i}^{-1}\left(\frac{1-z_i}{c_i\prod_{j=1}^{i}z_j}\right)\right)^{p_i}\right)\prod_{i=1}^{2}z_i^{a+i-2}d\,z_2dz_1.$$

$$=\frac{\Gamma(a+2)}{\Gamma(a)}\int_0^1\int_0^1\left(\prod_{i=1}^{2}\left(\left(\frac{1-z_i}{c_i\prod_{j=1}^{i}z_j}\right)\right)^{-\frac{p_i}{\alpha_i}}\right)\prod_{i=1}^{n}z_i^{a+i-2}d\,z_2dz_1$$

$$=\frac{(a+1)a\Gamma(a)}{\Gamma(a)}\int_0^1\int_0^1\left(\prod_{i=1}^{2}\left(\left(\frac{1-z_i}{c_i\prod_{j=1}^{i}z_j}\right)\right)^{-\frac{p_i}{\alpha_i}}\right)\prod_{i=1}^{n}z_i^{a+i-2}d\,z_2dz_1$$

$$=\prod_{i=1}^{2}(a+i-1)\int_0^1\int_0^1\left(\prod_{i=1}^{2}\left(\left(\frac{1-z_i}{c_i\prod_{j=1}^{i}z_j}\right)\right)^{-\frac{p_i}{\alpha_i}}\right)\prod_{i=1}^{n}z_i^{a+i-2}d\,z_2dz_1$$

$$E(t_1^{p_i},t_2^{p_j})=\prod_{i=1}^{2}(a+i-1)\beta\left(a+i-1+\sum_{j=1}^{n}\frac{p_j}{\alpha_j},1-\frac{p_i}{\alpha_i}\right)c_i^{\frac{p_i}{\alpha_i}}, \tag{9}$$

the mixed moments of order $p=\sum_{i=1}^{2}p_i$ for BBIII distribution can be shown, using (9), to be given by

$$E(t_1^{p_i},t_2^{p_j})=a(a+1)c_i^{p_i/\alpha_i}c_j^{p_j/\alpha_j}\beta\left(a+\frac{p_i}{\alpha_i}+\frac{p_j}{\alpha_j},1-\frac{p_i}{\alpha_i}\right)\beta\left(a+1+\frac{p_j}{\alpha_j},1-\frac{p_j}{\alpha_j}\right), \tag{10}$$

where $\beta(a,b)=\int_0^1 z^{a-1}(1-z)^{b-1}dz$ is the beta function. It follows that, for univariate case,

$$E(t_i^{p_i})=ac_i^{p_i/\alpha_i}\beta\left(a+\frac{p_i}{\alpha_i},1-\frac{p_i}{\alpha_i}\right). \tag{11}$$

The correlation coefficient $\rho(t_i,t_j)$ can be computed by using (10) and (11) with the appropriate choices of p_i and p_j.

3. Maximum Likelihood Method

In this section, the ML estimation of the vector of parameters $\underline{\varphi}=(a,b,\alpha_1,\alpha_2,\delta_1,\delta_2)$ for BBIII distribution is introduced.

3.1 Maximum likelihood estimators of the parameters

The likelihood function of BBIII distribution can be derived directly using the pdf in (4) but compounding $\prod_{i=1}^{k} f\left(t_i \middle| \underline{\varphi}\right)$ and $g(b)$can be applied to make the ML estimation easier, hence

$$L\left(\underline{\varphi}\middle|t_1,t_2,\theta\right)=\prod_{j=1}^{n} f(t_{1j},t_{2j},\theta_j)=\prod_{j=1}^{n}\left[\prod_{i=1}^{2} f(t_{ij}|\theta_j)g(\theta_j)\right]$$

$$=\frac{b^{na}}{(\Gamma(a))^n}\left(\prod_{j=1}^{n}\theta_j\right)^{a+1}\alpha_1^n\alpha_2^n\delta_1^n\delta_2^n\left(\prod_{j=1}^{n}t_{1j}\right)^{-(\alpha_1+1)}\left(\prod_{j=1}^{n}t_{2j}\right)^{-(\alpha_2+1)}$$

$$\times\ exp\left[-\sum_{j=1}^{n}\theta_j(\delta_1 t_{1j}^{-\alpha_1}+\delta_2 t_{2j}^{-\alpha_2}+b)\right]. \quad (12)$$

The log likelihood function is given by

$$\ell\left(\underline{\varphi}\middle|t_1,t_2,\theta\right)=na\ \ln b-n\ln\Gamma(a)+(a+1)\left(\sum_{j=1}^{n}\ln\theta_j\right)+n\ln\alpha_1+n\ln\alpha_2+n\ln\delta_1+n\ln\delta_2$$

$$-(\alpha_1+1)\left(\sum_{j=1}^{n}\ln t_{1j}\right)-(\alpha_2+1)\left(\sum_{j=1}^{n}\ln t_{2j}\right)-\left[\sum_{j=1}^{n}\theta_j(\delta_1 t_{1j}^{-\alpha_1}+\delta_2 t_{2j}^{-\alpha_2}+b)\right].(13)$$

The ML estimators of the parameters are obtained by differentiating (13) with respect to the parameters, setting to zero and then solving the resulting system of likelihood equations, given by

$$\frac{\partial\ell}{\partial a}=0=n\ln\hat{b}-n\psi(\hat{a})+\sum_{j=1}^{n}\ln\theta_j\ ,$$

$$\frac{\partial\ell}{\partial b}=0=\frac{n\hat{a}}{\hat{b}}-\sum_{j=1}^{n}\theta_j,$$

$$\frac{\partial\ell}{\partial\alpha_1}=0=\frac{n}{\hat{\alpha}_1}-\sum_{j=1}^{n}\ln t_{1j}+\sum_{j=1}^{n}\theta_j\hat{\delta}_1 t_{1j}^{-\hat{\alpha}_1}\ln t_{1j},$$

$$\frac{\partial \ell}{\partial \alpha_2} = 0 = \frac{n}{\hat{\alpha}_2} - \sum_{j=1}^{n} \ln t_{2j} + \sum_{j=1}^{n} \theta_j \hat{\delta}_2 t_{2j}^{-\hat{\alpha}_2} \ln t_{2j},$$

$$\frac{\partial \ell}{\partial \delta_1} = 0 = \frac{n}{\hat{\delta}_1} - \sum_{j=1}^{n} \theta_j t_{1j}^{-\hat{\alpha}_1},$$

and

$$\frac{\partial \ell}{\partial \delta_2} = 0 = \frac{n}{\hat{\delta}_2} - \sum_{j=1}^{n} \theta_j t_{2j}^{-\hat{\alpha}_2},$$

where $\psi(a) = \acute{\Gamma}(\hat{a})/\Gamma(\hat{a})$.

The previous equations cannot be solved analytically. It can be evaluated numerically to obtain the ML estimators.

The invariance property of the ML estimators can be applied to obtain the ML estimators for the $R(t_1, t_2)$ and $h(t_1, t_2)$by replacing the parameters in (7) and (8) by their ML estimators

$$\hat{R}(t_1, t_2) = \left[1 + \frac{\hat{\delta}_1}{\hat{b}} t_1^{-\hat{\alpha}_1} + \frac{\hat{\delta}_2}{\hat{b}} t_2^{-\hat{\alpha}_2}\right]^{-\hat{a}},$$

and

$$\hat{h}(t_1, t_2) = \hat{a}(\hat{a}+1) \frac{\hat{\alpha}_1 \hat{\alpha}_2 \hat{\delta}_1 \hat{\delta}_2}{\hat{b}^2} t_1^{-(\hat{\alpha}_1+1)} t_2^{-(\hat{\alpha}_2+1)} \left[1 + \frac{\hat{\delta}_1}{\hat{b}} t_1^{-\hat{\alpha}_1} + \frac{\hat{\delta}_2}{\hat{b}} t_2^{-\hat{\alpha}_2}\right]^{-2}, \qquad (15)$$

hence the $\hat{R}(t_1, t_2)$ and $\hat{h}(t_1, t_2)$ can be calculated numerically.

4 Numerical Illustrations

This section aims to investigate the precision of the theoretical results of estimation on the basis of simulation study and example data set.

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates on the basis of generated data from the BBIII distribution. The ML averages of the estimates, rf and hrf based on complete sample are computed. Moreover, *confidence intervals* (CIs) of the parameters, rf and hrf are calculated. Simulation studies are performed using Mathematica 11 for illustrating the obtained results.

The steps of the simulation procedure based on complete sample data are as follows:

a) For given values of $\underline{\varphi}$(where $\underline{\varphi} = (a, b, \alpha_1, \alpha_2, \delta_1, \delta_2)$), random samples of size *n* are generated from the BBIII distribution.

b) For each sample size sort the t'_{ij}s, such that $(t_{11}, t_{21}), (t_{12}, t_{22}), \ldots, (t_{1n}, t_{2n})$.

c) Repeat the previous two steps N times where N represents a fixed number of simulated samples.

The Newton-Raphson method can be applied; the ML averages and the CIs of the parameters are obtained. Also, the rf, hrf and their CIs are calculated using the ML averages of the parameters.

Evaluating the performance of the estimates is considered through some measurements of accuracy. In order to study the precision and variation of the estimates, it is convenient to use the *estimated risk*

$$(ER) = \frac{\sum_{i=1}^{N}(\text{estimator} - \text{true value})^2}{\text{N}}.$$

Simulation results of ML estimates are displayed in Tables 1 and 2, where N = 10000 is the number of repetitions and samples of size (n=30, 50, 100), in the complete sample case, and the population parameter values

$(a = 1.1, b = 2, \alpha_1 = 1.5, \alpha_2 = 2.2, \delta_1 = 1.5, \delta_2 = 1.9)and(a = 1.3, b = 1.2, \alpha_1 = 1.6, \alpha_2 = 2.3, \delta_1 = 2.2, \delta_2 = 3.5)$

are.

Tables 1 and 2 give the ML averages, ERs, and CIs of the unknown parameters. While Tables 3 and 4 present the ML averages, ERs and CIs of the rf and hrf for different values of time t_{01}, t_{02}.

Table 1

ML averages, estimated risk, variances, estimated risks and 95% confidence intervals of the parameters

$(N = 10000, a = 1.1, b = 2, \alpha_1 = 1.5, \alpha_2 = 2.2, \delta_1 = 1.9, \delta_2 = 3.1)$

n	parameters	Averages	Var	ER	UL	LL	Length
30	a	1.2627	0.0005	0.0270	1.3075	1.2179	0.0895
	b	2.3198	0.0015	0.1038	2.3958	2.2437	0.1521
	α_1	1.5182	0.0001	0.0004	1.5389	1.4974	0.0414
	α_2	2.4492	0.0035	0.0657	2.5666	2.3318	0.2347
	δ_1	1.9232	0.0000	0.0006	1.9314	1.9149	0.0165
	δ_2	3.1319	0.0002	0.0012	3.1582	3.1058	0.0525
50	a	1.1951	0.0002	0.0093	1.2272	1.1631	0.0640
	b	2.1744	0.0016	0.0319	2.2523	2.0965	0.1558
	α_1	1.5287	0.0009	0.0018	1.5905	1.4669	0.1235
	α_2	2.2602	0.0006	0.0042	2.3089	2.2113	0.0977
	δ_1	1.9104	0.0003	0.0004	1.9442	1.8768	0.0674
	δ_2	3.1087	0.0002	0.0003	3.1372	3.0801	0.0570
100	a	1.2794	0.0001	0.0323	1.2945	1.2643	0.0303
	b	2.3849	0.0038	0.1519	2.5063	2.5063	0.2427
	α_1	1.5481	0.0047	0.0069	1.6822	1.6822	0.2631
	α_2	2.5434	0.0108	0.1287	2.7466	2.7466	0.4064
	δ_1	1.9398	0.0009	0.0025	1.9982	1.9986	0.1168
	δ_2	3.2093	0.0047	0.0166	3.3435	3.0752	0.2683

Table 2

ML averages, estimated risk, variances, estimated risks and 95% confidence intervals of the parameters

$$(N = 10000, a = 1.3, b = 2.2, \alpha_1 = 1.6, \alpha_2 = 2.3, \delta_1 = 2.2, \delta_2 = 3.5)$$

n	parameters	Averages	Var	ER	UL	LL	Length
30	a	1.1818	0.0018	0.0158	1.2651	1.0986	0.1665
	b	2.0442	0.0034	0.0277	2.1595	1.9289	0.2305
	α_1	1.6009	0.00003	0.00003	1.6121	1.5898	0.0222
	α_2	2.3087	0.00004	0.0001	2.3209	2.2966	0.0244
	δ_1	2.2224	0.0001	0.0006	2.2446	2.2003	0.0442
	δ_2	3.5347	0.0003	0.0015	3.5666	3.5027	0.0638
50	a	1.1756	0.0009	0.0164	1.2361	1.1152	0.1209
	b	2.0315	0.0021	0.0305	2.1211	1.9420	01791
	α_1	1.5998	0.00002	0.00002	1.6093	1.5903	0.0190
	α_2	2.3019	0.00001	0.00002	2.3092	2.2947	0.0144
	δ_1	2.2221	0.0001	0.0005	2.2369	2.2073	0.0296
	δ_2	3.5348	0.0001	0.0013	3.5568	3.5129	0.0439
100	a	1.1860	0.0001	0.0131	1.2063	1.1657	0.0405
	b	2.0463	0.0001	0.0238	2.0737	2.0189	0.0548
	α_1	1.5995	9.0931e-6	9.3773e-6	1.6054	1.5935	0.0118
	α_2	2.2986	0.00001	0.00001	2.3053	2.2919	0.0134
	δ_1	2.2186	4.8589e-6	0.0004	2.2229	2.2144	0.0086
	δ_2	3.5295	6.6373e-6	0.0009	3.5346	3.5245	0.0101

Table 3

ML averages, relative absolute biases, variances, estimated risks and 95% confidence intervals of the reliability and hazard rate functions

$(N = 10000, a = 1.1, b = 2, \alpha_1 = 1.5, \alpha_2 = 2.2, \delta_1 = 1.9, \delta_2 = 3.1, t_{01} = 2, t_{02} = 3)$

n	Rf and hrf	Averages	RAB	Var	ER	UL	LL	Length
30	$R(t_0)$	0.6557	0.0048	0.00002	0.0087	0.6649	0.6465	0.0185
	$h(t_0)$	0.0262	0.0369	1.1389e-6	0.0539	0.0282	0.0242	0.0042
50	$R(t_0)$	0.6536	0.0059	0.00001	0.0059	0.6609	0.6463	0.0146
	$h(t_0)$	0.0261	0.0446	5.4239e-7	0.0446	0.0276	0.0247	0.0029
100	$R(t_0)$	0.6541	0.0024	2.8845e-6	0.0036	0.6574	0.6507	0.0067
	$h(t_0)$	0.0267	0.0152	1.3802e-7	0.0204	0.0275	0.0260	0.0015

Table 4

ML averages, relative absolute biases, variances, estimated risks and 95% confidence intervals of the reliability and hazard rate functions

$(N = 10000, a = 0.6, b = 1.2, \alpha_1 = 0.75, \alpha_2 = 1.1, \delta_1 = 0.95, \delta_2 = 1.55, t_{01} = 3, t_{02} = 4)$

n	Rf and hrf	Averages	RAB	var	ER	UL	LL	Length
30	$R(t_0)$	0.8986	0.2579	0.0021	0.0113	0.9879	0.8093	0.1785
	$h(t_0)$	0.0003	0.9185	1.4194e-7	0.2667	0.0011	0.0000	0.0011
50	$R(t_0)$	0.8623	0.2072	0.0015	0.0073	0.9386	0.7859	0.1527
	$h(t_0)$	0.0002	0.9450	7.6895e-8	0.2454	0.0008	0.0000	0.0008
100	$R(t_0)$	0.8444	0.1820	0.00004	0.0056	0.8568	0.8319	0.0249
	$h(t_0)$	0.0006	0.8579	1.2118e-8	0.2353	0.0008	0.0004	0.0004

4.2 Example of data set

In this example, a data set is analyzed from a Sankaran-Nair bivariate Pareto distribution [see Sankaran-Nair (1993) and Sankaran and Kundu (2014)]. The generated data set for n=30 is:

(0.252, 8.400), (1.105, 0.458), (0.427, 1.602), (12.491, 2.383), (0.260, 0.106), (0.240, 1.769), (4.888, 0.758), (0.870, 0.572), (0.036, 0.254), (1.537, 0.023), (1.508,0.535), (0.239, 1.4120), (0.173, 0.011), (1.090, 1.278), (6.002, 0.017), (0.897, 2.032), (0.690, 0.138), (1.883, 0.398), (0.960, 0.257), (0.561, 0.573), (5.370, 0.325), (0.167, 0.260), (13.602, 0.364), (3.922, 0.938), (0.132, 0.547), (0.603,0.102), (0.226, 0.481), (0.143, 0.779), (0.643, 0.071), (0.349, 1.586).

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively 0.808 and 0.393. The p value showed that the model fits the data very well.

Tables 5 and 6 display the ML estimates, ERs of the unknown parameters. While Table and 8 presents ML estimates, ERs and CIs of the rf and hrf for different values of time t_{01}, t_{02}.

Table 5

ML estimates and estimated risks for the parameters for the data set

n	Parameters	Estimate	ER
30	a	0.9734	0.0160
	b	1.9088	0.0083
	α_1	2.0172	0..2676
	α_2	2.2851	0.0072
	δ_1	2.1185	0.0477
	δ_2	3.2789	0.0320

Table 6

ML estimates and estimated risks for the parameters for the data set

n	Parameters	Estimate	ER
30	a	1.3178	0.0003
	b	2.4302	0.0529
	α_1	2.4194	0.6714
	α_2	2.3623	0.0039
	δ_1	2.5304	0.1092
	δ_2	3.7374	0.0564

Table 7
ML estimates and estimated risks of the reliability and hazard rate functions for the data set

n	Rf and hrf	Estimates	ER
30	$R(t_0)$	0.7139	0.0094
	$h(t_0)$	0.0285	0.4758

Table 8
ML estimates and estimated risks of the reliability and hazard rate functions for the data set

n	Rf and hrf	Estimates	ER
30	$R(t_0)$	0.7009	0.008
	$h(t_0)$	0.0379	0.4419

4.3 Concluding remarks

From the tables we notice that:

1- It is noticed, from the Tables 1 and 2 that the ML averages are very close to the population parameter values as the sample size increases. Also, ER is decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the true parameter values as the sample size increases.

2- The lengths of the CIs of the parameters become narrower as the sample size increases.

3- The ML averages for the rf and hrf performs better as the sample size increases, [see Tables 3 and 4].

4- The ML interval includes the estimates [between the *lower limit* (LL) and *upper limit* (UL)].

5. Bayesian Method

In this section Bayesian estimation and prediction for the vector of the parameters $\underline{\varphi} = (a, b, \alpha_1, \alpha_2, \delta_1, \delta_2)$ for the BBIII distribution will be derived under the SEL function using informative prior.

5.1 Bayes estimators of the parameters

Assuming that $(a,b), (\alpha_1, \delta_1)$ and (α_2, δ_2) are independent, then a prior density function of $\underline{\varphi}$ is given by

$$\pi\left(\underline{\varphi}\right) \propto \pi_1(a,b)\pi_2(\alpha_1,\delta_1)\pi_3(\alpha_2,\delta_2),$$

$$\pi_1(a,b) = \pi_{11}(a|b)\pi_{12}(b), \quad a|b \sim Gamma(c_1, b) and\ b \sim Gamma(c_2, c_3),$$

$$\pi_2(\alpha_1,\delta_1) = \pi_{21}(\alpha_1|\delta_1)\pi_{22}(\delta_1), \quad \alpha_1|\delta_1 \sim Gamma(c_4, \delta_1) and\ \delta_1 \sim Gamma(c_5, c_6)$$

and

$$\pi_3(\alpha_2,\delta_2) = \pi_{31}(\alpha_2|\delta_2)\pi_{32}(\delta_2), \quad \alpha_2|\delta_2 \sim Gamma(c_7, \delta_2)\ and\ \delta_2 \sim Gamma(c_8, c_9).$$

So that

$$\pi_1(a,b) \propto b^{c_1+c_2-1} a^{c_1-1} e^{-b(c_3+a)}, \tag{17}$$

$$\pi_2(\alpha_1,\delta_1) \propto \delta_1^{c_4+c_5-1} \alpha_1^{c_4-1} e^{-\delta_1(c_6+\alpha_1)}, \tag{18}$$

and

$$\pi_3(\alpha_2,\delta_2) \propto \delta_2^{c_7+c_8-1} \alpha_2^{c_7-1} e^{-\delta_2(c_9+\alpha_2)}, \tag{19}$$

Substituting from (17-19) in (16) and using the likelihood function (12) the posterior density function will separate into three posteriors, which are

$$\pi_1^*(a,b|t_1,t_2,\theta) \propto \frac{b^{na+c_1+c_2-1}}{(\Gamma(a))^n}\left(\prod_{j=1}^{n}\theta_j\right)^{a+1} a^{c_1-1} exp\left[-b\left(\sum_{j=1}^{n}\theta_j + c_3 + a\right)\right], \tag{20}$$

$$\pi_2^*(\alpha_1,\delta_1|t_1,t_2,\theta) \propto \delta_1^{c_4+c_5+n-1} \alpha_1^{c_4+n-1}\left(\prod_{j=1}^{n} t_{1j}\right)^{-(\alpha_1+1)} exp\left[-\delta_1\sum_{j=1}^{n}\theta_j t_{1j}^{-\alpha_1} + c_6 + \alpha_1\right], \tag{21}$$

and

$$\pi_3^*(\alpha_2,\delta_2|t_1,t_2,\theta) \propto \delta_2^{c_7+c_8+n-1}\alpha_2^{c_7+n-1}\left(\prod_{j=1}^{n} t_{2j}\right)^{-(\alpha_2+1)} exp\left[-\delta_2\sum_{j=1}^{n}\theta_j t_{2j}^{-\alpha_2}+c_9+\alpha_2\right], (22)$$

by using (20-22), assuming that (a^*,b^*),(α_1^*,θ_1^*) and (α_2^*,θ_2^*) are independent, posterior density function of $\underline{\varphi}^* = (a^*,b^*,\alpha_1^*,\delta_1^*,\alpha_2^*,\delta_2^*)$ is given by

$$\pi^*\left(\underline{\varphi}|t_1,t_2,\beta\right) \propto \pi_1^*(a,b|t_1,t_2,\theta)\pi_2^*(\alpha_1,\delta_1|t_1,t_2,\theta)\pi_3^*(\alpha_2,\delta_2|t_1,t_2,\theta)$$

The Bayes estimators of the parameters are the posterior means

$$\varphi_{j(SE)}^* = E(\varphi_j|t_1,t_2) = \int_{\underline{\varphi}} \varphi_j\,\pi^*\left(\underline{\varphi}|t_1,t_2,b\right)d\underline{\varphi}\ , \qquad j=1,2,3,4.$$

Which can be evaluate numerically to obtain the Bayes estimates for the parameters.

The Bayes estimators of the $R(t_1,t_2)$ and $h(t_1,t_2)$, can be obtained using (7), (8) and (23), respectively, as given below

$$R_{SE}^*(t_1,t_2) = E\left(R(t_1,t_2)\middle|\underline{\varphi}\right) = \int_{\underline{\varphi}} R(t_1,t_2)\,\pi^*\left(\underline{\varphi}\middle|t_1,t_2\right)d\underline{\varphi}, \qquad (24)$$

and

$$h_{SE}^*(t_1,t_2) = E\left(h(t_1,t_2)\middle|\underline{\varphi}\right) = \int_{\underline{\varphi}} h(t_1,t_2)\,\pi^*\left(\underline{\varphi}\middle|t_1,t_2\right)d\underline{\varphi}\,. \qquad (25)$$

Equations (24) and (25) can be calculated numerically to obtain the Bayes estimates of rf and hrf based on SEL function.

5.2 Two-sample Bayesian prediction

Considering two-sample prediction, the two samples are assumed to be independent and drawn from the same distribution. In univariate case, the density of the *s-th* order statistic in the future sample is used to obtain the predictive density function of the *s-th* ordered statistic. In bivariate case where the first variable in the vector of bivariate distribution is the ordered observation and the second variable is their concomitant. Therefor the joint pdf of the ordered observations and the concomitants is needed to obtain the joint predictive density function of the future ordered observations and their concomitants. For a future bivariate sample of size m, the joint pdf of future *s-th* ordered observation and its *s-th* concomitant denoted by $(y_{1(s:m)}, y_{2(s:m)})$, $s = 1,2,\ldots,m$, has the joint pdf which is given by (4) after replacing t_1 by $y_{1(s:m)}$ and t_2 by $y_{2(s:m)}$. For simplicity, it can be written as $(y_{1(s)}, y_{2(s)})$ instead of $(y_{1(s:m)}, y_{2(s:m)})$. Then the joint order statistic pdf of $(y_{1(s)}, y_{2(s)})$ can be derived as follows:

$$f_{s:m}\left(y_{1(s)}, y_{2(s)}; \underline{\varphi}\right) = \frac{m!}{(s-1)!\,(m-s)!} f\left(y_{1(s)}, y_{2(s)}; \underline{\varphi}\right) \left[F(y_{1(s)}, y_{2(s)})\right]^{s-1} \left[1 - F(y_{1(s)}, y_{2(s)})\right]^{m-s},$$

using the binomial expansion to simplify the last term in the previous equation, one gets

$$\left[1 - F(y_{1(s)}, y_{2(s)})\right]^{m-s} = \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \left[F(y_{1(s)}, y_{2(s)})\right]^j.$$

Thus, the joint pdf of $(y_{1(s)}, y_{2(s)})$ is

$$\begin{aligned} f_{s:m}\left(y_{1(s)}, y_{2(s)}; \underline{\varphi}\right) &= \frac{m!}{(s-1)!\,(m-s)!} f\left(y_{1(s)}, y_{2(s)}; \underline{\omega}\right) \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \left[F(y_{1(s)}, y_{2(s)})\right]^{s+j-1} \\ &= f\left(y_{1(s)}, y_{2(s)}; \underline{\varphi}\right) \sum_{j=0}^{m-s} \frac{m!}{(s-1)!\,(m-s-j)!\,(j)!} (-1)^j \left[F(y_{1(s)}, y_{2(s)})\right]^{s+j-1}, \\ &= f\left(y_{1(s)}, y_{2(s)}; \underline{\varphi}\right) \sum_{j=0}^{m-s} C_{m,s,j} \left[F(y_{1(s)}, y_{2(s)})\right]^{s+j-1}, \end{aligned} \tag{26}$$

where

$$C_{m,s,j} = \frac{m!}{(s-1)!\,(m-s-j)!\,(j)!} (-1)^j . \tag{27}$$

The Bayesian predictive density of ordered observations and their concomitants is given by (26). Substituting $f(t_1, t_2)$ given in (4) and $F(t_1, t_2)$ as in (5) in (26) after replacing t_1 by $y_{1(s)}$ and t_2 by $y_{2(s)}$ then

$$f_{s:m}\left(y_{1(s)},y_{2(s)};\underline{\varphi}\right)=$$
$$\frac{a(a+1)}{b^2}\alpha_1\alpha_2\delta_1\delta_2 y_{1(s)}^{-(\alpha_1+1)} y_{2(s)}^{-(\alpha_2+1)}\left[1+\frac{\delta_1}{b}y_{1(s)}^{-\alpha_1}+\frac{\delta_2}{b}y_{2(s)}^{-\alpha_2}\right]^{-(a+2)},$$

$$\times\sum_{j=0}^{m-s}C_{m,s,j}\left[1-\left[1+\frac{\delta_1}{b}y_{1(s)}^{-\alpha_1}\right]^{-a}-\left[1+\frac{\delta_2}{b}y_{2(s)}^{-\alpha_2}\right]^{-a}+\left[1+\frac{\delta_1}{b}y_{1(s)}^{-\alpha_1}+\frac{\delta_2}{b}y_{2(s)}^{-\alpha_2}\right]^{-a}\right]^{(s+j-1)},$$

$$\left(y_{1(s)},y_{2(s)}\right)>0,(a,b,\alpha_1,\alpha_2\delta_1,\delta_2)>0, \quad (28)$$

the joint Bayesian predictive density of ordered observations and their concomitants is given by

$$h\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)=\int_{\underline{\varphi}} g\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)d\underline{\varphi}, \quad (29)$$

where

$$g\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)=f\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)\pi^*\left(\underline{\varphi}|y_1,y_2\right), \quad (30)$$

and

$$\int_{\underline{\varphi}}=\int_a\int_b\int_{\alpha_1}\int_{\alpha_2}\int_{\delta_1}\int_{\delta_2}, \qquad d\underline{\varphi}=da\,db\,d\delta_2 d\alpha_2 d\alpha_1 d\delta_1,$$

Substituting (23) and (28) in (30), yields the joint Bayesian predictive density of $\left(y_{1(s)},y_{2(s)}\right)$ is

$$h\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)=\int_{\underline{\varphi}} I_1I_2I_3I_4I_5\,d\underline{\varphi}, \quad (32)$$

where

$$I_1=\frac{(a+1)a^{c_1}}{(\Gamma(a))^n}b^{na+c_1+c_2-3}\delta_1^{c_4+c_5+n}\alpha_1^{c_4+n}\delta_2^{c_7+c_8+n}\alpha_2^{c_7+n},$$

$$I_2=e^{-\left[b\left(\sum_{j=1}^{n}\theta_j+c_3+a\right)+\delta_1(c_6+\alpha_1)+\delta_2(c_9+\alpha_2)\right]}y_{1(s)}^{-(\alpha_1+1)}y_{2(s)}^{-(\alpha_2+1)},$$

$$I_3=\left[1+\frac{\delta_1}{b}y_{1(s)}^{-\alpha_1}+\frac{\delta_2}{b}y_{2(s)}^{-\alpha_2}\right]^{-(a+2)}\left(\prod_{j=1}^{n}\theta_j\right)^{a+1},$$

$$I_4=\left(\prod_{j=1}^{n}y_{1j}\right)^{-(\alpha_1+1)}\left(\prod_{j=1}^{n}y_{2j}\right)^{-(\alpha_2+1)}e^{-\left[\delta_1\sum_{j=1}^{n}\theta_j y_{1j}^{-\alpha_1}+\delta_2\sum_{j=1}^{n}\theta_j y_{2j}^{-\alpha_2}\right]},$$

and

$$I_5=\sum_{j=0}^{m-s}C_{m,s,j}\left[1-\left[1+\frac{\delta_1}{b}y_1^{-\alpha_1}\right]^{-a}-\left[1+\frac{\delta_2}{b}y_2^{-\alpha_2}\right]^{-a}+\left[1+\frac{\delta_1}{b}y_1^{-\alpha_1}+\frac{\delta_2}{b}y_2^{-\alpha_2}\right]^{-a}\right]^{(s+j-1)}d\underline{\varphi}$$

The point predictors of future ordered observation and their concomitants $(Y_{1(s)},Y_{2(s)}), s=1,2,\dots,m$, under SEL function can be equivalently obtained as follows

$$Y_1^*=E\left(y_{1(s)}\middle|\underline{\varphi}\right)=\int_{y_{1(s)}=0}^{\infty}y_{1(s)}\int_{y_{2(s)}}^{\infty}f\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)dy_{2(s)}dy_{1(s)}, \quad (34)$$

and

$$Y_2^*=E\left(y_{2(s)}\middle|\underline{\varphi}\right)=\int_{y_{2(s)}=0}^{\infty}y_{2(s)}\int_{y_{1(s)}}^{\infty}f\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)dy_{1(s)}dy_{2(s)}. \quad (35)$$

From (34) and (35), it is clear that the point predictors Y_1^* and Y_2^* cannot be obtained in closed from, and then the joint Bayesian points predictors of future ordered observation is

$$Y_1^*,Y_2^*=E\left(y_{1(s)},y_{2(s)}\middle|\underline{\varphi}\right)=\int_0^{\infty}\int_0^{\infty}y_{1(s)}y_{2(s)}f\left(y_{1(s)},y_{2(s)}|\underline{\varphi}\right)dy_{1(s)}dy_{2(s)}. \quad (36)$$

6. Numerical Illustration

This section aims to investigate the precision of the theoretical results of Bayesian estimation and prediction on the basis of simulated and a data set.

6.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates on the basis of generated data from the BBIII distribution. Bayes averages of the parameters, rf and hrf based on complete sample are computed. Moreover, credible intervals of the parameters, rf and hrf are calculated, Bayes point predictors for a future observation from the BBIII distribution are computed for the two-sample case. All simulation studies are performed using R programming language.

Simulation algorithm

I. In similar manner to the steps used in Subsection 4.1, different samples can be generated.

II. Bayes estimates of $a, b, \alpha_1, \alpha_2, \delta_1,$ and δ_2 are obtained by following the given steps:

1. Assuming the initial values of the distribution parameters and the value of the sample size *n*.

2. Generate random samples of size (30, 50, and 100) from the population distribution under study as.

3. Repeat Step 2 N times, where N=10000.

4. If φ_j^* is an estimate of $\omega,$ based on sample $j,\ j = 1,2,\dots,N,$ then the average estimate over the samples is given by $\overline{\varphi_j^*} = \frac{1}{N}\sum_{j=1}^{N}\varphi_j^*$.

5. The estimated risk of φ^* over the *N* samples is given by

$$ER(\varphi^*) = \frac{1}{N}\sum_{j=1}^{m}(\varphi_j^* - \varphi)^2.$$

Using Steps (4) and (5), compute $\overline{a^*}, \overline{b^*}, \overline{\alpha_1^*}, \overline{\alpha_2^*}, \overline{\delta_1^*}, \overline{\delta_2^*}$, $ER(a^*), ER(b^*),\ ER(\alpha_1^*), ER(\delta_1^*)$ and $ER(\delta_2^*)$.

In the case of two-sample Bayesian prediction

1. Assuming the initial values of the distribution parameters and the value of the sample size *n*.

2. Generate a bivariate random sample of size n, say $(T_1, Y_1), (T_2, Y_2)$ as shown in the beginning of this algorithm.

3. Follow steps in Subsection (5.2).

Table 9displays the average estimates, ERs and variances of the Bayes case based on samples of different size n and N=10000 repetitions with informative prior. The generated population parameters are $(a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \delta_1 = 1.2, \delta_2 = 1.7)$. The vector of hyper parameters is $(c_1 = 0.1, c_2 = 0.2, c_3 = 0.3, c_4 = 0.4, c_5 = 0.5, c_6 = 0.6)$.

Table 10 present the Bayes averages, ERs and credible intervals of rf and hrf for different values of the time t_{01}, t_{02} based on informative priors.

The Bayes two-sample predictors under informative priors is presented in Table 13.

In Table 9, the hyper parameters are $(c_1 = 0.1, c_2 = 0.2, c_3 = 0.3, c_4 = 0.4, c_5 = 0.5, c_6 = 0.6)$ and the population parameters are $(a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \delta_1 = 1.2, \delta_2 = 1.7)$, in case of the two sample prediction and using informative samples of different sizes.

Table 9

Bayes averages, relative absolute biases, estimated risks and 95% credible

Intervals for the parameters, of BBIII using informative prior

$(\mathbf{N = 10000, a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \delta_1 = 1.2, \delta_2 = 1.7})$

n	parameters	Average	RAB	bias	ER	UL	LL	Length
30	a	0.5985	0.0026	0.0015	2.6840e-6	0.5993	0.5974	0.0019
	b	0.7979	0.0026	0.0020	4.9248e-6	0.7995	0.7966	0.0029
	α_1	1.4991	0.0004	0.0004	5.2852e-7	1.2003	1.1982	0.0021
	α_2	1.1995	0.0006	0.0009	1.3222e-6	1.5001	1.4976	0.0025
	δ_1	1.0991	0.0008	0.0008	1.2149e-6	1.1004	1.0980	0.0023
	δ_2	1.6983	0.0010	0.0017	5.4785e-6	1.7008	1.6961	0.0047
50	a	0.6012	2.004e-3	1.2024e-3	1.8591e-6	0.6020	0.5998	0.0022
	b	0.7986	1.8096e-3	1.4477	2.2964e-6	0.7992	0.7974	0.0019
	α_1	1.4991	1.7369e-5	2.0843e-5	3.0462e-7	1.2011	1.1989	0.0022
	α_2	1.1999	5.3692e-4	8.0539e-4	8.0821e-7	1.4999	1.4982	0.0017
	δ_1	1.0986	1.2346e-3	1.3581	2.3690e-6	1.0999	1.0969	0.0031
	δ_2	1.7004	2.3871e-4	4.0581e-4	3.8302e-7	1.7011	1.6990	0.0021
100	a	0.6005	8.5666e-4	5.1399e-4	7.8194e-7	0.6016	0.5992	0.0024
	b	0.7999	1.8424e-4	1.4739e-4	1.6542e-7	0.8004	0.7987	0.0016
	α_1	1.5002	2.2129e-5	2.6554e-5	3.4582e-7	1.2009	1.1989	0.0020
	α_2	1.2000	1.2200e-4	1.8300e-4	5.5397e-7	1.5013	1.4989	0.0024
	δ_1	1.1000	2.6786e-5	2.9464e-5	1.8211e-7	1.1008	1.0989	0.0018
	δ_2	1.6994	3.3758e-4	5.7388e-4	8.1184e-7	1.7005	1.6979	0.0025

Table 10

Bayes averages, relative absolute biases, estimated risks and 95% credible intervals of the reliability and hazard rate functions, using informative prior

$(N = 10000, a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \delta_1 = 1.2, \delta_2 = 1.7, t_{01} = 2, t_{02} = 4)$

n	Rf and hrf	Averages	RAB	bias	ER	UL	LL	Length
30	$R(t_0)$	0.6664	0.0004	0.0003	1.2801e-6	0.6680	0.6647	0.0033
	$h(t_0)$	0.0083	0.1256	0.0012	1.9527e-6	0.0097	0.0071	0.0027
50	$R(t_0)$	0.6666	0.0002	0.0001	3.0931e-7	0.6673	0.6651	0.0022
	$h(t_0)$	0.0101	0.0598	0.0006	4.6083e-7	0.0108	0.0093	0.0015
100	$R(t_0)$	0.6669	0.0004	0.0015	2.2176e-7	0.6677	0.0002	0.0014
	$h(t_0)$	0.0098	0.0229	0.0002	1.9803e-7	0.0103	0.0086	0.0017

Table 13

Bayes predictive, bounds (informative prior), relative absolute biases and estimated risks of the future observations

$(N = 10000, a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \delta_1 = 1.2, \delta_2 = 1.7)$

n	s	$\hat{y}_{(s)}$	Averages	RAB	bias	ER	UL	LL	Length
30	1	$\hat{y}_{1(s)}$	4.0012	2.99062e-4	0.0012	2.3007e-6	4.0026	3.9995	0.0031
		$\hat{y}_{2(s)}$	6.9995	7.3615e-5	0.0005	8.1512e-7	7.0006	6.9979	0.0027
	12	$\hat{y}_{1(s)}$	4.0017	4.1629e-4	0.0017	3.6987e-6	4.0029	3.9997	0.0032
		$\hat{y}_{2(s)}$	7.0003	5.2687e-5	0.0004	2.5995e-6	7.0031	6.9982	0.0048
	18	$\hat{y}_{1(s)}$	4.0018	0.0005	0.0018	3.9678e-6	4.0031	3.9999	0.0032
		$\hat{y}_{2(s)}$	6.9975	0.0004	0.0025	8.0403e-6	7.0001	6.9953	0.0047
50	1	$\hat{y}_{1(s)}$	3.9988	3.0945e-4	0.0012	1.9226e-6	3.9997	3.9973	0.0024
		$\hat{y}_{2(s)}$	6.9994	8.5430e-5	0.0006	7.6591e-7	7.0003	6.9981	0.0022
	12	$\hat{y}_{1(s)}$	3.9986	0.0003	0.0014	2.3209e-6	3.9998	3.9973	0.0025
		$\hat{y}_{2(s)}$	7.0013	0.0001	0.0013	2.4401e-6	7.0025	6.9996	0.0029
	18	$\hat{y}_{1(s)}$	4.0012	0.0003	0.0013	2.4480e-6	4.0026	3.9997	0.0028
		$\hat{y}_{2(s)}$	7.0021	0.0003	0.0021	5.5599e-6	7.0037	6.9997	0.0039
100	1	$\hat{y}_{1(s)}$	3.9998	5.6324e-5	2.2529e-4	6.5285e-7	4.0013	3.9982	0.0031
		$\hat{y}_{2(s)}$	6.9999	6.8299e-6	4.7809e-5	4.2778e-7	7.0013	6.9987	0.0026
	12	$\hat{y}_{1(s)}$	7.0011	2.7225e-4	0.0011	1.4069e-6	4.0018	3.9999	0.0018
		$\hat{y}_{2(s)}$	7.0003	4.5632e-5	0.0003	1.4275e-6	7.0019	6.9983	0.0037
	18	$\hat{y}_{1(s)}$	4.0012	0.0003	0.0012	2.3173e-6	4.0025	3.9992	0.0033
		$\hat{y}_{2(s)}$	6.9983	0.0002	0.0018	3.7799e-6	7.0002	6.9968	0.0034

6.2 Example of a data set

The data set which was given in Subsection 4.2 are analyzed to illustrate the theoretical results in Bayesian inference.

Tables 11 and 12 present the Bayes averages and ERs, of the parameters, rf and hrf, for the example data based on complete sample under informative prior. Tables 13 and 14 display Bayes predictors, bounds, based on informative prior, also relative biases and estimated risks for the future observations for simulated samples and the data set.

Table 11

Bayes estimates, relative absolute biases, estimated risks and standard
Deviations of the parameters for BBIII using informative prior for the data set

parameters	Estimate	RAB	bias	ER	sd
a	1.4983	0.0011	0.0017	3.2914e-6	0.0008
b	0.5497	0.0005	0.0003	3.4607e-7	0.0005
α_1	5.7993	0.0001	0.0007	8.1274e-7	0.0006
α_2	3.4993	0.0002	0.0009	1.8762e-6	0.0005
δ_1	3.5991	0.0002	00007	8.4606e-7	0.0011
δ_2	2.5003	0.0001	0.0003	4.2252e-7	0.0006

Table 12

Bayes estimates, relative absolute biases, estimated risks and standard
deviations of the reliability and hazard rate functions for the data set

Rf and hrf	Estimate	RAB	bias	ER	sd
$R(t_0)$	0.6679	0.0019	0.0013	3.0417e-6	0.0012
$h(t_0)$	0.0098	0.0357	0.0003	2.9655e-7	0.0004

Table 14
Bayes predictive, bounds (informative prior) of the future observations, relative absolute biases, estimated risks, standard deviations, for the data set under two-sample prediction

s	$\hat{y}_{(s)}$	Estimate	RAB	bias	ER	sd
1	$\hat{y}_{(s)}$	4.0001	3.1085e-5	0.0001	1.5636e-7	0.0004
	$\hat{y}_{1(s)}$	7.0005	7.6529e-5	0.0005	5.8066e-7	0.0005
12	$\hat{y}_{2(s)}$	4.0019	0.0004	0.0019	4.8198e-6	00009
	$\hat{y}_{1(s)}$	6.9991	0.0001	0.0009	1.7419e-6	0.0009
18	$\hat{y}_{2(s)}$	3.9974	0.0006	0.0025	7.0428e-6	0.0008
	$\hat{y}_{1(s)}$	6.9988	0.0002	0.0012	2.0628e-6	0.0008

6.3 Concluding remarks

In this study, we observe the following

1. The variance of any of the estimates is inversely proportional to the sample size and that the variance of an estimate tends to zero as the sample size tends to infinity.
2. The lengths of the CIs of the parameters become narrower as the sample size increases.
3. The Bayes averages for the rf and hrf performs better as the sample size increases. Also, ER is decreasing when the sample size is increasing.
4. It is interesting to notice that if the variables of the prior density are independent and if the likelihood function factors out with respect to these variables then the variables of the posterior given data are also independent.

That if $\pi(\varphi_1, \ldots, \varphi_k) = \prod_{i=1}^{k} \pi(\omega_1)$ and if $L(\varphi_1, \ldots, \varphi_k | \underline{t}) = \prod_{i=1}^{k} L(\varphi_i | \underline{t})$,

then

$$\pi^*(\varphi_1, \ldots, \varphi_k | \underline{t}) \propto \pi(\varphi_1, \ldots, \varphi_k) L(\varphi_1, \ldots, \varphi_k | \underline{t}) = \prod_{i=1}^{k} \pi(\varphi_1)\, L(\varphi_i | \underline{t})$$

$= \prod_{i=1}^{k} \pi^*(\varphi_1 | \underline{t}) \Rightarrow (\varphi_1 | \underline{t}, \dots, \varphi_k | \underline{t})$ are independent, which the analysis will be easier.

5. The likelihood function of the BBIII distribution can be derived using the pdf in (2.4) directly, but compounding of $\prod_{i=1}^{k} f(t_i | \underline{\varphi})$ and $g(b)$ can be applied to make the ML estimation easier.

The results become better as the informative sample size gets larger. In all cases, the simulated percentage coverages are at least 95%.

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دراسة للتوزيع الثنائى لتوزيع BURR TYBE III

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ملخص باللغة العربية

تم تكوين توزيع ثنائي لتوزيع BurrIII حيث كان الأساس تركيب معكوس وايبل (Inverse Weibull) مع Gamma وذلك بإستخدام أسلوب مشابه لأسلوب AL-Hussaini (2005) حيث تناول هذا العمل عرضا لبعض الخصائص التى تم إشتقاقها من خلال التوزيع الثنائي لبير من النوع الثالث (BBurrIII) وكذلك تم تقدير معالم التوزيع الجديد بطريقتى الإمكان الأعظم وبييز. كذلك تم التنبؤ بالمشاهدات المستقبلية المرتبة والمشاهدات المصاحبة لها بإستخدام التنبؤ بطريقة بييز Bayesian prediction وقد تم إفتراض أن جميع المعالم مجهولة ومستقلة وأن التوزيع القبلى prior distribution لكل منها يتبع توزيع جاما وقد تم إيجاد التوزيع البعدى Posterior distribution وأيضاً دالة التنبؤ البيزية Bayesian predictive density فى حالة التنبؤ بعينتين Two-sample prediction. ثم عمل التنبؤ للمشاهدات البيزية. كما تم عمل دراسة عددية لدراسة أداء مقدرات دالة الإمكان الأعظم وبييز وتم تطبيقها على بيانات حقيقية لمعرفة مدى تطبيقه عملياً وذلك بإستخدام بيانات إستخدمها Sankaran and Kundu (2014) .