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AN IMPROVED DYNAMIC MODELLING OF FIBER REINFORCED LAMINATED COMPOSITE STRUCTURES

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ABSTRACT

In the present paper modified modelling techniques are generated for computing dynamic characteristics of laminated composite structures with more accurate results and with relatively small time of computation.

A comparison between the traditonal finite element model and the present improved dynamic models is introduced through a case of longitudinal vibration of fixed - free composite structural rod. The results show the efficiency of the present techniques for dynamic modelling of composite structures.

Key words : Composite structures, Dynamic modelling, Matrix computer method.

1. INTRODUCTION

Modelling of complex large structures is one of the fundamental activities of researchers as analysis, synthesis and measurments. The development and improvement of dynamic models of complex structures of isotropic nature are subjects of numerous scientific work. They include economic and accurate solutions in terms of different advanced techniques such as condensation, finite element and finite difference. Unfortunately, the volume of references avaliable concerning modelling and applications of these advanced techniques to large composite structures implies a lack of adhesive structures in the methodology of developing and improving of

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modelling and analysis for various composite structural problems.

The predominant aim of the present work is to provide efficient and simple tools for the development and analysis of dynamic models of fiber reinforced laminated composite structures. The proposed modelling techniques are based on an improved finite dynamic composite element (F.D.E.) in conjunction with the dynamic correction technique to reach for hand accurate results and reduced time of computations. For illustrating the efficiency of the developed techniques, the longitudnal vibrating fixed - free composite rod is considered. The comparison between the computed results of the proposed and traditional techniques for various samples of fiber reinforced laminated composite rods proves the efficiency of the present techniques as regard to the proper selection of the prescribed degree of accuracy and the small time of computation.

2. PROBLEM STATEMENT

One of the major factors for the proper modelling of the dynamic behaviour of fiber reinforced composite structure is the proper simulation of the elastic characteristics of its building material. An algorithm has been proposed within the frame of the present work for specification elastic characteristics of composite structures. The algorithm includes modelling of a general longitudinal composite rod in terms of the extensional stress strain relations given as :

$$\sigma_x = E_c \zeta_x$$

where E_c is the equivalent modulus of extensional elasticity of composite rod.

The second algorithem is focused on modelling of extensional stress strain relation of longitudinal composite rod of symmetrical nature. For a general extensional state of elastic deformation, the equivalent apparent Youngs modulus E_c for various code numbers are also computed by using the proper condensation technique. The influences of the code numbers and particularily the lamina orientations on the elastic characteristics of structural composite models have been investigated theoretically [8]. The analysis have revealed that the change of outer lamina orientations for three layer's structures have the pre-dominant effects on the stiffness modulus and on the dynamic nature of composite structures compared with the orientation of inner lamina as represented in Figures 2 and 3 respectively. In Figure 3 the models have been studied for six code numbers [0 / 0 / 0], [0 / 30 / 0], [0 / 45 / 0] [0 / 90 / 0], [45 /- 45 / 0] and [45 / 0 / 45] for two levels of fiber volume fraction namely V_f = 15 % and 45 %. The interrelationships are presented by means of various curves to simulate the dynamic nature of fiber reinforced composite structural beam.

Three phases of mathematical models of composite structures are introduced herein as

(1) Finite element method (F.E.M) for composite structures.
(2) Dynamic element method of composite structures (D.E.M).
(3) Modified formula for composite element method (M.F.M).

To demonstrate the considerable improvement in the accuracy of the computed natural frequencies by the utilization of the present techniques the results are tabulated and listed out in various curves as shown later.

2.1. The first phase (Finite element method, F.E.M)

For computing the natural frequencies of the composite rod the total mass and stiffness materices are constructed using the traditional polynomial static displacements function to satisfy its static strain displacements and stress - strain relationships. The standard form to compute the eigen value is given by

$$\left([K_\circ] - \omega^2 [M_\circ] \right) \lceil q \rceil = \lceil o \rceil \qquad \text{.......(1)}$$

where $[M_\circ]$: is the global mass matrix of an assembly of composite structural rod.

$[K_\circ]$: is the global stiffness matrix of an assembly of composite structural rod.

$|q|$: is the global physical displacement vector.

The element stiffness matrix $[k_e]$ and mass matrix $[m_e]$ of the eth element of the assembly are constructed in terms of the equivalent apparent moduls (E_c) as

$$[k_e] = \int_v [b(x)]^T \cdot E_c \cdot [b(x)] \cdot dv \qquad \text{.........(2)}$$

$$[m_e] = \int_v [a(x)]^T \cdot \rho_c \cdot [a(x)] \cdot dv \qquad \text{.........(3)}$$

where e = 1, ..., n,

and n = number of elements of the assembly

$[b_{(x)}]$ is the strain displacement matrix of beam element.

$[a_{(x)}]$ is the matrix of shape function of beam element.

ρ_c is the mass density of composite rod element.

The proposed expression of the apparent Young's modulus of elasticity of axial loading composite beam is given by

$$E_c = \left(\frac{1}{t}\right) \cdot A_c^* \qquad \text{.........(4)}$$

Where A_c^* : is the condensed extensional stiffness modulus of fiber reinforced one-dimensional lamina

t : is the thickness of rod layers

The values of A_c^* and ρ_c depend on different parameters mainly fiber volume fraction (v_f) and variation in lamina orientation (Θ) as shown in Fig. 2 and the form [Ref. 1,2] can be expressed as

$$A^* = \left[A_{11} - \frac{A_{12}^2}{A_{22}} + \frac{(A_{12}A_{26} - A_{16}A_{22})^2}{A_{22}\left(A_{26}^2 - A_{22}A_{66}\right)}\right]$$

Note that :

$\rho_c = \rho_f \cdot v_f + \rho_m \cdot v_m$

ρ_f is the mass density of fibers material.

ρ_m is the density of matrix material.

v_f is the fiber volume fraction.

v_m is the matrix volume fraction.

2.2. The second phase (modelling dynamic element method technique of composite structures (D.E.M) :

To improve the accuracy of vibrating composite models the dynamic element method utilizes the dynamic displacement function instead of the static displacement function (considered in the first phase). This yields higher order dynamic correction terms for the element stiffness and mass matrices. The matrix frequency depends on the dynamic shape function [Ref. 3,4], which can be employed to develop the frequency dependent stiffness and mass matrices of vibrating structure composite rod and the eigenvalue problem is then modified as

$$\left([K_o] - \omega^2 [M_o] - [R_s]\right)\lceil q \rceil = \lceil o \rceil \qquad \text{............} \quad (5)$$

where $[M_o]$ represents the static mass matrix

$[K_o]$ represents the static stiffness matrix

$[R_s]$ represents the dynamic correction matrix.

The general form of the dynamic correction matrix can be written (as shown later) as

$$[R_s] = \sum_{s=1}^{\infty} \left(\omega^{2s}[m_{2s}] - \omega^{4s}[k_{4s}]\right) \qquad \text{............} \quad (6)$$

where s = 0 is associated with the static displacement function and s >0 is the number of the retained dynamic corrections parameters.

2.3. The Third phase (Modified formula method for longitudinal vibrating composite structures M.F.M).

The third phase can be considered as a modification of the natural frequencies formula of an uniform isotropic sandwich rod (by introducing parameters concerning fiber orientation of composite lamina). It is known that the traditional form of the ith frequency [Ref. 5] may be given by

$$f_i = \frac{\lambda_i}{2\pi L}\left(\frac{E}{\rho}\right)^{\frac{1}{2}} \text{ for } i = 1,2, ...m \qquad (7)$$

where λ_i is the dimensionless parameter depending on boundary conditions.

E : is the modulus of elasticity of isotropic structural rod.

L : is the length of the rod.

ρ : is the mass density of the rod material,

m : is the number of modes.

For modifications of the modulus of elasticity and mass density of composite rod E_c and ρ_c are introduced in the later expression and the modified form is then expressed as

$$f_i = \frac{\lambda_i}{2\pi L \sqrt{\tau}}\left[\frac{E_c}{\rho_c}\right]^{\frac{1}{2}} \qquad [8]$$

It is obvious that the previous equation of the ith natural frequency of composite structural rod depends on different parameters mainly fiber volume of fraction (v_f), variation in lamina orientation (θ) and order of the mode of vibration.

3. A CASE STUDY : Computation of longitudinal natural frequencies of composite rod.

For illustrating the accuracy of the developed technique the longitudinal vibration of fixed free structural composite rod of an uniform cross section is considered. In view of the first phase the structural composite rod of length

L is discretized into eight elements interconnected at seven nodes as shown in Figure 4. Each node is represented by a single degree of freedom for describing the longitudinal vibration of the assembly as shown in the figure. q_1 and q_2 represent the nodal translatory displacements in the X-direction at nodes 1 and 2 of the element respectively. Assuming that the displacement $U_{(x)}$ of a field point within the element varies linearly in X direction we have:

$$U_{(x)} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{.......... (9)}$$

Thus

$$U_x\,(x) = [a\,(x)] \begin{vmatrix} q_1 \\ q_2 \end{vmatrix}$$

Applying the boundary conditions we have

$$[a\,(x)] = \left[\left(1 - \frac{x}{l}\right) \left(\frac{x}{l}\right) \right], \qquad \text{.......... (10)}$$

Note that, the strain - displacement relationship for axial element is given by

$$\zeta_x = \frac{\partial U_x\,(x)}{\partial\, x}, \qquad \text{.......... (11)}$$

, thus

$$\zeta_x = [b\,(x)] \,.\, |q| = \frac{1}{l}(-1 \quad 1)\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \qquad \text{.......... (12)}$$

In view of equation (2) the element stiffness matrix $[K_e]$ can then be expressed as

$$[k_e] = \frac{E_c\,.\,A}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{.......... (13)}$$

Similarly the element mass matrix $[m_e]$ can be calculated with the help of Eqn. (3) and the following equation :

$$[m_e] = \frac{\rho_c\,AL}{3} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \qquad \text{.......... (14)}$$

where A is the cross - sectional area of the composite rod. The natural frequencies of the assembly can then be calculated by Eqn. (1).

In view of the phase 2 the dynamic element modelling of the element is identified by two harmonic end displacements :

$$U_1(t) = q_1 . e^{i\omega t} \quad \text{and} \quad U_2(t) = q_2 . e^{i\omega t} \qquad (15)$$

The nodal displacement vector can then be expressed as

$$\left[U_{(t)}\right] = \begin{bmatrix} U_1(t) \\ U_2(t) \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} . e^{i\omega t} \qquad (16)$$

The axial displacement of a field point $U_x(x, \omega, t)$ is expressed in an expanded form of circular frequency ω, so that

$$U_x(x, \omega, t) = \left[a(x, \omega) \right] \left[U_{(t)}\right] \qquad (17)$$

in which $[a(x, \omega)]$ represents the matrix of frequency dependent shape functions [Ref. 3] given by

$$\left[a(x, \omega)\right] = \left[a_o(x) + \omega . \left[a_1(x)\right] + \omega^2 . \left[a_2(x)\right] + \ldots\right]$$

$$= \sum_{r=0}^{\infty} \omega^r . \left[a_r(x)\right] \qquad (18)$$

By substituting Eqn. (18) into the equation of motion of the uniform extensional continuous rod, (Appendix), the element mass and stiffness matrices of the composite rod can then be given respectively as

$$\left[m_e\right] = (m_o) + \omega^2 (m_2) + \omega^4 (m_4) + \ldots \qquad (19)$$

$$\left[k_e\right] = \int_V \left(\left[b(x, \omega)\right]^T . E_c . \left[b(x, \omega)\right]\right) d_v$$

$$= (k_o) + \omega^4 (k_4) + \omega^8 (k_8) + \ldots \qquad (20)$$

in which

$$(m_r) = \rho_c \int \left(\sum_{r=o,2,4}^{N} [a_r(x)]^T \cdot [a_{N-r}(x)] \right) .dv \qquad \text{.......................} (21)$$

$$(k_r) = E_c \int_v \left(\sum_{r=0,4,8}^{N} (b_r(x)^T \cdot (b_{N-r}(x) \right) . dv \qquad \text{.......................} (22)$$

The latter expressions can be then substituted in the correction matrix given by Eqn. (6) to improve the accuracy of the computed natural frequencies.

4. RESULTS AND DISCUSSION

In the numerical work the elastic parameters and natural frequencies of longitudinal vibration of fixed free glass fiber reinforced 3-laminated composite beam have been computed for various case studies. Three dynamic modelling techniques F.E.M, D.E.M and M.F.M are applied for computing the modulus of elasticity and natural frequencies for different six composite structure with code numbers [(0 / 0 / 0), (0 / 30 / 0), (0 / 45 / 0), (0 / 90 / 0), (45 / 0 / 45) and (45 / - 45 / 0)] and for two fiber volume fractions V_f = 15 % and 45 %.

The nondimensional results of the first four modes of vibration are listed out in Table 1, and compared with modified formula technique, for illustrating the efficiency of the proposed techniques as shown in Figs. (5, 6, 7). The percentage errors with respect to the modified formula are plotted in terms of the number of elements and of the various order of dynamic corrections, which are defined by the pair (a, b), respectively, versus the computing time.

From the stand point view of computational time it can be noticed that the consumed time of computation either for the pair (1, 5) or (2, 0) is the same and equal to 5 sec. while the ultilization of the pair (1, 5) yields more accurate results compared with the pair (2, 0). Similarly, the computing time for the pairs (2, 1) and (3, 0) is the same and equals to (13 sec.), while the error percentage of (2, 1) is less by 10 times of the error associated with the pair (3, 0). In that way the pairs (2, 5), (3, 1) and (4, 0) consume the same computing time (66 sec) taking into account that the pair (2, 5) offers the lowest error percentage compared with the other two cases.

It means that the influence of increasing the order of dynamic correction parameters is pre-dominant with respect to the refinement of the mesh or the increasing number of elements. Hence the required accuracy for each case can be chosen with the shortest computing time. As example, for the specific percentage error from 1 % to 10 %, the pair (1, 1) is preferred compared with the pair (2, 0) and (3, 0) due to the shortest time of compution as illustrated in Fig. 5. Without loss the generality the shortest time of computation for the specified accuracy, vice versa, can be then selected by listing the proper pair for the specified mode.

From qualitative and quantative analysis of the computed values of stiffness parameters versus the natural frequencies three parameters are to be considered, here as.

1. Type of dynamic modelling technique
2. Code number of rod
3. Fiber volume fraction

In view of dynamic modelling technique the modified formula shows more accurate values than those obtained from the two other F.E.M and D.E.M as listed in Table (1). This is expected since the formula is derived on the base of eigen solution of continuous media and with the avoidence of matrices manipulators needed by F.E.M and D.E.M. Unfortunatdy, the applications of the formula are restricted to simple structures subjected to basic boundary conditions.

As stated before, the D.E.M shows good results, and nearly closed to those obtained by the modified formula, in comparison to the results obtained by F.E.M as shown in Table (1).

With respect to the code number, the rod orientation sequence plays important factor in computed values of the stiffness modulus A*, as illustrated in Fig. 2. The variations in either outer or inner lamina orientation have effective influences. It is noticed that A* decreases curvilinearly as either outer or inner lamina orientation increases in the region $0 \leq \theta \leq 64.2°$, while it increases curvilinearly as the orientations increase in the region $115.8° \leq \theta \leq 180°$. In the region $64.2° \leq \theta \leq 115.8°$ the changes of angles of orientations have small effect on variation of A* (nearly constant).

In Fig. 3 it is noticed that the rod coded (0 / 0 / 0) has the highest longitudinal natural frequencies due to the great stiffness at such orientation sequence. The result is that an ascending linear relations exist between natural frequencies and equivalent stiffness modulus for various modes as shown in Fig. (3). Also it is shown that as the inner lamina deviates from 0° to 90° the computed natural frequencies decrease as the deviations increase as indicated in the case of the samples of the coded numbers (0 / 0 / 0), (0 / 30 / 0), (0 / 45 / 0) and (0 / 90 / 0). It is evident that the lowest natural frequencies are associated with the lamina orientation of angle 45° since the shear stress reaches its maximum values.

Refferred to the last parameter, the spectrum of the results of natural frequencies increases by increasing the rod fiber volume of fraction from 15 % to 45 %. This is expected since the highest volume fractions, the highest stiffness of composite rod will be reached.

5. CONCLUSION

1. From the accuracy and consumed time of computation stand point of view it is noticed that the high order of dynamic correction parameters has a dominant effect on the efficiency of the simulation of the eigen nature of composite structures compared with the fine mesh or the increase number of finite elements of the assembly.

Thus, without loss the generality the (M.F.M) provides more accurate results for a broad class of simple composite structures.

2. In the case of lamina of three layers Fig. 2 the variations in the orientations of the outer lamina have great influences on the values of the

stiffness and of eigen frequencies of the assembly compared with the variations in the orientations of the inner lamina.

3. As the current angle of orientations of lamina increases from zero as the natural frequencies decrease. It means that as the number of lamina of zero orientation increases as both the stiffness and frequencies increase, Fig. 3.

4. There exists a stationary state at which the variations of angles of orientations of either outer or inner lamina have negliable effects on both the stiffness and the natural frequencies. This statement is bounded by the angles of orientation $62° < \theta < 120°$ as shown in Fig. 2.

6. REFERENCES :

1. Jack R. Vinson and Tsu-Wei Chou, "Composite materials and their use in structuers", Applied science publishers Ltd., 1975.
2. Robert M. Jones, "Mechanics of composite materials", Scriptabook company, 1975.
3. J.S. Przemieniecki, "Theory of Matrix structural Analysis", New York : McGraw - Hill, 1968.
4. Singireus S. Rao, "Mechanical vibration", Adisson - Welsey publishing company, 1989.
5. Robert D. Blevins Ph.D., "Formula for Natural Frequency and Mode Shape", Litton Educational Publishing, Inc., 1979.
6. S. Krishnaswamy, K. Chandrashekhara and W.Z.B. Wu, "Analytical Solutions to vibration of Generally Layered Composite Beams" Journal of Sound and Vibration 159, 85 - 99, 1992.
7. R.F. Gibson and R. Plunket, "Dynamic, Stiffness and Damping of Fiber-Reinforced Composite Materials" Shock and Vibration Digest 9(2) (February 1977).
8. A. Maher, F.R. Gomaa and M. FERRA "On the Modelling of composite structures" A Thesis for M.Sc. Degree in Engineering Menoufia University Faculty of Engineering (1998).

Table (1) : Percentage errors for different mathematical dynamic models

Refared value of the first natural frequency (1.570796327)

Mode	Truncated order b / No. of element a	Static R_o	R_1	R_2	R_3	R_4	R_5
		Mathematical model					
			Dynamic				
First mode	1	(10.2)	(1.905)	(0.438)	(0.107)	(2.647x10^{-2})	(6.603x10^{-3})
		[1.5]	[2]	[3]	[3.5]	[4]	[5]
	2	(2.586)	(0.149)	(9.023x10^{-3})	(5.632x10^{-4})	(3.52x10^{-5})	(2.165x10^{-6})
		[5]	[13]	[18]	[27]	[38]	[50]
	3	(1.146)	(3.014x10^{-2})	(8.34x10^{-4})	(2.317x10^{-5}	(6.366x10^{-6})	(0)
		[13]	[34]	[66]	[109]	[163]	[232]
	4	(0.644)	(9.694x10^{-3})	(1.51x10^{-4})	(2.355x10^{-6})	(0)	(0)
		[23.5]	[66]	[141]	[233]	[353]	[501]
		Refared value (4.71238898) at the second natural frequency					
Second mode	2	(19.458)	(6.874)	(3.178)	(1.608)	(0.84)	(0.459)
		[5]	[13]	[18]	[28]	[38]	[53]
	3	(10.266)	(1.905)	(0.438)	(0.108)	(2.647x10^{-2})	(6.6x10^{-2})
		[13]	[35]	[66]	[109]	[163]	[232]
	4	(5.832)	(0.674)	(9.145x10^{-2})	(1.277x10^{-2})	(1.794x10^{-3})	(2.522x10^{-4})
		[25]	[66]	[143]	[233]	[355]	[501]
		Refared value (7.853981634) of the third natural frequency					
Third mode	3	(20.023	(8.272)	(4.650)	(2.863)	(1.830)	(1.196x10^{-2})
		[13]	[37]	[69]	[109]	[165]	[233]
	4	(15.348)	(4.053)	(1.370)	(0.503)	(0.191)	(7.369x10^{-2})
		[26]	[68]	[143]	[236]	[356]	[503]
		Refared value (10.99557429) of the fourth natural frequency					
Fourth mode	4	(19.154)	(8.296)	(5.121)	(3.491)	(2.461)	(1.765)
		[27.5]	[68]	[146]	[239]	[360]	[513]

Values in ( ) represent the percentage errors

Values in [ ] represent the computing time.

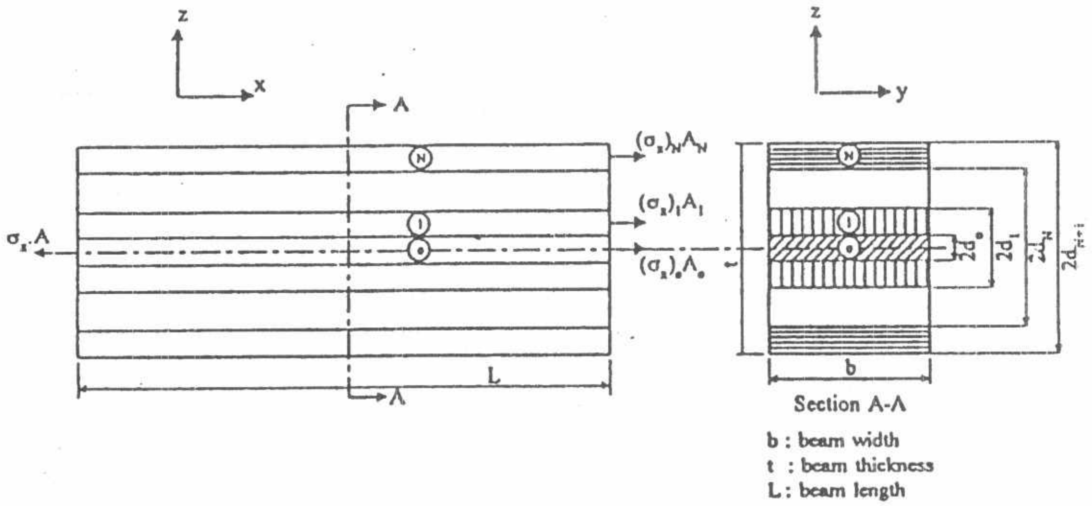

Fig. (1): laminated composite beam subjected to tensile stress, σ_x

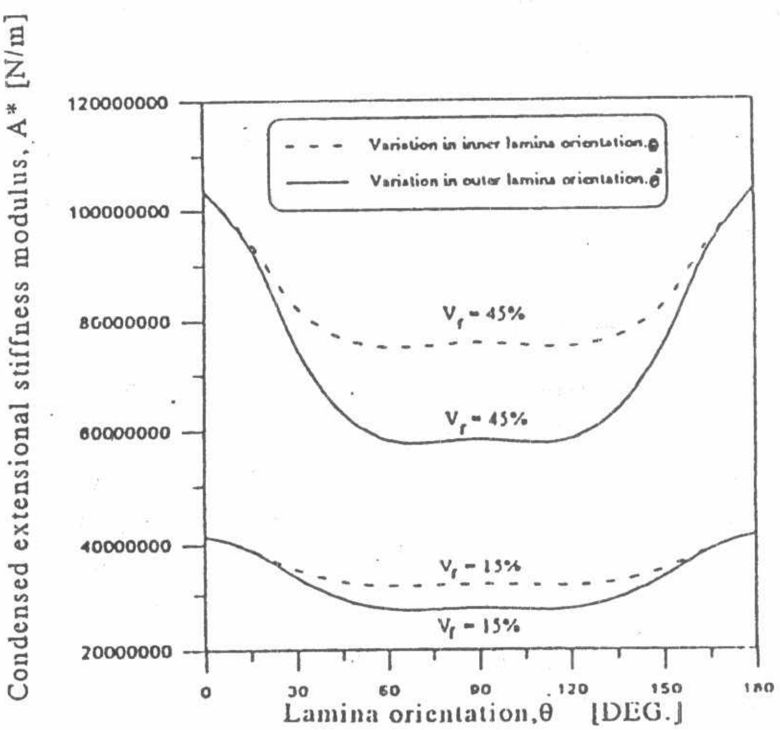

Fig. (2) : Effect of variation in lamina orientation on the condensed extensional stiffness modulus of the code no. $[\theta_c^* / \theta_c / \theta_c^*]$ at two levels of fiber volume fraction; V_f - 15% and 45%.

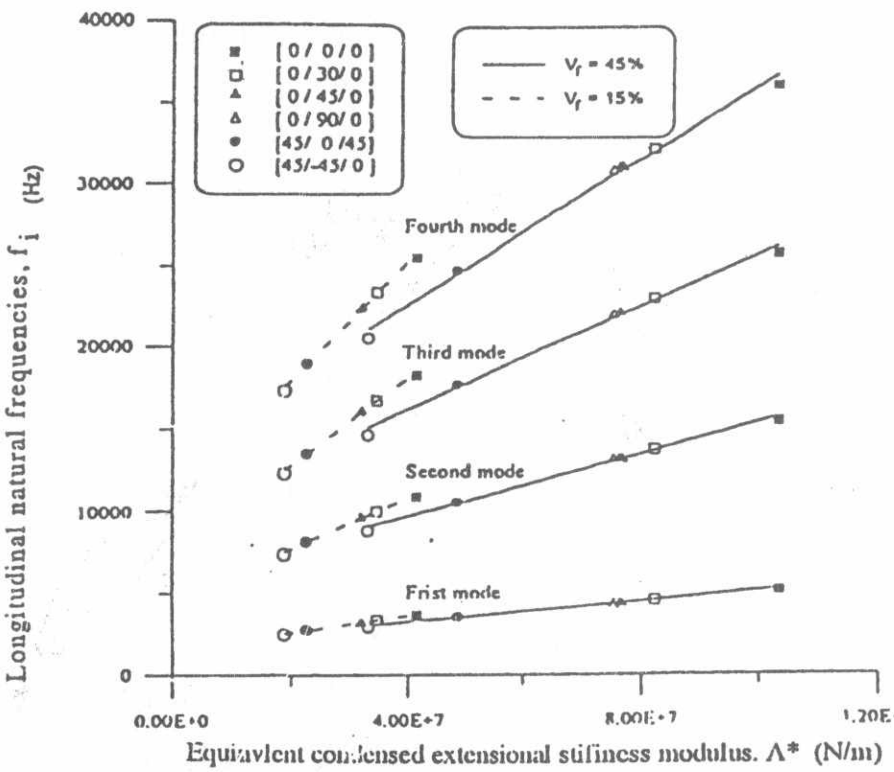

Fig. (3) Quasilinear relationships between the condensed extensional stifness modulus A* and the first four longitudinal natural frequencies (by modified formula) for fixed-free GRP 3-laminated composite beams.

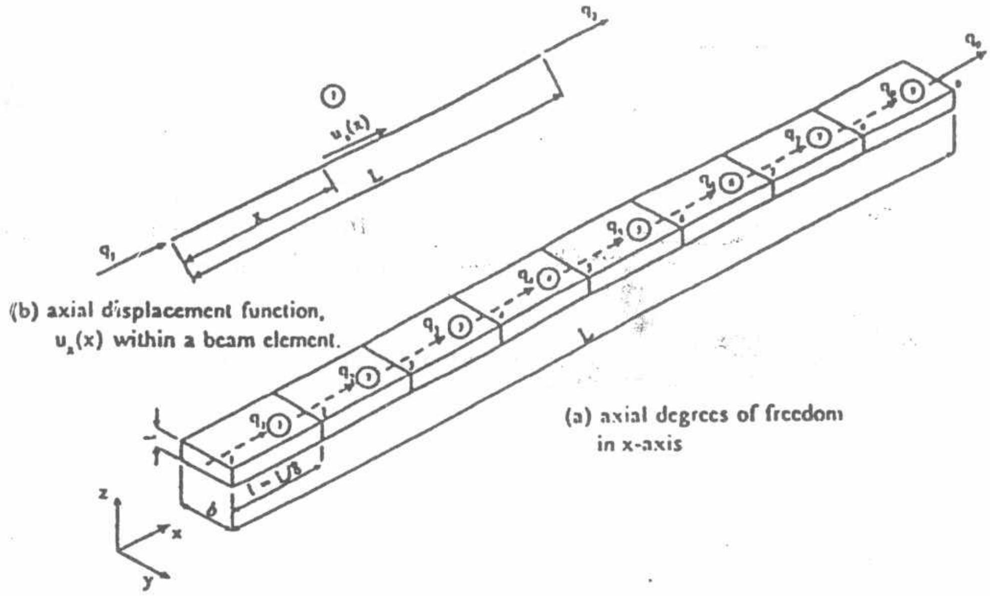

Fig. (4) : Discretized free-free extensional beam.

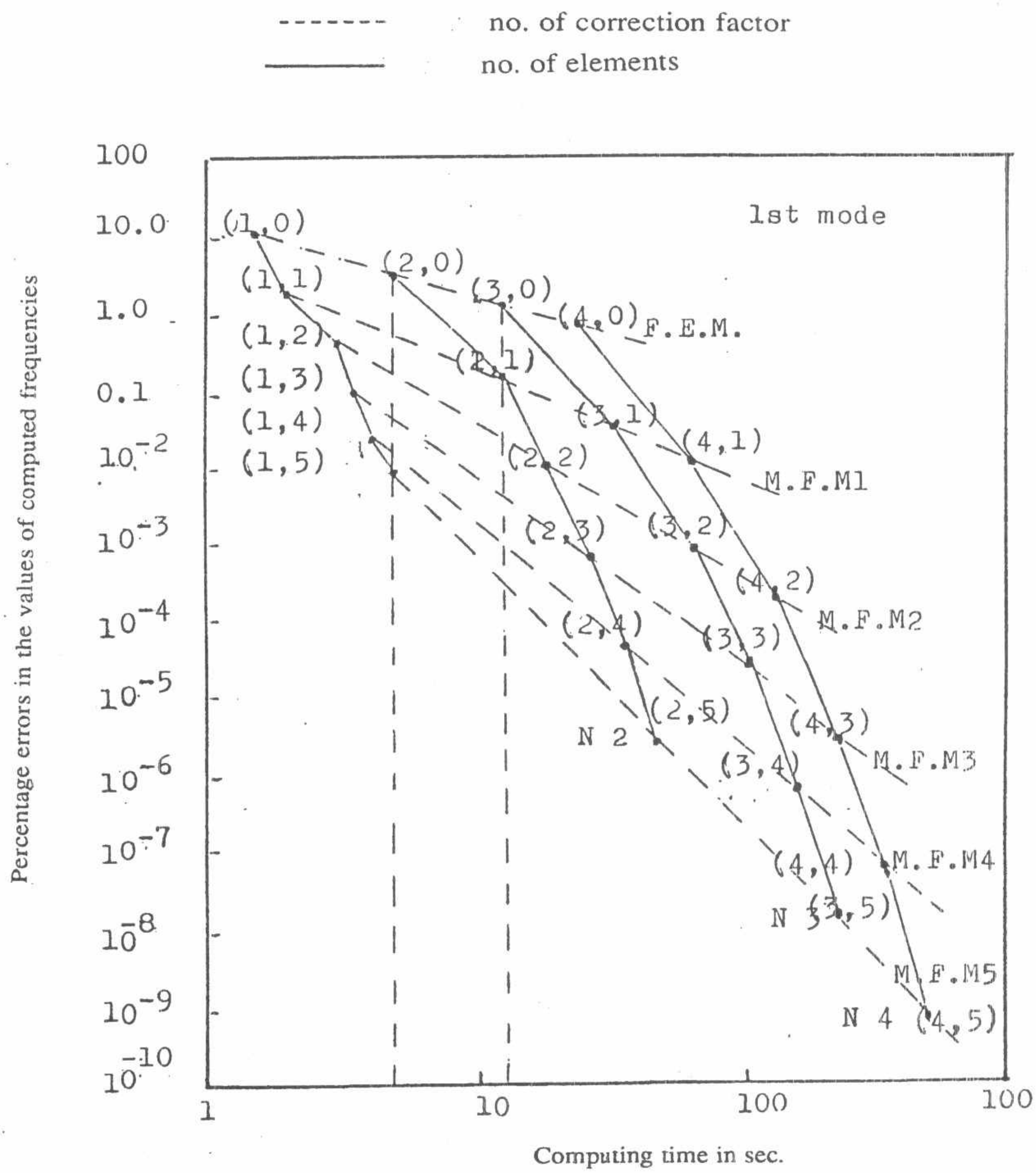

Fig. (5) : The accuracy of the computed first natural frequency by the utilization of the present technique.

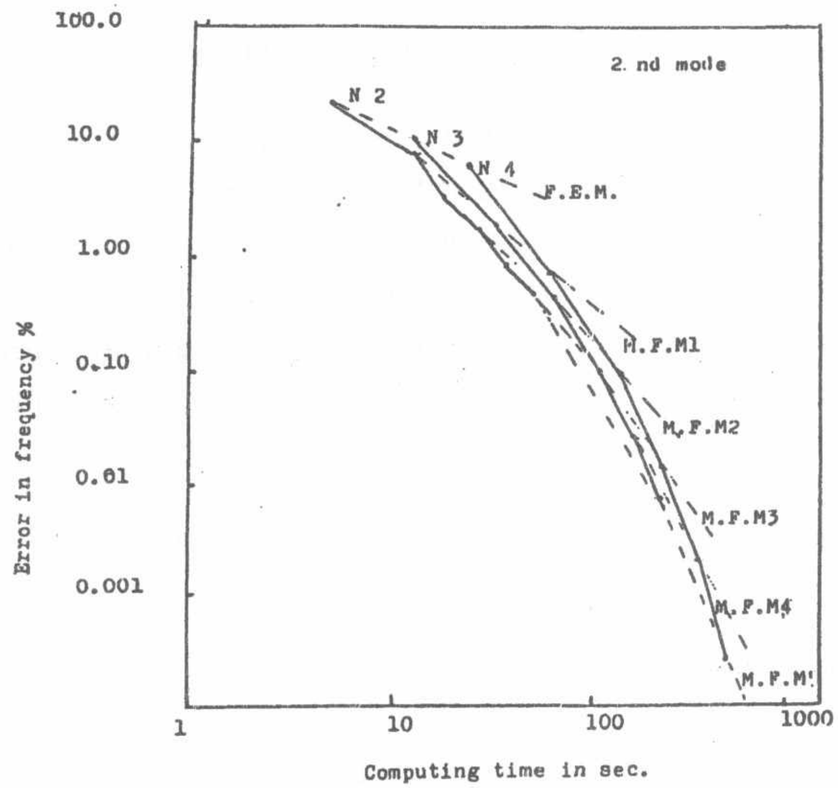

Fig (6) The accuracy of the computed second natural frequency by utilization of the present technique

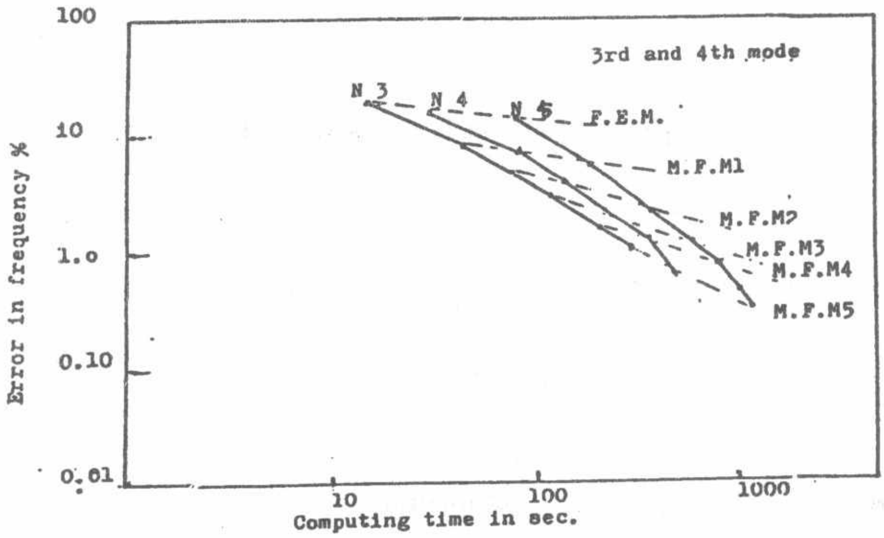

Fig. (7) The accuracy of the computed third and fourth natural frequency by utilization of the present technique

APPENDIX

Modelling of Extensional stress - strain Relations of LCB

The fiber - reinforced shown in Fig. (1) at which the distance from the neutral axis to the face between the K<u>th</u> and K+1<u>th</u> layers in the rod is denoted by d_K (do = 0). The rod is subjected to a tensile force σ_X. A, where σ_X is the normal stress and A is the rod cross - sectional area. Each layer is then subjected to a tensile force of $(\sigma_X)_K$, with $(\sigma_X)_K$ and A_K are the normal stress and cross - sectional area of the K<u>th</u> layer respectively. The equilibrium of forces in the x-directional yields.

$$\sigma_x = (\sigma_x)_\circ \cdot A_\circ + 2\sum_{k=1}^{N} (\sigma_x)_k \cdot A_K \qquad \text{[A-1]}$$

The series of A_K is then expressed as

$$\left.\begin{aligned} A_K &= \frac{d_{K+1} - d_K}{2\, d_{N+1}} \\ A_\circ &= d_1 / d_{N+1} \end{aligned}\right\} \quad \text{for } k = 1,2 \ldots\ldots N \qquad \text{[A-2]}$$

Substituting eqn (A-2) into equation (A-1) it gives

$$\sigma_x = \sum_{K=0}^{N} (\sigma_x)_K \cdot \left(\frac{d_{K+1} - d_K}{d_{N+1}}\right) \qquad \text{[A-3]}$$

The normal stress of the K<u>th</u> lamina is then given by

$$(\sigma_x)_K = \left(\overline{Q}^*\right)_K \cdot (\zeta)_K \qquad k=1,N \ldots \qquad \text{[A-4]}$$

Substituting (Eq. A-4) into (Eq. A-3) we get

$$E_c \cdot \zeta_x \cdot d_{N+1} = \sum_{K=0}^{N} \left(\overline{Q}^*\right)_K \cdot (\zeta_x)_K \cdot \left(d_{k+1} - d_K\right) \qquad \text{[A-5]}$$

Assuming the strain for each layer is invariant, i.e.

$$\zeta_x = (\zeta_x)_\circ = (\zeta_x)_1 = (\zeta_x)_K$$

where, $\sigma_x = E_c\,\zeta$

The equation (A-5) is then reduced to

$$E_c \,.\, d_{N+1} = \sum_{K=0}^{N} \left(\overline{Q}^*\right)_K . \left(d_{k+1} - d_K\right) \qquad \text{[A-6]}$$

Thus

$$E_c = \frac{2}{t} \left(\overline{Q}^*\right)_K . \left(d_{k+1} - d_K\right) \qquad \text{[A-7]}$$

where $t = 2d_{N+1}$

Equation (A-7) represents the apparent modulus of elasticity, E_c of fiber - reinforced subjected to tensile force.

In dynamic state the equation of motion of an uniform extensional continuous rod is given by

$$\frac{C^2 \partial^2 U_x (x, \omega, t)}{\partial x^2} - \ddot{U}_x (x, \omega, t) = 0$$

In terms of generalized coordinates, the later equation becomes.

$$C^2 \sum_{r=0}^{\infty} \omega^r [\ddot{a} (x)] \{q\} . e^{i\omega t} + \omega^2 [a_r (x)] \{q\} = 0$$

where $C^2 = \frac{E_c}{\rho_c}$

Equating to zero the coefficient of the same power of ω in the last equation the result is that

$[\ddot{a}_{(x)}] = 0$

, $[a_1 (x)] = 0$

$C^2 [\ddot{a}_2 (x)] = - [a_o (x)]$

$C^2 [\ddot{a}_3 (x)] = - [a_1 (x)]$

Terms from $[a_o (x)]$ to $[a_{16} (x)]$ are derived in terms of dimensionless ratio x/l. The distribution of strain in the axial beam element is given by

$$\zeta_x = \frac{\partial U_x (x, \omega, t)}{\partial x}$$

$$= \sum_{r=0}^{\infty} \omega^2 \left[a_r (x)\right] \{q\} \cdot e^{i\omega t}$$

$$= \sum_{r=0}^{\infty} \omega^2 \left[b_r (x)\right] \{q\} \cdot e^{i\omega t}$$

$$= \left[b (x, \omega)\right] \cdot \{U_{(t)}\}$$

in which b (x, ω) represents the frequency dependent strain displacement matrix

The forgoing expression of dynamic shape function matrix [a (x, ω)] can be employed to develop the frequency dependent mass and stiffness matrices of the dynamic extensional beam element as follows.

$$[m_e] = \int \left(\left[a (x, \omega)\right] \right)^T \cdot \rho_c \cdot \left(\left[a(x, \omega)\right] \right) dv$$

$$[ke] = \int_v \left(\left[b (x, \omega)\right]^T \cdot E_c \left[b (x, \omega)\right] \right) dv.$$