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New Approach for Solving of Extended KdV Equation

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ABSTRACT

The Extended Korteweg de-Vries equation solved by using a finite element algorithm based on Bubnov-Galerkin's method using quintic B-spline functions. Crank–Nicolson approximation in time has been used for time discretezation.The method can faithfully simulate the physics of the Extended Korteweg de-Vries equation, according to simulations.

Keywords:
Extended KdV equation, Quintic B-spline, Bubnov-Galerkin's method, Finite element method

1. INTRODUCTION

Early in 1877, Joseph Valentin Boussinesq introduced the KdV equation. Then in 1895, Diederik Korteweg and Gustav de Vries have just been rediscovered and formed by

$$U_t + \varepsilon U U_x + \alpha U_{xxx} = 0, \qquad a \le x \le b, \qquad (1)$$

where $U(x,t)$ is a field variable, ε and α are positive constants, and t and x denote time and space differentiation, respectively.

The KdV Eq.(1) is a third-order one-dimensional nonlinear partial differential equation that is used in nonlinear dispersive wave analysis. The equation was constructed to describe the one-dimensional behaviour of solitary shallow water waves.

There are many forms for KdVequation like Rosenau– KdV [1], extended KdV [2], generalized KdV [3], Rosenau KdV-RLW [4], KdV-Burger [5], the coupled Schrödinger–KdV equation [6], ect..

The KdV equation arises as an approximate equation governing weakly nonlinear long waves when terms up to the second order in the (small) wave amplitude are retained and when the weakly nonlinear and weakly dispersive terms are in balance. If effects of higher order are of interest then retention of terms up to the third order in the (small) wave amplitude leads to the extended KdV equation. [7]

We chose Extended Kdv equation as one type of KdV equation, which in the form

$$U_t + U_x + \frac{3}{2}\alpha U U_x + \frac{1}{6}\beta U_{3x} - \frac{3}{8}\alpha^2 U^2 U_x + \alpha\beta\left(\frac{23}{24}U_x U_{2x} + \frac{5}{12}U U_{3x}\right) + \frac{19}{360}\beta^2 U_{5x} = 0 \quad (2)$$

where $U(x,t)$ is a field variable, ε , α and β are positive constants, and t and x denote time and space differentiation, respectively

The extended Korteweg-de Vries equation, which includes terms of third order in wave amplitude, is derived in two ways; the first is an extension of the derivation of Whitham (1974) of the Korteweg-de Vries equation from the water-wave equations and the second is from the Lagrangian for the water-wave equations derived by Luke (1967). Since a Lagrangian for the extended Korteweg-de Vries equation is required to apply modulation theory, the second method of derivation is useful as it leads directly to this Lagrangian. Deriving the modulation equations for the full extended Korteweg-de Vries equation. [2]

In this work, we choose the Bubnov-Galerkin finite element method using quintic B-spline are the basis functions. Spline functions have highly desirable characteristics which have made them a powerful mathematical tool for numerical approximations, are employed to set up approximate functions. The quintic B-spline bases together with finite element methods are shown to provide very accurate solutions in solving some partial differential equations. For instance, quintic B-spline finite element method for the numerical solution of the Korteweg-de Vries equation is designed by Gardner.

If we want to talk about KdV applications, we need more than one paper, so we will name some of these aplications. Other features in the Jovian atmosphere, such as the Great Red Spot (GRS), as visible in cloud patterns, are perhaps the most daring use of KdV to date[8], both the cold and the hot plasma mathematically rigorously [9], Plasma physics should be mentioned in any list of KdV applications [10]. The soliton features of KdV were first confirmed using ion-acoustic waves [11], [12].

The KdV equation can be solved in many numerical ways like finite difference method [13], finite element method [14], [15], and collocation method [16], ect..

In this study, we'll examine one sort of KdV equation. the extended KdV equation [17], which would be solved by finite element method using Bubnov-Galerkin's with quintic b-spline. First, applying finite elemet method; second, studying Crank-Nicholson Approach; third, introduced the initial state; and then apply the algorithm in two experiments.

2. FINITE ELEMENT METHOD

The finite element method is a very successful application of classical methods such as: the Ritz method, the Galerkin method, the Least Squares method; for approximating the solutions of boundary value problems arising in the theory of elliptic partial differential equations.[18] giving specific applications of the finite element in the three major categories of boundary value problems, namely (1) equilibrium or steady state or time-independent problems, (2) eigenvalue problems, and (3) propagation or transient problems.

The extended KdV equation is solved numerically throughout $[a, b]$ that is a finite region with boundary conditions. Let a partition of $[a, b]$ be $a = x_0 < x_1 < \cdots < x_N = b$ by the equally spaced knots x_i and let quintic B-splines with knots at the points $x_i,\ 0 < i < N$ are $\phi_i(x)$ where $\{\phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1}, \phi_{i+2}, \phi_{i+3}\}$ are a collection of splines. serves as a foundation for functions that are sought within the finite region $[a, b]$. The $U(x,t)$ solution approximation $U_N(x,t)$ to is defined as follows:

$$U_N(x,t) = \sum_{i=-2}^{N+2} \phi_i(x) u_i(t), \tag{3}$$

where u_i are time-dependent parameters that can be calculated from boundary and condition conditions.

$$U(a,t) = U(b,t) = 0 \quad , \quad U_x(a,t) = U_x(b,t) = 0. \tag{4}$$

The intervals $[x_i, x_{i+1}]$ are used to identify finite elements with nodes at x_i and x_{i+1}. Each element $[x_i, x_{i+1}]$ is thus covered by six splines$(\phi_{i-2}, \phi_{i-1}, \phi_i, \phi_{i+1},\ \phi_{i+2}, \phi_{i+3})$, which are represented as a local coordinate system ζ given by $h\zeta = (x - x_i)$ where $h = x_{i+1} - x_i$ and $0 \le \zeta \le 1$. The expressions for all of these splines over through the element $[x_i, x_{i+1}]$ are as follows [19]

$$\begin{aligned}
\phi_{i-2} &= 1 - 5\zeta + 10\zeta^2 - 10\zeta^3 + 5\zeta^4 - \zeta^5 \\
\phi_{i-1} &= 26 - 50\zeta + 20\zeta^2 + 20\zeta^3 - 20\zeta^4 + 5\zeta^5 \\
\phi_i &= 66 - 60\zeta^2 + 30\zeta^4 - 10\zeta^5 \\
\phi_{i+1} &= 26 + 50\zeta + 20\zeta^2 - 20\zeta^3 - 20\zeta^4 + 10\zeta^5 \\
\phi_{i+2} &= 1 + 5\zeta + 10\zeta^2 + 10\zeta^3 + 5\zeta^4 - 5\zeta^5 \\
\phi_{i+3} &= \zeta^5.
\end{aligned} \tag{5}$$

Outside the interval $[x_{i-3}, x_{i+3}]$, the spline $\phi_i(x)$ and its fifth derivatives equal zero. When we use Eq.(4) to formulate equations based on the element parameters. u_i^e, these curves operate as "shape" functions for the element. The $U_N(x,t)$ variation across the element $[x_{i-3}, x_{i+3}]$ is provided by

$$u^e(x,t) = \sum_{j=i-2}^{i+3} \phi_j(x) u_i(t). \tag{6}$$

The derivatives at the knots and the nodal value of $U_N(x,t)$ are expressed in terms of the element parameters as shown below.

$$\begin{aligned}
U_i &= u_{i-2} + 26u_{i-1} + 66u_i + 26u_{i+1} + u_{i+2}, \\
hU_i' &= 5(u_{i+2} + 10u_{i+1} - 10u_{i-1} - u_{i-2}), \\
h^2U_i'' &= 20(u_{i-2} + 2u_{i-1} - 6u_i + 2u_{i+1} + u_{i+2}), \\
h^3U_i''' &= 60(u_{i+2} - 2u_{i+1} + 2u_{i-1} - u_{i-2}), \\
h^4U_i'''' &= 120(u_{i-2} - 4u_{i-1} + 6u_i - 4u_{i+1} + u_{i+2}),
\end{aligned} \tag{7}$$

The dashes imply differentiation in regard to x. When the Bubnov-Galerkin method is applied in Eq.(2) using weight functions $W(x)$, then the result is

$$\int_a^b W\left(U_t + U_x + \frac{3}{2}\alpha U U_x + \frac{3}{2}\beta U_{3x} - \frac{3}{8}\alpha^2 U^2 U_x + \alpha\beta\left(\frac{23}{24}U_x U_{2x} + \frac{5}{12}U U_{3x}\right) + \frac{19}{360}\beta^2 U_{5x}\right) dx = 0. \tag{8}$$

We'll now set up the appropriate element matrices. We have the contribution for the typical element

$[x_i, x_{i+1}]$, we obtain

$$\int_e W\left(u_t^e + u_x^e + \alpha\frac{3}{2}u^e u_x^e + \beta\frac{3}{2}u_{3x}^e - \frac{3}{8}\alpha^2(u^e)^2u_x^e + Z^e\right)dx, \qquad (9)$$

where Z^e is

$$Z^e = \alpha\beta\left(\frac{23}{24}u_x^e u_{2x}^e + \frac{5}{12}u^e u_{3x}^e\right) + \frac{19}{360}\beta^2 u_{5x}^e \qquad (10)$$

$$\begin{aligned}
&\sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i dx\right)\dot{u}_i^e + \sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i' dx\right)u_i^e + \\
&\frac{1}{6}\beta\sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i''' dx\right)u_i^e \\
&+\frac{19}{360}\beta^2\sum_{i=l-2}^{l+3}\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i'^5 dx\right)u_i^e + \\
&\frac{3}{2}\alpha\sum_{j=l-2}^{l+3}\sum_{i=l-2}^{l+3}\left(\left(\int_{x_l}^{x_{l+1}}\phi_i\phi_j'\phi_k dx\right)u_i^e\right)u_j^e \\
&+\frac{5}{12}\alpha\beta\sum_{j=l-2}^{l+3}\sum_{i=l-2}^{l+3}\left(\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i\phi_j''' dx\right)u_i^e\right)u_j^e + \\
&\frac{23}{24}\alpha\beta\sum_{j=l-2}^{l+3}\sum_{i=l-2}^{l+3}\left(\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i'\phi_j'' dx\right)u_i^e\right)u_j^e \\
&-\frac{3}{8}\alpha^2\sum_{j=l-2}^{l+3}\sum_{i=l-2}^{l+3}\left(\left(\int_{x_l}^{x_{l+1}}\phi_k\phi_i^2\phi_j' dx\right)(u^2)_i^e\right)u_j^e = 0, \qquad (11)
\end{aligned}$$

which the matrix form is formed by

$$\begin{aligned}
&A^e\dot{u}^e + B^e u^e + \frac{1}{6}\beta C^e u^e + \frac{19}{360}\beta^2 D^e u^e + \frac{3}{2}\alpha u^{eT}E^e u^e \\
&+\frac{5}{12}\alpha\beta u^{eT}F^e u^e + \frac{23}{24}\alpha\beta u^{eT}G^e u^e - \frac{3}{8}\alpha^2(u^{eT})^2 H^e u^2 = 0, \qquad (12)
\end{aligned}$$

where the dot is the differentiation with respect to the time t, and

$$u^e = (u_{l-2}, u_{l-1}, u_l, u_{l+1}, u_{l+2}, u_{l+3})^T. \qquad (13)$$

The element matrices are given by

$$\begin{aligned}
&A_{ij}^e = \int_{x_l}^{x_{l+1}}\phi_k\phi_i dx, B_{ij}^e = \int_{x_l}^{x_{l+1}}\phi_k\phi_i' dx, C_{ij}^e = \int_{x_l}^{x_{l+1}}\phi_k\phi_i''' dx, \\
&D_{ij}^e = \int_{x_l}^{x_{l+1}}\phi_k'^5\phi_i dx, E_{ij}^e = \int_{x_l}^{x_{l+1}}\phi_i\phi_j'\phi_k dx, F_{ijk}^e = \int_{x_l}^{x_{l+1}}\phi_k\phi_i\phi_j''' dx, \\
&G_{ijk}^e = \int_{x_l}^{x_{l+1}}\phi_k\phi_i'\phi_j'' dx, H_{ijk}^e = \int_{x_l}^{x_{l+1}}\phi_k\phi_i^2\phi_j' dx, \qquad (14)
\end{aligned}$$

where i, j, k take only $l-2, l-1, l, l+1, l+2, l+3$ for this element $[x_l, x_{l+1}]$. The matrices A^e, B^e, C^e and D^e are therefore 6×6 and F^e, G^e, E^e and H^e are $6\times 6\times 6$. Instead of F^e, G^e, H^e, E^e we utilise the associated 6×6 matrix f^e, g^e, h^e and e^e in our algorithm. In our algorithm, we use

$$\begin{aligned}
&F_{ij}^e = \sum_{k=l-2}^{l+3} f_{ijk}^e u_k^e, G_{ij}^e = \sum_{k=l-2}^{l+3} g_{ijk}^e u_k^e \\
&H_{ij}^e = \sum_{k=l-2}^{l+3} h_{ijk}^e (u^2)_k^e, E_{ij}^e = \sum_{k=l-2}^{l+3} A_{ijk}^e (u)_k^e, \qquad (15)
\end{aligned}$$

This is dependent on the variables u_k^e. The matrices of elements A^e, B^e, C^e and D^e are determined algebraically from Eq.(13), which u_k^e is given by Eq.(12). The equation below is obtained by assembling the elements Eq.(11).

$$A\dot{u} + \left(B + \frac{1}{6}\beta C + \frac{19}{360}\beta^2 D + \frac{3}{2}\alpha E + \frac{5}{12}\alpha\beta F + \frac{23}{24}\alpha\beta G - \frac{3}{8}\alpha^2 H\right)u = 0, \ (16)$$

where the matrices A, B, C, D, E, F, G are constructed from the element matrices $A^e, B^e, C^e, D^e, E^e, F^e, G^e, H^e$ respectively in the usual way and

$$u = (u_{-2}, u_{-1}, u_0, \cdots, u_{N+1}, u_{N+2})^T. \quad (17)$$

3. CRANK-NICHOLSON APPROACH

The Crank–Nicolson technique is a finite difference method for numerically solving the heat equation and other partial differential equations in numerical analysis [20]. Time centre on $(n+\frac{1}{2})\Delta t$, where Δt is the time step, then utilise Crank- Nicholson method [21], with

$$u = \frac{1}{2}(u^n + u^{n+1}) \quad , \quad \dot{u} = \frac{1}{\Delta t}(u^{n+1} - u^n). \quad (18)$$

Substituting Eq.(17) into Eq.(15), we obtainthe recurrence relationship

$$\frac{A}{\Delta t}(u^{n+1} - u^n) + \frac{1}{2}(B + \frac{1}{6}\beta C + \frac{19}{360}\beta^2 D + \frac{3}{2}\alpha E + \frac{5}{12}\alpha\beta F + \frac{23}{24}\alpha\beta G - \frac{3}{8}\alpha^2 H)(u^{n+1} + u^n) = 0, \quad (19)$$

and then

$$(A + \frac{\Delta t}{2}(B + \frac{1}{6}\beta C + \frac{19}{360}\beta^2 D + \frac{3}{2}\alpha E + \frac{5}{12}\alpha\beta F + \frac{23}{24}\alpha\beta G - \frac{3}{8}\alpha^2 H))u^{n+1} = (A - \frac{\Delta t}{2}(B + \frac{1}{6}\beta C + \frac{19}{360}\beta^2 D + \frac{3}{2}\alpha E + \frac{5}{12}\alpha\beta F + \frac{23}{24}\alpha\beta G - \frac{3}{8}\alpha^2 H))u^n, \quad (20)$$

where the time labels are represented by the superscripts n and $n+1$. The system (19) is made up of $N+1$ linear equations with $N+5$ variables. Four more conditions must be met in order to obtain a unique solution to the system. These are derived from the boundary conditions and may be utilised to exclude $u_{-2}, u_{-1}, u_0, \cdots, u_{N+1}, u_{N+2}$ from the recurrence relationships (19), resulting in an 11 banded $(N+5)\times(N+5)$ matrix equation. At each time step, an inner iteration is performed to verify that the nonlinear term converges. The following is the iteration algorithm [22]:

First, u^0 is known. The first approximation which is derived from Eq.(17), is calculated u_1^1 to u using $u = u^0$. The second approximation u_2^1 is found with $u = \frac{1}{2}(u^0 + u_1^1)$, and the third u_3^1 with $u = \frac{1}{2}(u^0 + u_2^1)$. We found that 10 rounds are usually enough to get a fair approximation for u^1 in this first stage.

To find a first approximation in general u_1^{n+1} to u^{n+1}, we use $u = u^n + \frac{1}{2}(u^n + u^{n-1})$, A second approximation u_2^{n+1} is then found from, $u = \frac{1}{2}(u^n + u^{n+1})$ and so on. Convergence is normally achieved after two or three iterations [23].

The time evolution of u^n is determemined by a system of the decadiagonal [See Appendix A], and as a result, after the initial vector of the parameters u^0 is obtained, $U_N(x,t)$ can be begun.

4. THE INITIAL STATE

We using recurrence relationships (19) to begin the time assessment of u^{n+1} by determining the vector u^0 from the starting condition. From Eq.(6), if we rewrite the global trial functions as follows

$$U_N(x,0) = \sum_{i=-2}^{N+2} \phi_i(x) u_i^0 \quad (21)$$

u_i^0 denotes unknown parameters that must be determined. U_N must meet the following requirements in order to determine the initial vector u_i^0 [24]. At the knots x_j, it agrees with the analytical initial condition; applying Eq.(6), it leads to $N+1$ conditions. The solution of matrix equations is then used to get the start up vector u^0 ,

$$Mu^0 = b, \qquad (22)$$ (21)

where

$$M = \begin{bmatrix} 3 & 30 & 27 & & & & & & \\ 1 & 18 & 33 & 8 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & & \cdot & & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & & & \\ & & & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 8 & 33 & 18 & 1 \\ & & & & & & 27 & 30 & 3 \end{bmatrix} \qquad (23)$$ (22)

$$b = (U''(x_0), U'(x_0), U(x_0), U(x_1), \cdots, U(x_N), U'(x_N), U''(x_N))^T,$$
$$u^0 = (u_{-2}^0, u_{-1}^0, u_0^0, \cdots, u_N^0, u_{N+1}^0, u_{N+2}^0)^T. \qquad (24)$$ (23)

After determining the initial vector u^0 as the solution of the undecadiagonal matrix Eq.(19), the system is solved using a Thomas algorithm.

The numerical algorithm developed in Section 3 will be validated by studying test problems concerned with the migration and interaction of solitons. We use the L_2 and L_∞ error norms to measure the difference between the numerical and analytical solutions and hence to show how well the scheme predicts the position and amplitude of the solution as the simulation proceeds. The and error norms of the solution are defined by

$$L_2 = \left\| U^{exact} - U^n \right\|_2 = \left[h \sum_{i=1}^{N} \left| U_i^{exact} - U_i^n \right|^2 \right]^{\frac{1}{2}}, \qquad (25)$$

$$L_\infty = \left\| U^{exact} - U^n \right\|_\infty = \max_i \left| U_i^{exact} - U_i^n \right|$$

5. EXTENDED KDV SIMULATIONS: SINGLE SOLITARY WAVE SIMULATION

5.1 Experiment 1

Subject to the boundary conditions [25]

$$U(18,t) = U(70,t) = 0 \quad , \qquad t > 0. \qquad (26)$$

The analytical solution of of the extended KdV Eq.(2) is as follows

$$U(x,0) = A_1 \mathrm{sech}^2 \left[\sqrt{\frac{3\alpha}{4\beta}} (x - x_0) \right] \quad , \qquad 18 \leq x \leq 70. \qquad (27)$$

The analytic solution for the initial condition:

$$U(x,t) = A_1 \mathrm{sech}^2 \left[\sqrt{\frac{3\alpha}{4\beta}} (x - x_0 - (1 + \frac{\alpha}{2} A)t) \right] \quad , \qquad 18 \leq x \leq 70 \qquad (28)$$

Figure 1 comparison between the numerical solution and the exact solution at the same time, which agree with the exact solution.In order to determine the accuracy of the current scheme, we used $\alpha = \beta = 0.1, x_0 = 18, \Delta t = 0.05, \Delta x = 0.01$ and $A_1 = 1$ complete the simulation up to $t = 5$.

Figure 2 Numerical solution at different time [t=0, 5, 10, 15, 20 seconds], which agree with the exact solution. In order to determine the accuracy of the current scheme, and shows that the wave moves when the time changed. The wave acts like a pulse.

The L_2 and L¥ error norms are also recorded and the L_2 norm is less than $3*10^{25}$, while the L_∞ norm is less than 6_10^{19} .

t	L_2 $*10^{25}$	$L_\infty$$*10^{25}$
1.0	3.86481324	3638.57688
2.0	60.22283148626536	233.5064587820526
3.0	938.4125966520094	14.98532752901747
4.0	146.2266352846622	96.16866373770172*10
5.0	2278.553052565474	6.171644808688053

Table 1: Error norms for the single solitary wave of the extended KdV equation at t = 1,2, .. ,5
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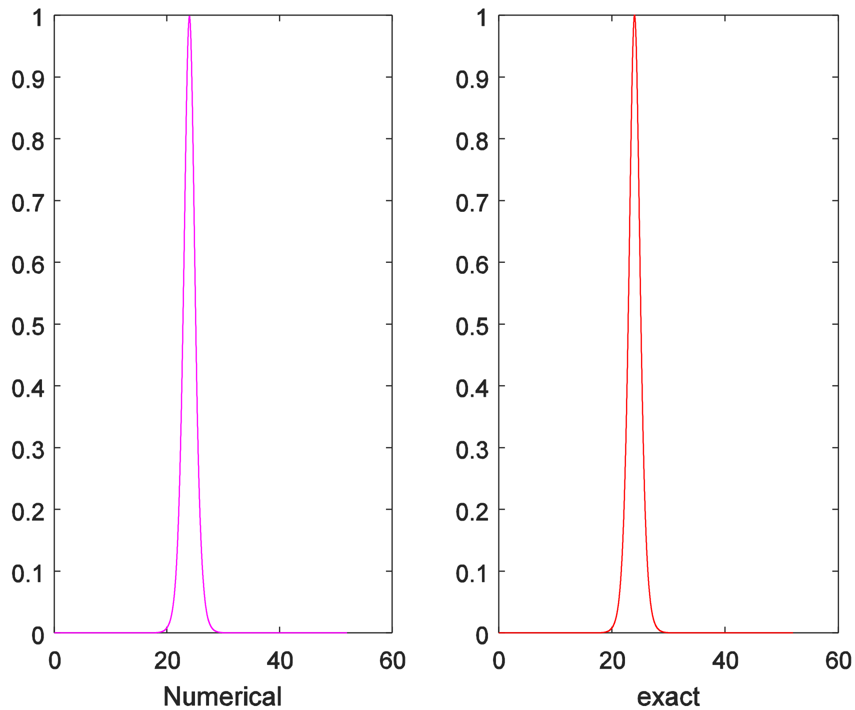
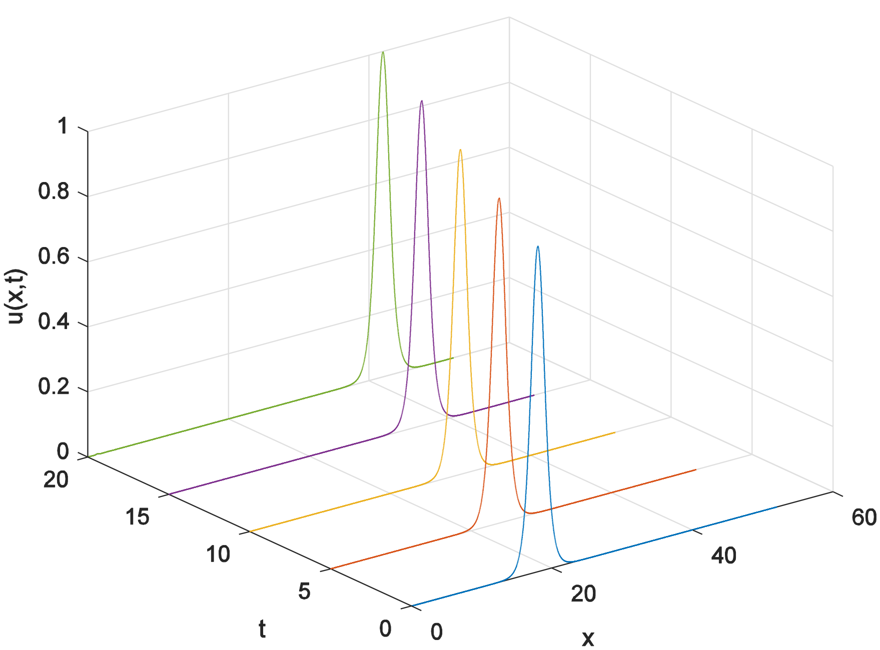


Fig(1): Comparison between numerical solution and the exact solution at time 5 seconds



Fig(2): Numerical solution at time t=0, 5, 10, 15, 20 seconds

6. EXTENDED KDV SIMULATIONS: TOW SOLITARY WAVES

6.1 Experiment 1

Subject to the boundary conditions

$$U(18,t) = U(70,t) = 0 \quad , \qquad t > 0. \tag{29}$$

The analytical solution of of the extended Extended KdV Eq.(2) is as follows

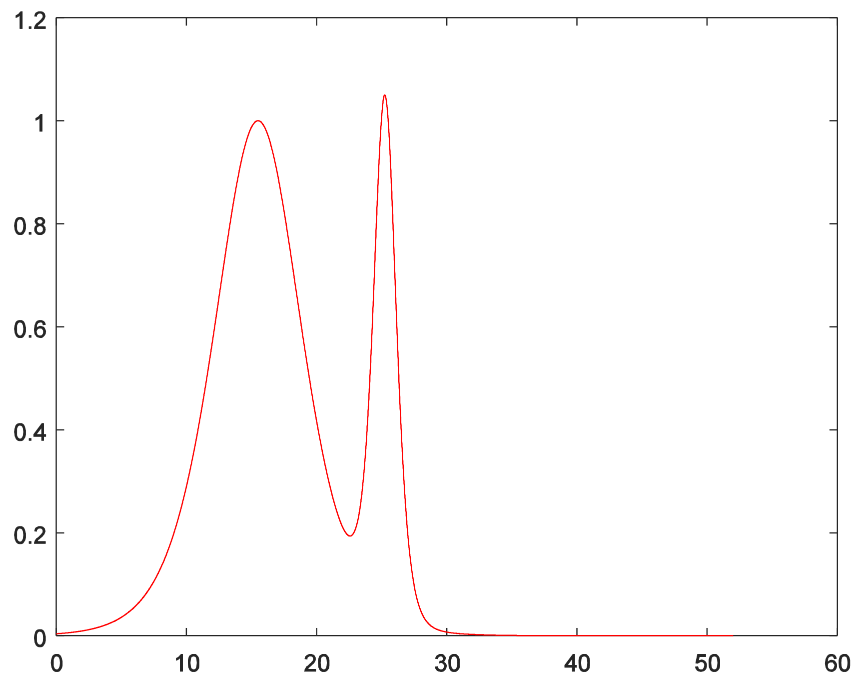
$$U(x,0) = A_1\left[\operatorname{sech}^2\left[\sqrt{\frac{3\alpha}{4\beta}}(x-x_o)\right] + \operatorname{sech}^2\left[\sqrt{\frac{3\alpha}{4\beta}}(x-x_o)\right]\right], 18 \le x \le 70 \tag{30}$$

The analytic solution for the initial condition:

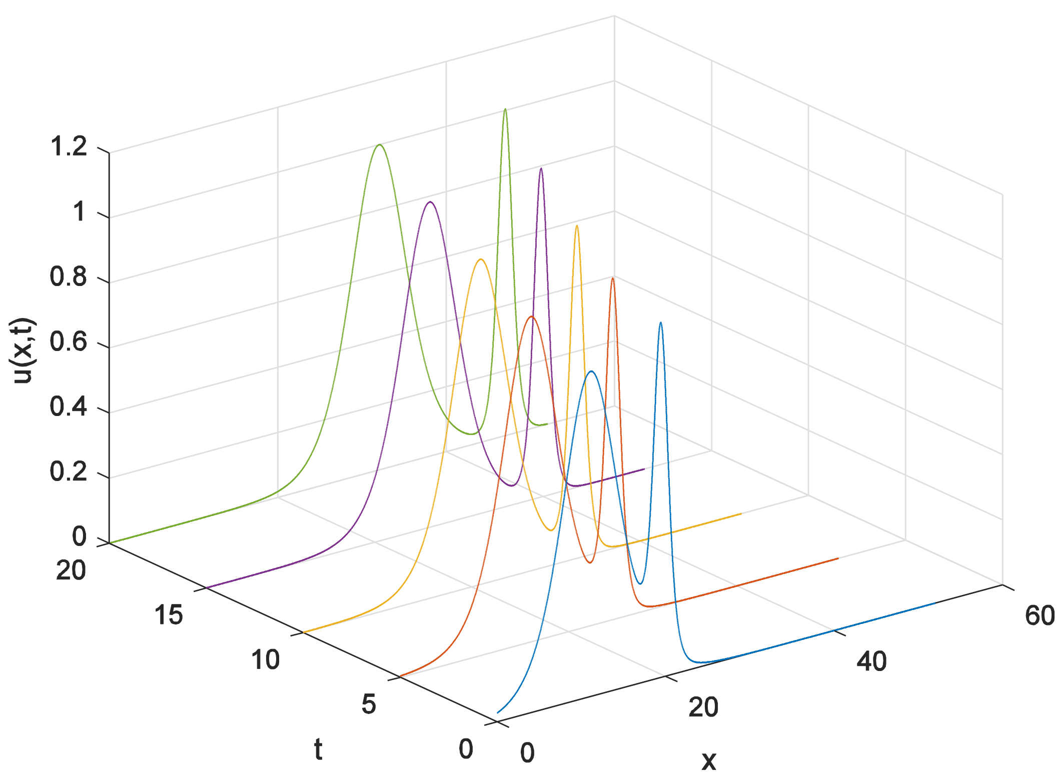
$$U(x,0) = A_1\left[\operatorname{sech}^2\left[\sqrt{\frac{3\alpha}{4\beta}}\left(x-x_o-\left(1+\frac{\alpha}{2}A_1\right)t\right)\right] + \operatorname{sech}^2\left[\sqrt{\frac{3\alpha}{4\beta}}(x-x_o)-\left(1+\frac{\alpha}{2}A_1\right)t\right]\right] \quad , \quad 18 \le x \le 70 \tag{31}$$

Figure3 comparison between the numerical solution at at time t=5 seconds, which agree with the exact solution.In order to determine the accuracy of the current scheme, we used $\alpha=\beta=0.9$, in the fisrt term, and $\alpha=0.2$, $\beta=3$ in the second term $x_0=18$, , $\Delta t = 0.05, \Delta x = 0.01$ and $A_1=1$. .

Figure 4 Numerical solution at time t=0, 5, 10, 15, 20 seconds, which agree with the exact solution. In order to determine the accuracy of the current scheme, we used $\alpha=\beta=0.9$, in the first term, and $\alpha = \beta = 0.1, x_0 = 18, \Delta t = 0.05, \Delta x = 0.01$ and $A_1 = 1$ complete the simulation up to t=20.0 seconds,
shown in the figure that the two solitons merge and dismerge when the time changed .



Fig(3): The numerical solution at time t=5 seconds



Fig(4): Numerical solution at time t=0, 5, 10, 15, 20 seconds

7. CONCLUSIONS

The Extended KdV equation is a nonlinear transient dispersive equation, any numerical system that replicates it must accurately reflect all of its properties. To deal with the fifth derivative of the extended KdV equation, Based on Galerkin and quintic B-spline shape and weight functions, we developed a one-dimensional B-spline finite element method. The Crank Nicholson technique is used to create time discretization. This results in a nonlinear equation system with 11 diagonal matrices. Matlab code was used to complete all calculations. To solve the equation in this work, we employed the finite element method with a quintic B-spline. We used several functions to study extended KdV equation and the wave form in each function in diffierent times. To solve the equation in this work, we employed the finite element method with a quintic B-spline. We used several functions to study extended KdV equation and the wave form in each function in diffierent times. From experiment 1 and 2 (one solitary wave and two solitons), the results obtained proved the method to be reliable, accurate and efficient through the calculated error norms. We believe that the technique given here could be applicable in other situations where derivative continuity is required. We can say that our numerical method can be reliably used to obtain the numerical solution of the Extended KdV equation and similar type non-linear equations.

Appendix A: An Undecadiagonal Matrix Algorithm

Consider the problem of solving N simulation equations, which can be stated as follows:

$$[T]\{u\} = \{y\}$$

where $[T]$ is a matrix of known coefficients and $\{u\}$ and $\{y\}$ represent unknown and known equations, respectively, and $[T]$ is a vector of unknown and known equations.[26].

$$M =$$

$$\begin{bmatrix}
a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
g_2 & a_2 & b_2 & c_2 & d_2 & e_2 & f_2 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
h_3 & g_3 & a_3 & b_3 & c_3 & d_3 & e_3 & f_3 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
m_4 & h_4 & g_4 & a_4 & b_4 & c_4 & d_4 & e_4 & f_4 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\
n_5 & m_5 & h_5 & g_5 & a_5 & b_5 & c_5 & d_5 & e_5 & f_5 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
p_6 & n_6 & m_6 & h_6 & g_6 & a_6 & b_6 & c_6 & d_6 & e_6 & f_6 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & p_7 & n_7 & m_7 & h_7 & g_7 & a_7 & b_7 & c_7 & d_7 & e_7 & f_7 & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & p_{N-5} & n_{N-5} & m_{N-5} & h_{N-5} & g_{N-5} & a_{N-5} & b_{N-5} & c_{N-5} & d_{N-5} & e_{N-5} & f_{N-5} \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & p_{N-4} & n_{N-4} & m_{N-4} & h_{N-4} & g_{N-4} & a_{N-4} & b_{N-4} & c_{N-4} & d_{N-4} & e_{N-4} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & p_{N-3} & n_{N-3} & m_{N-3} & h_{N-3} & g_{N-3} & a_{N-3} & b_{N-3} & c_{N-3} & d_{N-3} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & p_{N-2} & n_{N-2} & m_{N-2} & h_{N-2} & g_{N-2} & a_{N-2} & b_{N-2} & c_{N-2} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & p_{N-1} & n_{N-1} & m_{N-1} & h_{N-1} & g_{N-1} & a_{N-1} & b_{N-1} \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & p_N & n_N & m_N & h_N & g_N & a_N
\end{bmatrix}$$

The 11-diagonal matrix is decomposed into two tridiagonal matrices using LU decomposition. When a result, defining the following parameters using forward recursion as needed,

$$q_1 = a_1 \quad , \quad r_1 = b_1 \quad , \quad s_1 = c_1 \quad , \quad t_1 = d_1 \quad , \quad v_1 = e_1 \quad , \quad z_2 = \frac{g_2}{q_1},$$

$$q_2 = a_2 - z_2 r_1 \quad , \quad r_2 = b_2 - z_2 S_1 \quad , \quad s_2 = c_2 - z_2 t_1,$$
$$t_2 = d_2 - z_2 v_1 \quad , \quad v_2 = e_2 - z_2 f_1 \quad , \quad \alpha_3 = \frac{h_3}{q_1} \quad , \quad z_3 = \frac{g_3 - \alpha_3 r_1}{q_2};$$

$q_3 = a_3 - \alpha_3 s_1 - z_3 r_2$, $r_3 = b_3 - \alpha_3 t_1 - z_3 s_2$, $s_3 = c_3 - \alpha_3 v_1 - z_3 t_1$,
$t_3 = d_3 - \alpha_3 f_1 - z_3 v_2$, $v_3 = e_3 - z_3 f_2$,

$\beta_4 = \frac{m_4}{q_1}$, $\alpha_4 = \frac{h_4 - \beta_4 r_1}{q_2}$, $z_4 = \frac{g_4 - \beta_4 s_1 - \alpha_4 r_2}{q_3}$, $q_4 = a_4 - \beta_4 t_1 - \alpha_4 s_2 - z_4 r_3$,
$r_4 = b_4 - \beta_4 v_1 - \alpha_4 t_2 - z_4 s_3$, $t_4 = d_4 - \alpha_4 f_2 - z_4 v_3$, $v_4 = e_4 - z_4 f_3$,

$\gamma_5 = \frac{v_5}{q_1}$, $\beta_5 = \frac{m_5 - \gamma_5 r_1}{q_2}$, $\alpha_5 = \frac{h_5 - \gamma_5 s_1 - \beta_5 r_2}{q_3}$,
$z_5 = \frac{1}{q_4}(g_5 - \gamma_5 t_1 - \beta_5 s_2 - \alpha_5 r_3)$, $q_5 = a_5 - \gamma_5 v_1 - \beta_5 t_2 - \alpha_5 s_3 - z_5 r_4$,
$r_5 = b_5 - \gamma_5 f_1 - \beta_5 v_2 - \alpha_5 t_3 - z_5 t_4$, $s_5 = c_5 - \beta_5 f_2 - \alpha_5 v_3 - z_5 t_4$,
$t_5 = d_5 - \alpha_5 f_3 - z_5 v_4$, $v_5 = e_5 - z_5 f_4$,

and $i = 6 \to N$

$\epsilon_i = \frac{p_i}{q_{i-5}}$, $\gamma_i = \frac{v_i - \epsilon_i r_{i-5}}{q_{i-4}}$,
$\beta_i = \frac{1}{q_{i-3}}(m_i - \epsilon_i s_{i-5} - \gamma_i r_{i-4})$, $\alpha_i = \frac{1}{q_{i-2}}(h_i - \epsilon_i t_{i-5} - \gamma_i s_{i-4} - \beta_i r_{i-3})$,
$z_i = \frac{1}{q_{i-1}}(g_i - -\epsilon_i v_{i-5} \gamma_i t_{i-4} - \beta_i s_{i-3} - \alpha_i r_{i-2})$,
$q_i = a_i - \epsilon_i f_{i-5} - \gamma_i v_{i-4} - \beta_i t_{i-3} - \alpha_i s_{i-2} - z_i r_{i-1}$,
$r_i = b_i - \gamma_i f_{i-4} - \beta_i v_{i-3} - \alpha_i t_{i-2} - z_i s_{i-1}$,
$s_i = c_i - \beta_i f_{i-3} - \alpha_i v_{i-2} - z_i t_{i-1}$, $t_i = d_i - \alpha_i f_{i-2} - z_i v_{i-1}$, $v_i = e_i - z_i f_{i-1}$.

As a result, the lower tringular matrix can be used to deduce the following vector $\{x\}$.

$x_1 = y_1$, $x_2 = y_2 - z_2 x_1$, $x_3 = y_3 - \alpha_3 x_1 - z_3 x_2$,
$x_4 = y_4 - \beta_4 x_1 - \alpha_4 x_2 - z_4 x_3$, $x_5 = y_5 - y_5 x_1 - \beta_5 x_2 - \alpha_5 x_3 - z_5 x_4$,

$i = 6 \to N$:

$$x_i = y_i - \epsilon_i x_{i-5} - \gamma_i x_{i-4} - \beta_i x_{i-3} - \alpha_i x_{i-2} - z_i x_{i-1}.$$

Those calculations then use reverse recursion to angender the computation of the unknown vector $\{u\}$from the higher triangular matrix.

$u_N = \frac{x_N}{q_N}$,
$u_{N-1} = \frac{1}{q_{N-1}}(x_{N-1} - r_{N-1} u_N)$,
$u_{N-2} = \frac{1}{q_{N-2}}(x_{N-2} - s_{N-2} u_N - r_{N-2} u_{N-1})$,
$u_{N-3} = \frac{1}{q_{N-3}}(x_{N-3} - t_{N-3} u_N - s_{N-3} u_{N-3} - r_{N-3} u_{N-2})$,
$u_{N-4} = \frac{1}{q_{N-4}}(x_{N-4} - v_{N-4} u_N - t_{N-4} u_{N-1} - s_{N-4} u_{N-2} - r_{N-4} u_{N-3})$.

$i = (N-5) \to 1$

$$u_i = \frac{1}{q_i}(x_i - f_i u_{i+5} - v_i u_{i+4} - t_i u_{i+3} - s_i u_{i+2} - r_i u_{i+1}).$$

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