Bayesian analysis of the Truncated Power Lindley under different loss functions for censored data

Mahmoud.A.Eltehiwy[1] , Mohamed. S. Hamouda[2]

[1]*Faculty of the Politics and Economics, Department of Statistics, Beni-suef University, Egypt.*

[2]*Faculty of Commerce, Department of Quantitative Techniques, Aswan University, Egypt*

[1]Email: Mahmoud.abdelmone@eps.bsu.edu.com
[2]Email: m.sattar@com.aswu.edu.eg

Abstract

We perform a Bayesian analysis of the upper truncated power Lindley distribution based on type II censored data. Using various loss functions, including the generalized quadratic, entropy and Linex loss functions, we obtain Bayes estimators and their corresponding posterior risks. As tractable analytical forms of these estimators are out of reach, we propose Markov chain Monte-Carlo (MCMC) based simulation approach to study their performance. Moreover, given initial values for the parameters of the model, we obtain maximum likelihood estimators. Furthermore, we compare their performance with that of the Bayesian estimators using Pitman's closeness criterion and integrated mean square error. Finally, we illustrate our approach through an example with real data.

Key Words: Truncated power Lindley distribution, Bayes estimators, loss function, Pitman criterion, Metropolis-Hastings algorithm.

1. Introduction

Truncated statistical distributions occur when a random variable X follows a known distributional model and a portion of the sample space is unavailable for observation. If no values of the random variable are observed below a certain lower limit, say ζ, the distribution is said to be truncated on the left at ζ. A truncated distribution is a conditional distribution that occurs when the statistical distribution's domain is limited. As a consequence, truncated distributions are used when occurrences are restricted to values above or below a given threshold or within a given range. When occurrences are restricted to values less than a certain threshold, the lower (left) truncated distribution is obtained. Similarly, if occurrences are restricted to values above a given threshold, the upper (right) truncated distribution rises up. (See, for example, Dusit and Cohen(1994)). Wingo (1988) proposed parameter point estimation for a doubly truncated Weibull distribution. Martinez (1991) investigated the maximum likelihood estimation (MLE) of parameters in the upper truncated Weibull distribution. Shalaby and El-Yousef (1993) presented Bayesian parameter estimators for a doubly truncated Weibull distribution, and Shalaby (1993) discussed the estimation's Bayesian risk. Seki and Yokoyama (1993) addressed a robust estimation method for the Weibull and truncated Weibull parameters. Balakrishnan and Mitra (2012) used the EM algorithm to estimate the parameters of the Weibull distribution when the model is truncated on the left and the data is censored on the right. Zhang and Xie (2011) investigated the properties of the truncated Weibull distribution and demonstrated its applicability to modelling lifetime data. Ahmed et al. (2010) proposed a truncated version of the Birnbaum-Saunders (BS) distribution and demonstrated that the truncated BS distribution is more appropriate than the classical BS model for describing commercial bank financial loss data. On the basis of real-world data, Singh et al. (2014) introduced the truncated version of the Lindley distribution and discussed the statistical properties of the proposed distribution, demonstrating that the truncated version of the Lindley distribution provides better modelling than the Weibull, Lindley, and exponential distributions.

Eltehiwy (2020) proposed the truncation in the power Lindley distribution proposed by Ghitany et al. (2013). Boudjerda et al. (2016) investigated the Bayesian analysis of the right truncated Weibull distribution with type II censored data and derived Bayes estimators and corresponding risks using symmetric and asymmetric loss functions. Aouf and Chadli (2017) investigated the Bayesian analysis of the generalised Lindley distribution with type II censored data, calculating Bayes estimators and corresponding risks with symmetric and asymmetric loss functions. Hamida and Hiba (2021) performed a Bayesian analysis of the upper truncated Zeghdoudi distribution based on type II censored data using various loss functions including the generalized quadratic, entropy and Linex functions. They obtained Bayes estimators and the corresponding posterior risks.

In this paper, we investigate the estimation of the upper-truncated power Lindley distribution, which is dependent on three parameters. There are two methods suggested. The first is the traditional maximum likelihood estimation method. The second is the Bayesian procedure, which is carried out using the generalised quadratic (GQ), entropy, and Linex loss functions. We compare the Bayesian estimators with respect to the posterior risks using an exhaustive Monte-Carlo study. Then, for each loss function, we choose the best estimator. Using the Pitman closeness criterion and the mean squared error (MSE), these three Bayesian estimators are compared to maximum likelihood estimators.

The rest of the paper is arranged into the following sections: In section 2, the truncated version of the power Lindley distribution, named the upper truncated power Lindley (UTPL), is introduced. Section 3 deals with the maximum likelihood estimation. In section 4, we propose the Bayesian estimators under various loss functions. A Monte-Carlo study is proposed in section 5. An application with real life data was provided in section 6.

2. The truncated power Lindley distributions

The probability density function of a random variable X having power Lindley distribution can be written as:

$$f(x;\theta,\beta) = \frac{\theta^2\beta(1+x^\beta)}{\theta+1}x^{\beta-1}e^{-\theta x^\beta}, \quad x>0,\theta,\beta>0. \tag{1}$$

It can easily be seen that at $\beta = 1$, the Equation (1) reduces to the Lindley distribution.

A distribution $G(x;\Theta)$ is said to be a double truncated distribution over the interval $[\nu,\zeta]$ if it has the cumulative distribution function (cdf) defined as

$$G(x;\Theta) = \frac{F(x;\Theta)-F(\nu;\Theta)}{F(\zeta;\Theta)-F(\nu;\Theta)}, \quad \nu \le x \le \zeta \qquad -\infty < \nu < \zeta < \infty \tag{2}$$

and probability density function (pdf) is

$$\mathrm{g}(x;\Theta) = \frac{f(x;\Theta)}{F(\zeta;\Theta)-F(\nu;\Theta)} \quad \nu \le x \le \zeta \qquad -\infty < \nu < \zeta < \infty \tag{3}$$

where, $f(x;\Theta)$ and $F(x;\Theta)$ are the pdf and cdf of the baseline model and $\Theta \in \mathcal{R}^{\mathrm{n}}$ denotes the vector parameter of base line model. Here, three cases can be recognized as

I. When $\nu = 0$ and $\zeta \to \infty$, it reduces to baseline model.
II. When $\nu = 0$, it is called the upper truncated distribution of the baseline model.
III. When $\zeta \to \infty$, it is called the lower truncated distribution of the baseline model.

In this article, we consider the power Lindley distribution as baseline model with the following distribution function :

$$F(x) = \left[1-\left(1+\frac{\theta x^\beta}{\theta+1}\right)e^{-\theta x^\beta}\right] \tag{4}$$

Using (2) and (4), the double truncated power Lindley distribution is defined as

$$\mathrm{g}_D(x;\theta,\beta,\zeta,\nu) = \frac{\theta^2}{\theta+1}\frac{\beta(1+x^\beta)x^{\beta-1}e^{-\theta x^\beta}}{F(\zeta;\Theta)-F(\nu;\Theta)} \quad , \quad 0 \le \nu \le x \le \zeta < \infty$$

In the following sections, we will only discuss the upper truncated power Lindley distribution. The upper truncated power Lindley distribution has the following pdf given by

$$g_u(x;\theta,\beta,\zeta) = \frac{\theta^2\beta(1+x^\beta)x^{\beta-1}\exp\left(-\theta\left(x^\beta-\zeta^\beta\right)\right)}{(\theta+1)\left(\exp(\theta\zeta^\beta)-1\right)-\theta\zeta^\beta}; \qquad 0 \le x \le \zeta \qquad (5)$$

It is denoted by UTPL(θ,β,ζ), and its cumulative distribution function is

$$G_u(x;\theta,\beta,\zeta) = \frac{\exp(\theta\zeta^\beta)-\left(1+\frac{\theta x^\beta}{\theta+1}\right)\exp\left(-\theta\left(x^\beta-\zeta^\beta\right)\right)}{\left(\exp(\theta\zeta^\beta)-1\right)-\theta\zeta^\beta}, \qquad 0 \le x \le \zeta < \infty.$$

The corresponding hazard function at epoch *t* is given by

$$h(t;\theta,\beta,\zeta) = \frac{\theta^2}{\theta+1}\frac{\beta(1+t^\beta)t^{\beta-1}e^{-\theta t^\beta}}{F(\zeta;\Theta)-F(t;\Theta)}, \qquad 0 \le t \le \zeta < \infty.$$

3. Maximum likelihood estimation

Suppose that $X_1 < X_2 < \cdots < X_m$ is a type-II cencored sample of size n observed from lifetime testing experiment whose lifetime have the UTPL(θ,β,ζ) model. It assumed that parameters θ,β and ζ are unknown, the likelihood function for the parameters is then

$$L\left(\underline{X}\middle|\theta,\beta,\zeta\right) = \frac{n!}{(n-m)!}\left[\frac{\theta^2\beta}{(\theta+1)}\right]^m \prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\sum_{i=1}^m\left(x_i^\beta\right)}A^{n-m}B^{-n}, \quad 0 \le x \le \zeta \quad (6)$$

where, $\underline{X} = (x_1, x_2, \dots, x_m)$, $A = \left[\left(1+\frac{\theta(x_m)^\beta}{\theta+1}\right)e^{-\theta(x_m)^\beta} - \left(1+\frac{\theta\zeta^\beta}{\theta+1}\right)e^{-\theta\zeta^\beta}\right]$ and $B = \left[1-\left(1+\frac{\theta\zeta^\beta}{\theta+1}\right)e^{-\theta\zeta^\beta}\right]$. If $n = m$ then, equation (6) reduces to complete samples. By taking logarithm of Eq. (6), the log-likelihood function is

$$\ln L = \ln(n!) - \ln\left((n-m)!\right) + 2m\ln\theta + m\ln\beta - m\ln((\theta+1)) + \sum_{i=1}^m \ln\left(1+x_i^\beta\right)$$
$$+(\beta-1)\sum_{i=1}^m \ln(x_i) - \theta\sum_{i=1}^m x_i^\beta + (n-m)\ln A - n\ln B, \quad (7)$$

The maximum likelihood estimators $\hat{\theta},\hat{\beta}$ and $\hat{\zeta}$of θ,β and ζ are then the solutions of the following non-linear equations:

$$\frac{\partial}{\partial\beta}\ln L = \frac{m}{\beta} + \sum_{i=1}^n \frac{x_i^\beta\ln(x_i)}{1+x_i^\beta} + \sum_{i=1}^n \ln(x_i) - \theta\sum_{i=1}^m x_i^\beta\ln(x_i) + (n-m)\frac{A1}{A} - n\frac{B1}{B} = 0, \quad (8)$$

where $A1 = \theta(x_m)^\beta e^{-\theta(x_m)^\beta}\ln(x_m)\left[\frac{1-\theta(x_m)^\beta}{\theta+1} - 1\right] + \theta\zeta^\beta e^{-\theta\zeta^\beta}\ln(\zeta)\left(\frac{\theta\zeta^\beta-1}{\theta+1}+1\right)$,

$B1 = \theta\zeta^\beta e^{-\theta\zeta^\beta}\ln(\zeta)\left(\frac{\theta\zeta^\beta-1}{\theta+1}+1\right)$.

$$\frac{\partial}{\partial\theta}\ln L=\frac{2m}{\theta}-\frac{m}{(\theta+1)}-\sum_{i=1}^{m}x_i^{\beta}++(n-m)\frac{A2}{A}-n\frac{B2}{B}=0, \tag{9}$$

where $A2=(x_m)^{\beta}e^{-\theta(x_m)^{\beta}}\left(\frac{1}{(\theta+1)^2}-\frac{\theta(x_m)^{\beta}}{\theta+1}-1\right)+\zeta^{\beta}e^{-\theta\zeta^{\beta}}\left(\frac{\theta\zeta^{\beta}}{\theta+1}-\frac{1}{(\theta+1)^2}+1\right)$,

$B2=\zeta^{\beta}e^{-\theta\zeta^{\beta}}\left(\frac{\theta\zeta^{\beta}}{\theta+1}-\frac{1}{(\theta+1)^2}+1\right)$.

$$\frac{\partial}{\partial\zeta}\ln L=(n-m)\frac{A3}{A}-n\frac{B3}{B}, \tag{10}$$

where $A3=B3=\frac{\theta^2}{\theta+1}\beta\left(1+\zeta^{\beta}\right)\zeta^{\beta-1}e^{-\theta\zeta^{\beta}}$.

There is no analytical solution of this system. Then, we need numerical methods, such as Newton-Rahphson method to obtain approximate values of the maximum likelihood estimators’θ_{MLE}, β_{MLE}, and ζ_{MLE} of the parameters $\theta,\beta$and ζ respectively. If $n=m$, the normal equation in (8), (9) and (10) will reduce to the normal equations from complete sample in Eltehiwy (2020).

4. Bayesian estimation under different loss functions

Now, we deal with the problem of estimating the parameters θ,β and ζ under the generalized quadratic (GQ), the Linex and the entropy loss functions. As the name suggests, informative priors are more informative than the non-informative priors and convey specific and definite information about the parameters. In our study, we assumed that the prior distributions of θ,β and ζ are independent. This assumption of independence is not new in the Bayesian literature, e.g., Punt and Walker (1998), Punt and Butterworth (2000) and Kundu and Mitra (2016). Since the parameters β and θ are assumed to be unknown, the prior distributions for β and θ are taken to be Gamma (b_1,a_1) and Gamma (b_2,a_2) respectively of the following forms:

$$\pi_1(\beta)=\frac{a_1^{b_1}}{\Gamma(b_1)}\beta^{b_1-1}e^{-a_1\beta}, \qquad \beta>0,\ a_1,b_1>0, \tag{11}$$

$$\pi_2(\theta)=\frac{a_2^{b_2}}{\Gamma(b_2)}\theta^{b_2-1}e^{-a_2\theta}, \qquad \theta>0,\ a_2,b_2>0, \tag{12}$$

where $a_1,b_1,a_2$and b_2 are called hyper-parameters. Moreover, we choose the improper prior of ζ, which not depend on $(\theta,\ \beta)$ and given by $\pi(\zeta)$=1/ ζ. There is no objective motivation for choosing the gamma family as prior distributions, except for their exibility, tractability and for being natural conjugate priors for the exponential distributions. Other prior distribution may well be used. Then the joint prior distribution of β,θ and ζ is given by

$$\pi(\beta,\theta,\zeta) = \frac{a_1^{b_1} a_2^{b_2}}{\zeta\Gamma(b_1)\Gamma(b_2)} \beta^{b_1-1}\theta^{b_2-1}e^{-a_1\beta-a_2\theta}.$$

The joint posteriors of β, θ and ζ is obtained as follows,

$$\pi\left(\beta,\theta,\zeta|\underline{x}\right) = K\frac{\beta^{m+b_1-1}\theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}, \qquad (13)$$

where K is the normalizing constant. We consider the generalized quadratic, the Linex and the entropy loss functions. The following table presents these loss functions and the expressions of the Bayesian estimators with the corresponding posterior risks (PR).

Table 1: the loss functions and the corresponding Bayesian estimators and the posterior risk of the parameters.

Loss function	Expression	Bayes estimators	Posterior risk
Generalized quadratic	$L(\lambda,\delta) = \tau(\lambda)(\lambda-\delta)^2$	$\hat{\delta}_{GQ} = \frac{E_\pi(\tau(\lambda)\lambda)}{E_\pi\left(\tau(\lambda)\right)}$	$E_\pi\left(\tau(\lambda)\left(\lambda-\hat{\delta}_{GQ}\right)^2\right)$
Entropy	$L(\lambda,\delta) = \left(\frac{\delta}{\lambda}\right)^p - p\ln\left(\frac{\delta}{\lambda}\right) - 1$	$\hat{\delta}_E = [E_\pi(\lambda)^{-p}]^{-1/p}$	$p\left[E_\pi\left(\ln(\lambda)-\ln\left(\hat{\delta}_E\right)\right)\right]$
Linex	$L(\lambda,\delta) = e^{r(\delta-\lambda)} - r(\delta-\lambda) - 1$	$\hat{\delta}_L = \frac{-1}{r}\ln\left(E_\pi\left(e^{-r\lambda}\right)\right)$	$\mathrm{r}\left(\hat{\delta}_{GQ}-\hat{\delta}_L\right)$

Under the generalized quadratic loss function assuming $\tau(\lambda) = \lambda^{\alpha-1}$, the Bayes estimators are given by the formulas:

$$\hat{\beta}_{GQ} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1+\alpha}\ \theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta}{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-2+\alpha}\ \theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta}.$$

$$\hat{\theta}_{GQ} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1}\ \theta^{2m+b_2-1+\alpha}}{\zeta(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta.}{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1}\ \theta^{2m+b_2-2+\alpha}}{\zeta(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta.}$$

$$\hat{\zeta}_{GQ} = \frac{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\zeta^{\alpha-1}\beta^{m+b_1-1}\theta^{2m+b_2-1}}{(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta}{\int_0^\infty\int_0^\infty\int_0^\infty \frac{\zeta^{\alpha-2}\beta^{m+b_1-1}\theta^{2m+b_2-1}}{(1+\theta)^m}\prod_{i=1}^{m}\left(1+x_i^{\beta}\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^{n}x_i^{\beta}+a_2\right)-a_1\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta}.$$

The corresponding posterior risks are then

$PR\left(\beta_{GQ}\right) = E_\pi(\beta^{\alpha+1}) - 2\hat{\beta}_{GQ}E_\pi(\beta^{\alpha}) + \beta_{GQ}^2E_\pi(\beta^{\alpha-1})$.

$PR\left(\theta_{GQ}\right) = E_\pi(\theta^{\alpha+1}) - 2\hat{\theta}_{GQ}E_\pi(\theta^{\alpha}) + \theta_{GQ}^2E_\pi(\theta^{\alpha-1})$.

$PR\left(\zeta_{GQ}\right) = E_\pi(\zeta^{\alpha+1}) - 2\hat{\zeta}_{GQ}E_\pi(\zeta^{\alpha}) + \zeta_{GQ}^2E_\pi(\zeta^{\alpha-1})$.

Notice that, when α= 1, we have the basic quadratic loss. Under the entropy loss function, we obtain the following estimators

$$\hat{\beta}_E = \left[K\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1-p}\theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^n x_i^\beta+a_2\right)-a_1\beta}A^{n-m}B^{-n}\,d\beta d\theta d\zeta\right]^{\frac{-1}{p}}.$$

$$\hat{\theta}_E = \left[K\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1}\theta^{2m+b_2-1-p}}{\zeta(1+\theta)^m}\prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^n x_i^\beta+a_2\right)-a_1\beta}A^{n-m}B^{-n}\,d\beta d\theta d\zeta\right]^{\frac{-1}{p}}.$$

$$\hat{\zeta}_E = \left[K\int_0^\infty\int_0^\infty\int_0^\infty \frac{\zeta^{-p-1}\beta^{m+b_1-1}\theta^{2m+b_2-1}}{(1+\theta)^m}\prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^n x_i^\beta+a_2\right)-a_1\beta}A^{n-m}B^{-n}\,d\beta d\theta d\zeta\right]^{\frac{-1}{p}}.$$

The corresponding posterior risks are then

$$PR(\beta_E) = pE_\pi\left(\ln(\beta)-\ln\left(\hat{\beta}_E\right)\right).$$
$$PR(\theta_E) = pE_\pi\left(\ln(\theta)-\ln\left(\hat{\theta}_E\right)\right).$$
$$PR(\zeta_E) = pE_\pi\left(\ln(\zeta)-\ln\left(\hat{\zeta}_E\right)\right).$$

Following estimators under the Linex loss function, we obtain the

$$\hat{\beta}_L = \frac{-K}{r}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1}\theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^n x_i^\beta+a_2\right)-a_1\beta-r\beta}A^{n-m}B^{-n}d\beta d\theta d\zeta.$$

$$\hat{\theta}_L = \frac{-K}{r}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1}\theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^n x_i^\beta+a_2\right)-a_1\beta-r\theta}A^{n-m}B^{-n}d\beta d\theta d\zeta.$$

$$\hat{\zeta}_L = \frac{-K}{r}\int_0^\infty\int_0^\infty\int_0^\infty \frac{\beta^{m+b_1-1}\theta^{2m+b_2-1}}{\zeta(1+\theta)^m}\prod_{i=1}^m\left(1+x_i^\beta\right)x_i^{\beta-1}e^{-\theta\left(\sum_{i=1}^n x_i^\beta+a_2\right)-a_1\beta-r\zeta}A^{n-m}B^{-n}d\beta d\theta d\zeta.$$

The corresponding posterior risks are

$$PR(\beta_L) = r\left(\hat{\beta}_Q-\hat{\beta}_L\right);\ PR(\theta_L) = r\left(\hat{\theta}_Q-\hat{\theta}_L\right);\ PR(\zeta_L) = r\left(\hat{\zeta}_Q-\hat{\zeta}_L\right).$$

Since, we cannot calculate the analytical expressions of all these estimators; we will use MCMC procedures as Metropolis-Hastings algorithm in the following Monte Carlo section.

5. Monte Carlo Study

In this section, we present some simulation results to compare the performance of the different estimations that are proposed in this paper. We compare the performance of the MLE and the Bayes estimators of the unknown parameters for the UTPL(θ,β,ζ) distribution under type II censored data. In this section, we perform a Monte Carlo study assuming that $a_1 = b_1 = 2$, $a_2 = b_2 = 1$, $\theta = 1$, β =2 and $\zeta = 1.5$. Then, using N = 5000 samples of the upper truncated model and using 20% Censored Samples**,** we obtain the following results.

5.1. Likelihood estimation

Since analytical solutions are not available, to obtain the maximum likelihood estimators, we need to use numerical procedures. In this section, we will use the R package BB which is based on the Barzilai-Borwein gradient method developed by Raydan (1997) to derive the numerical values of the MLE estimators.The MLE estimates along with the quadratic error, $\left(\text{Bias of } \hat{\theta}\right)^2$, are summarized in table 2 as follows:

Table. 2: The MLE of the parameters with quadratic error (in brackets)

n	m	Parameter	MLE
10	8	θ	0.6841 (0.0828)
		β	1.8900 (0.004)
		ζ	1.9567(0.1770)
30	24	θ	0.9047 (0.0088)
		β	2,0006(0.0071)
		ζ	1.6302(0.0141)
50	40	θ	0.9463(0.0050)
		β	2.0465(0.0011)
		ζ	1.6854(0.0214)
100	80	θ	0.9834(0.00325)
		β	2.0634(0.0014)
		ζ	1.5258(0.0004)
200	160	θ	0.9386(0.0063)
		β	2.0867(0.0081)
		ζ	1.6546(0.0079)

We remark that the estimated values of θ and β are close to the true values of parameters. However, the estimation of ζ is somewhat close to the true value.

5.2. Bayesian estimation

The MCMC algorithm is used for computing the Bayes estimates of the parameters θ, β and ζ. We consider the Metropolis-Hastings algorithm, to generate samples from the conditional posterior distributions and then compute the Bayes estimates. For more details about the MCMC methods see, for example, Upadhyaya et al. (2001) and Upadhyaya and Gupta (2010). From (13), the marginal posterior density of θ is proportional to

$$\pi\left(\theta|\beta,\zeta,\underline{x}\right) \propto \frac{\theta^{2m+b_2-1}}{(1+\theta)^m} e^{-\theta\left(\sum_{i=1}^{n} x_i^\beta + a_2\right)} A^{n-m} B^{-n},$$

Similarly, the full posterior conditional distributions for β and ζ are given as follows:

$$\pi\left(\beta|\theta,\zeta,\underline{x}\right) \propto \beta^{m+b_1-1} \prod_{i=1}^{m}\left(1 + x_i^\beta\right) x_i^{\beta-1} e^{-\theta\left(\sum_{i=1}^{n} x_i^\beta\right) - a_1\beta} A^{n-m} B^{-n},$$

$$\pi(\zeta|\beta,\theta,\underline{x}) \propto \frac{1}{\zeta} A^{n-m} B^{-n},$$

respectively. The full conditional distributions are not in standard distributional forms; therefore, we propose the use of MH algorithm to draw the random sample from the full conditionals. A hybrid algorithm of Gibbs and MH samplers consists of the following steps:

Step 1. Set initial values θ^0, $\beta^0$and ζ^0 for θ, β and ζ.

Step 2. Using initial values θ^0, β^0 and ζ^0, generate candidate points $\{\theta^c, \beta^c \text{ and } \zeta^c\}$ respectively from the proposal densities $q_1(\theta^c|\theta^0)$, $q_2(\beta^c|\beta^0)$ and $q_3(\zeta^c|\zeta^0)$ where $q(\Theta^c|\Theta^0)$, $\Theta = \{\theta, \beta, \zeta\}$ is the probability of returning a value of Θ^c given a previous value of Θ^0. Here, we propose the use of asymptotic distributions of MLEs as proposal densities.

Step 3. Generate a uniform variate on range 0 to 1, i.e., $u \sim U(0,1)$.

Step 4. Calculate Hastings-ratio using candidate point θ^c and previous point θ^0 as given by

$$\rho_1(\theta^c|\theta^0) = \left[\frac{\pi(\theta^c|\beta^0,\zeta^0,\underline{x})q_1(\theta^0|\theta^c)}{\pi(\theta^0|\beta^0,\zeta^0,\underline{x})q_1(\theta^c|\theta^0)}\right]$$

Step 5. Accept the candidate point as

$$\theta = \begin{cases} \theta^c, & if\ u \le \min\left(1, \rho_1(\theta^c|\theta^0)\right) \\ \theta^0, & \text{otherwise} \end{cases},$$

and set $\theta^0 = \theta$.

Step 6. Now using the current point β^0, calculate Hastings-ratio for the parameter β as given by

$$\rho_2(\beta^c|\beta^0) = \left[\frac{\pi(\beta^c|\theta^0,\zeta^0,\underline{x})q_2(\beta^0|\beta^c)}{\pi(\beta^0|\theta^0,\zeta^0,\underline{x})q_2(\beta^c|\beta^0)}\right].$$

Step 7. Accept the candidate point as

$$\beta = \begin{cases} \beta^c, & if\ u \le \min\left(1, \rho_2(\beta^c|\beta^0)\right) \\ \beta^0, & \text{otherwise} \end{cases},$$

and set $\beta^0 = \beta$.

Step 8. Now using the current point ζ^0, calculate Hastings-ratio for the parameter ζ as given by

$$\rho_3(\zeta^c|\zeta^0) = \left[\frac{\pi(\zeta^c|\theta^0,\beta^0,\underline{x})q_3(\zeta^0|\zeta^c)}{\pi(\zeta^0|\theta^0,\beta^0,\underline{x})q_3(\zeta^c|\zeta^0)}\right].$$

Step 9. Accept the candidate point as

$$\zeta = \begin{cases} \zeta^c, & if\ u \leq \min\left(1, \rho_3(\zeta^c|\zeta^0)\right) \\ \zeta^0, & \text{otherwise} \end{cases},$$

and set $\zeta^0 = \zeta$.

Step 10. Repeat Steps 2-9, N=5000 times and obtain posterior sample of size N for the parameters θ, β and ζ.

Step 11. Bayes estimates of θ, β and ζ under generalized quadratic loss, can be obtained as the mean of the simulated sample from their posteriors. Thus, the formulae to obtain Bayes estimates of β, θ and ζ are given by

$$\hat{\beta}_{GQMC} = \frac{\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} \beta_i^{\alpha}}{\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} \beta_i^{\alpha-1}},$$

$$\hat{\theta}_{GQMC} = \frac{\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} \theta_i^{\alpha}}{\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} \theta_i^{\alpha-1}},$$

$$\hat{\zeta}_{GQMC} = \frac{\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} \zeta_i^{\alpha}}{\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} \zeta_i^{\alpha-1}}.$$

The approximate Bayes estimates for β, θ and ζ under entropy loss are given by

$$\hat{\beta}_{EMC} = \left(\frac{1}{N-N_0}\sum_{i=N_0+1}^{N}(\beta^{-p})\right)^{-\frac{1}{p}},$$

$$\hat{\theta}_{EMC} = \left(\frac{1}{N-N_0}\sum_{i=N_0+1}^{N}(\theta^{-p})\right)^{-\frac{1}{p}},$$

$$\hat{\zeta}_{EMC} = \left(\frac{1}{N-N_0}\sum_{i=N_0+1}^{N}(\zeta^{-p})\right)^{-\frac{1}{p}}.$$

Also, The approximate Bayes estimates for θ, β and ζ under Linex loss are given by

$$\hat{\beta}_{LMC} = \frac{-1}{r}\left[\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} e^{-r\beta_i}\right],$$

$$\hat{\theta}_{LMC} = \frac{-1}{r}\left[\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} e^{-r\theta_i}\right],$$

$$\hat{\zeta}_{LMC} = \frac{-1}{r}\left[\frac{1}{N-N_0}\sum_{i=N_0+1}^{N} e^{-r\zeta_i}\right].$$

Here, N_0 (burn-in period) is taken to be 1000.

It well known that rapid convergence is facilitated by choosing appropriate starting values. In order to guarantee the convergence and to remove the affection of the selection of initial value, the first N_0 simulated variates are discarded. Then the selected sample are β_i, θ_i and ζ_i, $i =$

$N_0, \dots, N$, for sufficiently large N, forms an approximate posterior sample which can be used to develop the Bayesian inference.

Table 3 presents the Bayesian estimations and the corresponding posterior risks (PR), in brackets, under the generalized quadratic loss function. We remark that the value $\alpha = -2$ gives us the best posterior risk and then improve the basic quadratic case. Also, we obtain the smallest suitable posterior risk when n is high.

Table 3: Bayes estimators and PR (in brackets) under generalized quadratic loss function

n	m	Parameter	α							
			-2	-1.5	-1	-0.5	0.5	1	1.5	2
10	8	θ	1.0903 (0.0068)	1.0934 (0.0081)	1.0982 (0.0086)	1.1036 (0.0091)	1.12112 (0.0097)	1.1254 (0.0101)	1.1298 (0.0106)	1.1248 (0.01102)
		β	1.1783 (0.0734)	1.2654 (0.0992)	1.3452 (0.1329)	1.4418 (0.1786)	1.7238 (0.3258)	1.8291 (0.3244)	1.8920 (0.3643)	1.9665 (0.3935)
		ζ	1.1952 (0.0448)	1.2286 (0.0531)	1.2765 (0.713)	1.3074 (0.0735)	1.3912 (0.0942)	1.4322 (0.1045)	1.4769 (0.1243)	1.5261 (0.1432)
30	24	θ	1.0748 (0.0056)	1.0827 (0.0059)	1.0847 (0.0073)	1.0893 (0.0075)	1.0961 (0.0076)	1.0986 (0.0082)	1.1028 (0.0087)	1.1062 (0.0092)
		β	1.1623 (0.0709)	1.2334 (0.0947)	1.3104 (0.1301)	1.3963 (0.1723)	1.6027 (0.2615)	1.7164 (0.3116)	1.8218 (0.3557)	1.9807 (0.3915)
		ζ	1.1736 (0.0435)	1.2175 (0.0524)	1.2346 (0.1607)	1.2852 (0.0728)	1.3672 (0.0919)	1.4156 (0.1032)	1.4334 (0.1182)	1.4892 (0.1287)
50	40	θ	1.0740 (0.0044)	1.0784 (0.0052)	1.0827 (0.0064)	1.0853 (0.0068)	1.0923 (0.0073)	1.0975 (0.0078)	1.0993 (0.0083)	1.1039 (0.0087)
		β	1.1442 (0.0675)	1.1998 (0.0852)	1.2731 (0.1229)	1.3580 (0.1623)	1.5689 (0.2534)	1.6834 (0.3065)	1.7834 (0.3485)	1.8742 (0.2403)
		ζ	1.1576 (0.0426)	1.1896 (0.0512)	1.2236 (0.0623)	1.2643 (0.0721)	1.3466 (0.0853)	1.3851 (0.1019)	1.4387 (0.1163)	1.4653 (0.1227)
100	80	θ	1.2012 (0.0016)	1.2042 (0.0018)	1.2045 (0.0019)	1.2056 (0.0021)	1.2079 (0.0026)	1.2090 (0.0038)	1.2101 (0.0045)	1.2113 (0.0066)
		β	2.2003 (0.0013)	2.2009 (0.0074)	2.2016 (0.0008)	2.2022 (0.0009)	2.2035 (0.0029)	2.2041 (0.0031)	2.2048 (0.0046)	2.2054 (0.0066)
		ζ	1.7017 (0.0016)	1.7025 (0.0008)	1.7033 (0.0009)	1.7041 (0.0012)	1.7058 (0.0023)	1.7066 (0.0028)	1.7173 (0.0037)	1.7081 (0.0048)
200	160	θ	1.2109 (0.0013)	1.2119 (0.0015)	1.2128 (0.0016)	1.2139 (0.0018)	2.1929 (0.0021)	2.1937 (0.0021)	2.1942 (0.0032)	2.1950 (0.0034)
		β	2.1895 (0.0003)	2.1902 (0.0055)	2.1909 (0.0007)	1.1916 (0.0009)	2.1930 (0.0024)	2.1937 (0.0028)	2.1942 (0.0045)	2.1951 (0.0048)
		ζ	1.7006 (0.0006)	1.7013 (0.0007)	1.7022 (0.0009)	1.7029 (0.0011)	1.7044 (0.0020)	1.7052 (0.0026)	1.7060 (0.0034)	1.7068 (0.0046)

With the entropy loss function, we obtain the following table where we can notice that the value $p = -0.5$ and the cases $n = 100$ and $n = 200$ provide the best posterior risk.

Table 4: Bayes estimators and PR (in brackets) under entropy loss function.

n	m	Parameter	p							
			-2	-1.5	-1	-0.5	0.5	1	1.5	2
10	8	θ	1.1198 (0.0154)	1.1177 (0.0088)	1.1156 (0.0036)	1.1134 (0.0009)	1.1091 (0.0008)	1.1070 (0.0039)	1.1048 (0.0087)	1.1027 (0.0155)
		β	1.8339 (0.2452)	1.7938 (0.1385)	1.7492 (0.0654)	1.6997 (0.0159)	1.5917 (0.0882)	1.5363 (0.0895)	1.4826 (0.1577)	1.4322 (0.2794)
		ζ	1.4679 (0.1093)	1.4508 (0.0653)	1.4324 (0.0292)	1.4239 (0.0073	1.3727 (0.0077)	1.3505 (0.0368)	1.3293 (0.0699)	1.3084 (0.1289)
30	24	θ	1.1019 (0.0132)	1.1002 (0.0074)	1.0984 (0.0032)	1.0966 (0.0008)	1.0934 (0.0007)	1.0917 (0.0031)	1.0898 (0.007)	1.0883 (0.0124)
		β	1.8036 (0.2285)	1.7617 (0.1359)	1.7147 (0.0637)	1.6634 (0.0147)	1.5531 (0.0701)	1.4978 (0.0717)	1.445 (0.1413)	1.3962 (0.2638)
		ζ	1.4425 (0.1078)	1.4251 (0.0626)	1.4066 (0.0287)	1.3871 (0.0074)	1.3462 (0.0077)	1.3253 (0.0318)	1.3046 (0.0695)	1.2844 (0.1246)
50	40	θ	1.0994 (0.0128)	1.0976 (0.0069)	1.0958 (0.0023)	1.0941 (0.0009)	1.0905 (0.0009)	1.0888 (0.0024)	1.0870 (0.0063)	1.0853 (0.0119)
		β	1.7745 (0.2135)	1.7298 (0.1243)	1.6803 (0.0622)	1.6269 (0.0125)	1.5150 (0.0629)	1.4604 (0.0702)	1.4092 (0.1333)	1.3626 (0.2549)
		ζ	1.4259 (0.1044)	1.4076 (0.0561)	1.3881 (0.0201)	1.3675 (0.0058)	1.3257 (0.0060)	1.3055 (0.309)	1.2839 (0.0518)	1.2639 (0.1229)
100	80	θ	1.2101 (0.0038)	1.2096 (0.0022)	1.2089 (0.0009)	1.2084 (0.0002)	1.2073 (0.0003)	1.2067 (0.0009)	1.2062 (0.0022)	1.2056 (0.0038)
		β	2.2048 (0.0012)	2.2045 (0.0007)	2.2042 (0.0008)	2.2038 (0.0008)	2.2032 (0.0003)	2.2038 (0.0003)	2.2025 (0.0007)	2.2022 (0.0012)
		ζ	1.7074 (0.0019)	1.7069 (0.0011)	1.7066 (0.0005)	1.7062 (0.0002)	1.7054 (0.0002)	1.7049 (0.0005)	1.7046 (0.0011)	1.7042 0.0021
200	160	θ	1.2177 (0.0032)	1.2172 (0.0018)	1.2168 (0.0008)	1.2163 (0.0002)	1.2153 (0.0002)	1.2148 (0.0008)	1.2144 (0.0018)	1.2139 (0.0032)
		β	2.1943 (0.0013)	2.1940 (0.0008)	1.1937 (0.0004)	1.1933 (0.0007)	1.1926 (0.0004)	1.1923 (0.0003)	1.1919 (0.0008)	1.1925 (0.0013)
		ζ	1.7060 (0.0018)	1.7056 (0.0012)	1.7052 (0.0005)	1.7048 (0.0002)	1.7041 (0.0002)	1.7037 (0.0005)	1.7034 (0.0012)	1.7029 (0.0018)

Table 5: Bayes estimators and PR (in brackets) under Linex loss function.

n	m	Parameter	r							
			-2	-1.5	-1	-0.5	0.5	1	1.5	2
10	8	θ	1.1253 (0.0196)	1.1229 (0.0108)	1.1204 (0.0049)	1.1178 (0.0013)	1.1132 (0.0013)	1.1108 (0.0049)	1.1082 (0.0108)	1.1059 (0.1192)
		β	1.9693 (0.4403)	1.9301 (0.2715)	1.8808 (0.1388)	1.8205 0.0387	1.6704 (0.0395)	1.5893 (0.1598)	1.5122 (0.3557)	1.4432 (0.7304)
		ζ	1.5206 (0.1765)	1.5017 (0.1039)	1.4806 (0.0482)	1.4572 (0.0125)	1.4056 (0.0151)	1.3789 (0.0536)	1.3516 (0.1213)	1.3249 (0.2151)
30	24	θ	1.1058 (0.0151)	1.1041 (0.0085)	1.1022 (0.0038)	1.1003 (0.0011)	1.0966 (0.0008)	1.0947 (0.0038)	1.0929 (0.0097)	1.0911 (0.1107)
		β	1.9447 (0.4398)	1.9036 (0.2634)	1.8519 (0.1373)	1.7886 (0.0371)	1.6341 (0.0304)	1.5534 (0.1623)	1.4763 (0.3277)	1.4095 (0.6107)
		ζ	1.4951 (0.1758)	1.4759 (0.1031)	1.4547 (0.0481)	1.4315 (0.0121)	1.3815 (0.0132)	1.3538 (0.0528)	1.3272 (0.119)	1.3014 (0.2105)
50	40	θ	1.1037 (0.0147)	1.1017 (0.0079)	1.0998 (0.0036)	1.0978 (0.0009)	1.0939 (0.0008)	1.0919 (0.0029)	1.0898 (0.0089)	1.0881 (0.0155
		β	1.9272 (0.3038)	1.8822 (0.2028)	1.8253 (0.1258)	1.7581 (0.0328)	1.5973 (0.0224)	1.5255 (0.1547)	1.4421 (0.2587)	1.3771 (0.6064)

		ζ	1.4821 (0.1685)	1.4616 (0.0509)	1.4389 (0.0309)	1.4142 (0.0031)	1.3609 (0.0036)	1.3336 (0.0344)	1.3069 (0.1018)	1.2813 (0.1137)
100	80	θ	1.2118 (0.0056)	1.2111 (0.0032)	1.2104 (0.0014)	1.2097 (0.0004)	1.2083 (0.0004)	1.2076 (0.0015)	1.2069 (0.0032)	1.2063 (0.0057)
		β	2.2070 (0.0057)	2.2063 (0.0032)	2.2056 (0.0025)	2.2049 (0.0014)	2.2035 (0.0024)	2.2029 (0.0015)	2.2021 (0.0032)	2.0213 (0.0056)
		ζ	1.7094 (0.0057)	1.7087 (0.0032)	1.7079 (0.0014)	1.7073 (0.0004)	1.7059 (0.0014)	1.7052 (0.0041)	1.7044 (0.0032)	1.7038 (0.0056)
200	160	θ	1.2191 (0.0048)	1.2185 (0.0027)	1.2179 (0.0012)	1.2174 (0.0003)	1.2162 (0.0003)	1.2156 (0.0012)	1.2144 (0.0027)	1.2144 (0.0047)
		β	2.1966 (0.0051)	2.1959 (0.0024)	2.1952 (0.0016)	2.1645 (0.0004)	2.1929 (0.0007)	2.1921 (0.0011)	2.1915 (0.0025)	2.1906 (0.0041)
		ζ	1.7078 (0.0052)	1.7072 (0.0031)	1.7065 (0.0013)	1.7059 (0.0004)	1.7016 (0.0004)	1.7038 (0.0012)	1.7032 (0.0028)	1.7026 (0.0052)

From table 5, one can notice that the value $r = -0.5$ provides the best PR**.** From tables 3-5, when the effective sample sizes (n, m) are increase the PR of the all estimates based on Type-II censored data are decrease. If we compare the three loss functions, we notice that the entropy loss function provides the best Bayesian estimator of θ, β and ζ. This is illustrated by the following table:

Table 6: Bayes estimators and PR (in brackets) under the three loss function

n	m	Parameter	Generalized quadratic $(\alpha = -2)$	Entropy $(p = -0.5)$	Linex $(r = -0.5)$
10	8	θ	1.0903 (0.0068)	1.1134 (0.0009)	1.1178 (0.0013)
		β	1.1783 (0.0734)	1.6997 (0.0159)	1.8205 0.0387
		ζ	1.1952 (0.0448)	1.4239 (0.0073	1.4572 (0.0125)
30	24	θ	1.0748 (0.0056)	1.0966 (0.0008)	1.1003 (0.0011)
		β	1.1623 (0.0709)	1.6634 (0.0147)	1.7886 (0.0371)
		ζ	1.1736 (0.0435)	1.3871 (0.0074)	1.4315 (0.0121)
50	40	θ	1.0740 (0.0044)	1.0941 (0.0009)	1.0978 (0.0009)
		β	1.1442 (0.0675)	1.6269 (0.0125)	1.7581 (0.0328)
		ζ	1.1576 (0.0426)	1.3675 (0.0058)	1.4142 (0.0031)
100	80	θ	1.2012 (0.0016)	1.2084 (0.0002)	1.2097 (0.0004)
		β	2.2003 (0.0013)	2.2038 (0.0008)	2.2049 (0.0014)
		ζ	1.7017 (0.0016)	1.7062 (0.0002)	1.7073 (0.0004)

200	160	θ	1.2109 (0.0013)	1.2163 (0.0002)	1.2174 (0.0003)
		β	2.1895 (0.0003)	1.1933 (0.0007)	2.1645 (0.0004)
		ζ	1.7006 (0.0006)	1.7048 (0.0002)	1.7059 (0.0004)

5.3. Comparison with the likelihood estimators

In this subsection, we propose to compare the best Bayesian estimators obtained above with the maximum likelihood estimator. For this, we propose to use the following criteria: the Pitman closeness (Pitman, (1937) Fuller, (1982) and Jozani, (2012)) and the mean squared error (MSE) defined as follows:

Definition 5.1 An estimator θ_1 of a parameter θ dominates in the sense of Pitman closeness criterion another estimator θ_2, if for all $\boldsymbol{\theta} \in \boldsymbol{\Theta}$

$$P_\theta[|\theta_1 - \theta| < |\theta_2 - \theta|] > 0.5.$$

Consider the estimates θ_i (*i=1… N*) Obtained with *N* samples of the model.

Definition 5.2 the mean square error is defined as

$$MSE = \frac{\sum_{i=1}^{N}(\theta_i - \theta)^2}{N}.$$

In the following, we present the values of the Pitman probabilities, which allow us to compare the Bayesian estimators with the MLE under the three loss functions where $\alpha = -2$, $p = -0.5$ and $r = -0.5$. Table 8 should be read as follows: when the probability is greater than 0.5, the Bayesian estimator is better than the MLE estimator. Then, we notice that, according to this criterion:

-When *n* is not high, the Bayesian estimators θ_B and ζ_B of θ and ζ are better than the MLE's θ_{MLE} and ζ_{MLE}. The generalized quadratic loss function provides the best values. However, β_{MLE} is closer to the true value than all the Bayesian estimators.

- When *n* is high, the MLE of the three parameters performs better than the Bayesian estimators.

Table 7. Pitman comparison of the estimators θ, β and ζ.

n	m	Parameter	Generalized quadratic ($\alpha = -2$)	Entropy ($p = -0.5$)	Linex ($r = -0.5$)
10	8	θ	0.689	0.636	0.651
		β	0.248	0.131	0.158
		ζ	0.656	0.587	0.611
30	24	θ	0.576	0.557	0.561
		β	0.274	0.111	0.152
		ζ	0.533	0.488	0.502
50	40	θ	0.536	0.516	0.519
		β	0.232	0.089	0.118

		ζ	0.424	0.395	0.398
100	80	θ	0.153	0.148	0.148
		β	0.131	0.137	0.162
		ζ	0.415	0.378	0.384
200	160	θ	0.098	0.097	0.006
		β	0.223	0.078	0.087
		ζ	0.279	0.252	0.259

Table 8 presents the values of the mean squared error of the Bayesian estimators of the parameters under the three loss functions, and the maximum likelihood estimators. According to this criterion, when n is small, the Bayesian estimators θ_B and ζ_B provide the smallest MSE for θ and ζ comparatively to θ_{MLE} and ζ_{MLE}. furthermore, the values provided by the generalized quadratic loss function are relatively equivalent to the entropy and linex. But, the MSE of β_{MLE} is smaller than the MSE of the Bayesian estimators. If n is high, then, all the Bayesian estimators perform better than the MLE estimators, and the generalized quadratic loss function provides the best values of the MSE.

Table 8. The MSE of the estimators θ, β and ζ.

n	m	Parameter	MLE	Generalized quadratic	Entropy	Linex
10	8	θ	0.2697	0.0081	0.0223	0.0191
		β	0.1186	0.4334	0.3667	0.3687
		ζ	0.3279	0.1036	0.1072	0.7039
30	24	θ	0.1032	0.0099	0.0169	0.0156
		β	0.0687	0.3923	0.3937	0.3906
		ζ	0.2578	0.1037	0.1108	0.1088
50	40	θ	0.0703	0.0116	0.0168	0.158
		β	0.0552	0.4182	0.4276	0.4238
		ζ	0.2438	0.1127	0.1192	0.1176
100	80	θ	0.0873	0.0382	0.0462	0.0446
		β	0.0528	0.0192	0.0439	0.0363
		ζ	0.2841	0.0322	0.0448	0.0423
200	160	θ	0.1909	0.1625	0.1673	0.1664
		β	0.5468	0.5288	0.5494	0.5434
		ζ	0.4839	0.3168	0.3264	0.3243

6. Application

In this section, we consider a real-world data set to demonstrate how Bayesian estimation works in practice. Aljuaid (2013) obtained Bayes and classical estimators for two parameters of the exponentiated inverted Weibull distribution when the sample is available from a complete and type II censoring scheme. They analyzed real data sets for the purpose of illustration using the non-informative gamma priors for the parameters, that is , when the hyperparameters are zero.

In this section, we use the MCMC technique to estimate the unknown parameters of the distribution using the non-informative gamma priors under different loss functions as discussed in Section 5, assuming that, $a_1 = b_1 = a_2 = b_2 =0$. R software was used to perform all computations. The uncensored data set below consists of 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by Dimitrakopoulou et al. (2007). The entire data set is provided below:

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0
1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3
4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0	24.5		

The truncation point is clearly known and equal 24.5 for complete data and 4.7 for censored data when the failure time, m=36. Estimates of the parameters of upper truncated power Lindley model by MLE and Bayesian with three loss functions are given in the table 9 for activity repair time data. The PRs under Generalized quadratic, Entropy and Linex loss functions are presented in Table 10.

Table 9**:** Likelihood and Bayesian estimation of the parameters under three loss of function.

n	m	parameter	MLE	Generalized quadratic	Entropy	Linex
46	46	$\hat{\theta}$	0.676	0.646	0.645	0.655
		$\hat{\beta}$	0.749	0.781	0.818	0.825
		$\hat{\zeta}$	2.488	2.531	2.501	2.605
46	36	$\hat{\theta}$	0.637	0.613	0.615	0.617
		$\hat{\beta}$	0.886	0.789	0.798	0.796
		$\hat{\zeta}$	2.548	2.563	2.516	2.616

Table 10: Posterior risk under the three loss function

n	m	Parameter	Generalized quadratic $(\alpha = -2)$	Entropy $(p = -0.5)$	$(r = -0.5)$
46	46	θ	0.0038	0.0014	0.0023
		β	0.0294	0.0072	0.0145
		ζ	0.0051	0.0042	0.0063
46	36	θ	0.0043	0.0016	0.0025
		β	0.0359	0.0087	0.0178
		ζ	0.0087	0.0043	0.0094

From table 10, it is clear that the result based on real life data has a smaller amount of posterior risks for uncensored data as compared to censored data. This is because of the loss of information during the censoring. When the three loss functions are compared, the entropy loss function estimates are associated with lower amounts of posterior risks and provide the best Bayesian estimator of θ, β and ζ. From tables 11, and 12, we conclude that the Bayesian performance by the three-loss functions method is better than the likelihood method.

Table 11. Pitman comparison of the estimators θ, β and ζ

n	m	Parameter	Generalized quadratic ($\alpha = -2$)	Entropy ($p = -0.5$)	Linex ($r = -0.5$)
46	46	θ	0.697	0.624	0.661
		β	0.638	0.741	0.854
		ζ	0.478	0.395	0.397
46	36	θ	0.586	0.577	0.571
		β	0.624	0.711	0.762
		ζ	0.348	0.351	0.328

Table 12 **:** The MSE of the estimators of θ, β and ζ

n	m	Parameter	MLE	Generalized quadratic ($\alpha = -2$)	Entropy($p = -0.5$)	Linex ($r = -0.5$)
46	46	θ	0.0214	0.0821	0.0738	0.0791
		β	0.1202	0.0111	0.0782	0.0956
		ζ	0.3857	0.3482	0.3693	0.3534
46	36	θ	0.0258	0.0234	0.0211	0.0231
		β	0.0625	0.0253	0.0230	0.0242
		ζ	0.3761	0.3351	0.3033	0.3040

7. Conclusion

In this study, the MLE estimation and the Bayesian estimation based on the generalized quadratic, entropy and Linex loss functions for the unknown parameters of the upper truncated power Lindley distribution have been discussed based on the Type-II censored samples. In the Bayesian estimation, for each loss function, we obtained the suitable parameter, which optimises the Bayesian estimation. Then, our Monte Carlo study showed that the entropy loss function provides the smallest posterior risks. These selected Bayesian estimators are compared to the maximum likelihood estimators of the parameters using the Pitman closeness criterion and the mean square error. Then, using our exhaustive Monte Carlo procedure, we showed that when n is small, the Bayesian estimators are better for θ and ζ and not for β. If n is high enough, the MLE's are closer to the true values but provide the highest MSE than the Bayesian estimators. Finally, we illustrate our study with an example of real life data.

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