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STOCHASTIC CONTROL OF BILINEAR SYSTEMS
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4 **ABSTRACT**

In this paper we present an alternative formulation for the control problem for a class of partially observed systems governed by bilinear stocahstic differential equations. The problem is described by three sets of stochastic differential equations: one for the system to be controlled, one for the observer and one for the control process which is driven by the observer. With this formulation, the stochastic control problem can be converted into an equivalent (deterministic) identification problem in which the controller parameters are the unknowns. Using variational arguments we derive the necessary conditions for optimal identification.Based on these necessary conditions, we propose an iterative algorithm for determining controller parameters along with some numerical simulations to illustrate the effectiveness of the proposed control scheme.

KEY WORDS: **Optimal Contorl, Stochastic Differential Equations, Parameter Identification.**

1. INTRODUCTION

The Hamilton-Jacobi-Belman (HJB) equation arising from the application of Belman's principle of optimality as well as Ito's formula to controlled stochastic (comletely or partially observed) systems has been the major tool for determining optimal control laws (see for example [1,2,3,4,5,6]). With this approach, one is required to solve a nonlinear partial differential equation (of parabolic type) on the state space R^n. This has, so far, posed a major stumbling block in its application to engineering problems.It seems almost impossible to avoid solving the HJB equation if one is interested in determining optimal controls for nonlinear, bilinear, or even linear stochastic systems with control constraints. There are, however, some techniques that has appeared recently in the literature with the

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help of which one can determine optimal controls without solving the HJB equation. These techniques can, however, be applied to a limited class of linear stochastic systems where no constraints are imposed on controls. In [7], for example, Ren and Kumar have considered the stochastic adaptive control problem for linear time invariant systems. For this class of systems, they provided several algorithms for adaptive filtering, adaptive control and for identification problems by employing an indirect or direct procedure and either least square or gradient based parameter estimators. In [8] Van Schuppen introduced the concept of tuning of a stochastic control system governed by linear time-invariant stochastic difference equation. The results showed however the limitations of the synthesis procedure of self-tuning regulation. The reader is also referred to the work of Aström, Kumar, Becker, Varaiya, and Goodwin (see [9-18]) where the authors mostly considered stochastic control or adaptive control, or identification problems for calsses of systems governed by linear (time varying or time-invariant) stochastic differential (or difference) equations with no constraints on controls. In [3], Dabbous and Ahmed considered the identification problem for a class of systems governed by nonlinear time varying partially observed stochastic differential equations of Ito-type. Using pathwise [3] formulation of Zakai equation, the problem was converted into an equivalent (deterministic) identification problem in which the solutions of Zakai equation is treated as the state and the unknown parameters as controls. Using variational arguments the necessary conditions for optimal identification were obtained.

In [1] Ahmed has proposed a new formulation for stochastic control problem of partially observed linear systems governed by stochastic differential equations driven by general martingles. With this formulation, the original stochastic control problem has been converted into an equivalent (deterministic) identification problem for which the corresponding necessary conditions of optimality can be obtained by direct use of variational arguments. In this paper we consider the stochastic control problem for a class of partially observed stochastic systems governed by bilinear stochastic differential equations with control constraints. Using similar control structure as that proposed in [1], we convert the original stochastic control problem into an equivalent (deterministic) identification problem. Using variational arguments, we develop the corresponding necessary conditions for optimal identification on the basis of which optimal al controls can be determined. The paper is organized as follows. In section 2, we formulate the stochastic control problem and present necessary notations and assumptions ions needed for the development of the necessary conditions. In section 3, we show how the stochastic control problem can be converted into an equivalent (deterministic) identification problem in which the control parameters are the unknowns. In section 4, we utilize variational arguments to develop the necessary conditions of optimality for the identification problem. Based on these necessary conditions, we propose in section 5, an algorithm for computing the unknowns along with some numerical simulations to illustrate some of the results of this paper.

2. PROBLEM STATEMENT, NOTATIONS, ASSUMPTIONS

Consider the following (bilinear) stochastic system

$$dx(t) = A(t)x(t)dt + B(t)du(t) + \sum_{i=1}^{M} \sigma_i(t)x(t)dw_i(t) \tag{2.1}$$

$$x(0) = x_0;\ t \in I \equiv [0,T],$$

where A, B and σ_i; $1 \le i \le M$, are (nxn), (nxr), and (nxn) matrix valued functions, respectively. Further, $W \equiv \{w_i; 1 \le i \le M\}$ is a standard Wiener process with values in R^M and the initial state x_0 is a random variable independent of W. The control process $u(t)$; $t \in I$, will be defined shortly. Let the observation process $y(t)$; $t \in I$, be related to the state process $x(t)$; $t \in I$, through the following (bilinear) stochcastic differential equation

$$dy(t) = \left(H_1(t)x(t) + H_2(t)y(t)\right)dt + \sum_{i=1}^{N} \tilde{\sigma}_i(t)y(t)dv_i(t) \tag{2.2}$$
$$y(0) = y_0 ;t \in I.$$

Here H_1, H_2 and $\tilde{\sigma}_i$; $1 \le i \le N$, are (mxn), (mxm), and (mxm) matrix valued functions on I. Further, $V \equiv \left\{v_i;\ 1 \le i \le N\right\}$ is an R^N-valued standard Wiener process independent of W and x_0.

Assuming that all the random processes and vectors described above are defined on some probability space $(\Omega, \mathbf{B}, \mu)$, we wish to design a control system having the following (linear) structure

$$du(t) = K_1(t)y(t)dt + K_2(t)dy(t), \tag{2.3}$$
$$u(0) = 0,\ t \in I,$$

where the control parameters $\{K_1, K_2\}$, are (rxm) matrix valued functions on I.

With this set up, we can now state the stochastic control problem as follows

<u>Problem (P1)</u>
Consider the system (2.1)-(2.2) and suppose the controller has the structure (2.3). Then the problem is to choose the parameter $K \equiv \{K_1, K_2\}$ such that

$$J(K) \equiv E\int_0^T (Q(t)y(t), y(t))\, dt \equiv \min. \tag{2.4}$$

Here (.,.) denotes the scalar product in R^m and Q is positive semidefinite symmetric (mxm) matrix valued function on I, and E(X) denotes the expectation of the random variable X.

<u>Remark 2.1</u>
Note that the integral in (2.4) is obtained by assuming $K(t)$; $0 \le t \le T$, and then solving the equations (2.1)-(2.2) to obtain $y(t)$; $0 \le t \le T$. Clearly, the solution y depends on the parameter K and that is why we have denoted the integral by $J(K)$.

<u>Remark 2.2</u>
One may consider several objective functionals related to the observed process y such as:

(i) $J_1(K) \equiv E\int_0^T \big(Q(t)(y(t)-y_d(t)), y(t)-y_d(t)\big)dt,$

(ii) $J_2(K) \equiv E\int_0^T \big(Q(t)(\hat{y}(t)-y_d(t)), \hat{y}(t)-y_d(t)\big)dt,$

(iii) $J_3(K) \equiv E\int_0^T \big(Q(t)(\hat{y}(t)-y(t)), \hat{y}(t)-y(t)\big)dt,$

where $\hat{y}(t) \equiv E\{y(t)\}$; $t \geq 0$, and y_d denotes the desired output to be tracked. In this paper we shall only deal with the cost functional (2.4), the others can be dealt with in a similar manner.

Remark 2.3

Note that in equation (2.4), the controller is required to regulate the process y about the origin whereas in J_1, the controller is required to track the desired output y_d. Similar arguments is used to explain the objective functionals J_1 and J_2.

For the solution of the problem (P1) we shall need the following notations and assumptions.

Notations:

Let $R^{(nxn)}$ denote the space of all (nxn) matrices.For any matrix $A \in R^{(nxn)}$, let A' denote the transpose of the matrix A and $\|A\|$ denote the norm of A. Let $C(I, R^n)$ denote the space of all continuous functions on I with values in R^n equipped with the usual sup norm. We use $L_\infty(I, R^{(mxm)})$ to denote the class of essentially bounded measurable functions on I having values in $R^{(mxm)}$ with the usual L_∞ norm. Let $\mathcal{F}_t^w \equiv \sigma\{w(\theta);\ \theta \leq t\}$ and $\mathcal{F}_t^y \equiv \sigma\{y(\theta);\ \theta \leq t\}$ denote the σ-algebras generated by the processes y and w up to time t, respectively. We use L_1^{loc} to denote the class of Lebesgue integrable functions on R such that $\int_I |f(t)|dt < \infty$, for any bounded interval $I \subset R$. Further notations will be introduced in the sequel as required.

Assumptions

(A1) All the matrix-valued functions A, B, σ_i, H_1, H_2 and $\tilde{\sigma}_i$ are measurable on $I \equiv [0, T]$.

(A2) The control parameter $K \equiv \{K_1, K_2\} \in \mathcal{K}$, where $\mathcal{K}$ is closed, bounded and convex subset of $L_\infty(I, R^{(rxm)}) x L_\infty(I, R^{(rxm)})$.

Remark 2.4

Under the assumptions (A1)-(A2) and for $E|x_0|^2 < \infty$, one can show that the stochastic differential equation (2.1) has a unique $\mathcal{F}_t^w$ - adapted solution which is continuous with probability one and that $E|x(t)|^2 < \infty$, for all $t \in I$ (see [2]).

3. FORMULATION OF DETERMINISTIC IDENTIFICATION PROBLEM

In this section we show how the stochastic control problem (P1) can be converted into an equivalent deterministic identification problem in which the control parameters $K \equiv \{K_1, K_2\}$ are the unknowns. Consider the system (2.1)-(2.2) and suppose the control process $u(t)$; $t \geq 0$, is given by equation (2.3). Then using (2.3) in (2.1), we have

$$dx(t) = \big(A(t) + B(t)K_2(t)H_1(t)\big)x(t)dt + \big(B(t)K_1(t) + B(t)K_2(t)H_2(t)\big)y(t)dt$$
$$+ \sum_{i=1}^{M}\sigma_i(t)x(t)dw_i(t) + \sum_{i=1}^{N}B(t)K_2(t)\tilde{\sigma}_i(t)y(t)dv_i(t), \tag{3.1}$$
$$x(0) = x_0;\ t \in I.$$

Defining $\xi \equiv (x, y)'$, it follows that the system (3.1) together with the observation dynamics, equation (2.2) can be written as

$$d\xi(t) = \mathcal{A}(t,K)\xi(t)dt + \sum_{i=1}^{M}C_i(t)\xi(t)dw_i(t) + \sum_{i=1}^{N}D_i(t,K)\xi(t)dv_i(t)$$
$$\xi(0) = \xi_0;\ t \in I, \tag{3.2}$$

where

$$\mathcal{A}(t,K) \equiv \begin{pmatrix} A(t) + B(t)K_2(t)H_1(t) & B(t)(K_1(t) + K_2(t)H_2(t)) \\ H_1(t) & H_2(t) \end{pmatrix}, \tag{3.3}$$

$$C_i(t) \equiv \begin{pmatrix} \sigma_i(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad D_i(t,K) \equiv \begin{pmatrix} 0 & B(t)K_2(t)\tilde{\sigma}_i(t) \\ 0 & \tilde{\sigma}_i(t) \end{pmatrix}. \tag{3.4}$$

For the formulation of the (sotchastic) control problem (P1) as a (deterministic) identification problem, we shall need the following result.

Lemma 3.1

Consider the system (3.1) and suppose ξ_0, W and V are statistically independent. Let

$$\hat{\xi}(t) \equiv E(\xi(t)),$$
$$P(t) \equiv E\Big\{(\xi(t) - \hat{\xi}(t))(\xi(t) - \hat{\xi}(t))'\Big\}, \tag{3.5}$$
$$\Lambda(t) \equiv P(t) + \hat{\xi}(t)\hat{\xi}'(t).$$

Then $\Lambda(t)$; $t \geq 0$, satisfies the following differential equation

$$\frac{d\Lambda(t)}{dt} = \mathcal{A}(t,K)\Lambda(t) + \Lambda(t)\mathcal{A}'(t,K) + \sum_{i=1}^{M}C_i(t)\Lambda(t)C_i'(t)$$
$$+ \sum_{i=1}^{N}D_i(t,K)\Lambda(t)D_i'(t,K);\ t \in I,\ K \in \mathcal{K}, \tag{3.6}$$
$$\Lambda(0) = \Lambda_0.$$

Proof

Let $\phi_k(t,\theta)$; $0 \le \theta \le t$, denote the transition operator associated with $\mathcal{A}(t,K)$ for any $K \in \mathcal{K}$. Then the solution of (3.2) is given by the solution of the following stochastic integral equation

$$\xi(t) = \phi_k(t,0)\xi_o + \sum_{i=1}^{M}\int_0^t \phi_k(t,\theta)C_i(\theta)\xi(\theta)dw_i(\theta) + \sum_{i=1}^{N}\int_0^t \phi_k(t,\theta)D_i(\theta)\xi(\theta)dv_i(\theta) \tag{3.7}$$

Taking the mathematical expectation for both sides of (3.7), we have

$$\hat{\xi}(t) = \phi_k(t,0)\hat{\xi}_o. \tag{3.8}$$

Since $P(t) \equiv E\left\{(\xi(t)-\hat{\xi}(t))(\xi(t)-\hat{\xi}(t))'\right\}$, it follows from (3.7) and (3.8) that $P(t)$ satisfies the following integral equation

$$P(t) = \phi_k(t,0)P(0)\phi_k'(t,0) + \sum_{i=1}^{M}\int_0^t \phi_k(t,\theta)C_i(\theta)(P(\theta)+\hat{\xi}(\theta)\hat{\xi}'(\theta))C_i'(\theta)\phi_k'(t,\theta)d\theta$$
$$+\sum_{i=1}^{N}\int_0^t \phi_k(t,\theta)D_i(\theta,K)(P(\theta)+\hat{\xi}(\theta)\hat{\xi}'(\theta))D_i'(\theta,K)\phi_k'(t,\theta)d\theta. \tag{3.9}$$

Using the fact that $\Lambda(t) \equiv P(t)+\hat{\xi}(t)\hat{\xi}'(t)$; $t \ge 0$, it follows that

$$\Lambda(t) = \phi_k(t,0)\Lambda(0)\phi_k'(t,0) + \sum_{i=1}^{M}\int_0^t \phi_k(t,\theta)C_i(\theta)\Lambda(\theta)C_i'(\theta)\phi_k'(t,\theta)d\theta$$
$$+\sum_{i=1}^{M}\int_0^t \phi_k(t,\theta)D_i(\theta,K)\Lambda(\theta)D_i'(\theta,K)\phi_k'(t,\theta)d\theta \tag{3.10}$$

Differentiating (3.10) with respect to t and noting that for each $K \in \mathcal{K}$,

$$\frac{\partial\phi_k(t,\theta)}{\partial t} = \mathcal{A}(t,K)\phi_k(t,\theta),$$
$$\phi_k(\theta,\theta) = I, \quad 0 \le \theta \le T,$$

equation (3.6) follows. This completes the proof. □

Define

$$\bar{Q}(t) \equiv \begin{pmatrix} 0 & 0 \\ 0 & Q(t) \end{pmatrix}.$$

Then, using the definition of Λ, one can easily verify that (2.4) can be written as

$$J(K) = \int_0^T \mathrm{tr}\,(\bar{Q}(t)\Lambda(t))dt, \tag{3.11}$$

where $\mathrm{tr}(A)$ denotes the trace of the matrix A.

With this preparartion, we can now state the following (deterministic) identification problem.

Problem (P2) (Deterministic Identification Problem)

Find $K^o \in \mathcal{K}$ so that $J(K^o) \le J(K)$ for all $K \in \mathcal{K}$ subject to the dynamic constraint (3.6), where J is given by (3.11).

4. NECESSARY CONDITIONS OF OPTIMALITY

In this section we make use of variational arguments to derive the necessary conditions for optimal identification for the problem P2. In our derivation we shall follow similar arguments as those of [14]. Let $K^o \in \mathcal{K}$ be the solution of the problem (P2), and let $K^\varepsilon(t) = K^o(t) + \varepsilon(K(t) - K^o(t))$; $t \in I$, $\varepsilon \in [0,1]$, $K \in \mathcal{K}$. Since the parameter set $\mathcal{K}$ is assumed to be convex, it is clear that K^ε is also an element of $\mathcal{K}$. Let $\Lambda^o(t) \equiv \Lambda(t, K^o)$ and $\Lambda^\varepsilon(t) \equiv \Lambda(t, K^\varepsilon)$, $t \ge 0$, be the solutions of (3.6) with K being replaced by K^o and K^ε , respectively. Let

$$\widetilde{\Lambda}(t) \equiv \widetilde{\Lambda}(t, K^o, K - K^o) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\Lambda^\varepsilon(t) - \Lambda^o(t)\right); \quad t \ge 0, \tag{4.1}$$

denote the Gateaux differential of Λ at K^o in the direction $(K - K^o)$. The following lemma claims that the Gateaux differential $\widetilde{\Lambda}$ exists and satifies a related differential equation.

Lemma 4.1

Consider the problem (P2) and suppose the parameter set $\mathcal{K}$ is convex. Then for each pair $K, K^o \in \mathcal{K}$, the Gateaux differential $\widetilde{\Lambda}$ of Λ exists and satisfies the following differential equation

$$\begin{aligned}\frac{d\widetilde{\Lambda}(t)}{dt} = {} & \mathcal{A}(t, K^o)\widetilde{\Lambda}(t) + \widetilde{\Lambda}(t)\,\mathcal{A}(t, K^o) + \sum_{i=1}^{M} C_i(t)\widetilde{\Lambda}(t)C_i'(t) \\ & + \sum_{i=1}^{N} D_i(t, K^o)\widetilde{\Lambda}(t)D_i'(t, K^o) + \widetilde{\mathcal{A}}(t, K - K^o)\Lambda^o(t) + \Lambda^o(t)\mathcal{A}'(t, K - K^o) \\ & + \sum_{i=1}^{N}\left(D_i(t, K^o)\Lambda^o(t)\widetilde{D}_i'(t, K - K^o) + \widetilde{D}_i(t, K - K^o)\Lambda^o(t)D_i'(t, K^o)\right)\end{aligned} \tag{4.2}$$

$\widetilde{\Lambda}(0) = 0, \quad t \in I.$

Here $\widetilde{\mathcal{A}}$ and $\widetilde{D}_i$; $1 \le i \le N$, are the (Gateaux) differentials of $\mathcal{A}$ and D_i; $1 \le i \le N$, in the direction $(K - K^o)$, respectively.

Proof

The proof follows from standard computations (see [14-16]). □

With the help of the above lemma, we can now present the necessary conditions for optimal identification for problem (P2).

Theorem 4.2 (Necessary Conditions of Optimality)

Consider the problem (P2) and suppose Lemma 4.1 hold. Then the optimal parameter $K^{o} \in \mathcal{K}$ can be determined by the simultaneous solutions of the differential equation

$$\frac{d\Lambda^{o}(t)}{dt} = \mathcal{A}(t,K^{o})\Lambda^{o}(t) + \Lambda^{o}(t)\mathcal{A}'(t,K^{o}) + \sum_{i=1}^{M} C_i(t)\Lambda^{o}(t)C_i'(t)$$
$$+ \sum_{i=1}^{N} D_i(t,K^{o})\Lambda^{o}(t)D_i'(t,K^{o}) \tag{4.3}$$
$$\Lambda^{o}(0) = \Lambda_0, \quad t \in I,$$

the adjoint equation

$$-\frac{d\Gamma^{o}(t)}{dt} = \mathcal{A}'(t,K^{o})\Gamma^{o}(t) + \Gamma^{o}(t)\mathcal{A}(t,K^{o}) + \sum_{i=1}^{M} C_i'(t)\Gamma^{o}(t)C_i(t)$$
$$+ \sum_{i=1}^{N} D_i'(t,K^{o})\Gamma^{o}(t)D_i(t,K^{o}) + \overline{Q}(t), \tag{4.4}$$
$$\Gamma^{o}(T) = 0, \quad t \in I,$$

and the inequality

$$\int_0^T \mathrm{tr}\left(\Gamma^{o}(t)\widetilde{\mathcal{A}}(t,K-K^{o})\Lambda^{o}(t) + \sum_{i=1}^{N} \Gamma^{o}(t)D_i'(t,K^{o})\Lambda^{o}(t)\widetilde{D}_i(t,K-K^{o}) \right) dt \geq 0, \tag{4.5}$$

for all $K \in \mathcal{K}$.

<u>Proof</u>

Define

$$J(K^{\varepsilon}) \equiv \int_0^T \mathrm{tr}\left(\overline{Q}(t)\Lambda^{\varepsilon}(t)\right)dt,$$
$$J(K^{o}) \equiv \int_0^T \mathrm{tr}\left(\overline{Q}(t)\Lambda^{o}(t)\right)dt, \tag{4.6}$$
$$\widetilde{J}_o(K-K^{o}) \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(K^{\varepsilon}) - J(K^{o})\right),$$

where $\widetilde{J}_o(K-K^{o})$ denotes the Gateaux differential of J at K^{o} in the direction $(K-K^{o})$. In order that J attains its minimum at K^{o}, it is necessary that

$$\widetilde{J}_o(K-K^{o}) = \int_0^T \mathrm{tr}\left(\overline{Q}(t)\widetilde{\Lambda}(t)\right)dt \geq 0, \tag{4.7}$$

where $\widetilde{\Lambda}$ satisfies (4.2). The inequality (4.7) can be further simplified by introducing the adjoint variable $\Gamma^{o}(t)$; $t \geq 0$, which satisfies the (backward) differential equation (4.4). Using (4.2), (4.4) and (4.7) and noting that

$$\int_0^T \frac{d}{dt}\left(\widetilde{\Lambda}(t)\eta, \Gamma^{o}(t)\eta\right) dt = 0, \quad \text{for all } \eta \in R^n,$$

one can easily verify that

$$\int_0^T \mathrm{tr}\left(\overline{Q}(t)\widetilde{\Lambda}(t)\right)dt = 2\int_0^T \mathrm{tr}\left(\Gamma^o(t)\widetilde{A}(t, K-K^o)\Lambda^o(t) + \sum_{i=1}^{N}\Gamma^o(t)D_i'(t, K^o)\Lambda^o(t)\widetilde{D}_i(t)\right)dt. \tag{4.8}$$

The inequality (4.5) now follows from (4.7) and (4.8). This completes the proof. □

In the following section we propose a numerical algorithm, which is based on Theorem 4.2, for determining the optimal parameter K^o along with some numerical simulations to illustrate some of the results of the paper.

5. ALGORITHM AND NUMERICAL SIMULATIONS

In this section we utilize the necessary conditions of optimality obtained in the previous section (see Theorem 4.2) to device an iterative scheme for determining the (optimal) control parameters K_1 and K_2. We also present a worked out example to illustrate the effectiveness of the proposed control scheme.

Algorithm

1. set n=1 and guess the control parameter $K^{(n)} \equiv \left\{K_1^{(n)}, K_2^{(n)}\right\}$.
2. Solve the differential equation (4.3) and get $\Lambda^{(n)}(t) \equiv \Lambda(t, K^{(n)})$; $t \geq 0$.
3. Solve the adjoint equation (4.4) and get $\Gamma^{(n)}(t) \equiv \Gamma(t, K^{(n)})$; $t \geq 0$.
4. Using the inequality (4.5), obtain the gradient vector $g^{(n)}(t) \equiv g(t, K^{(n)})$.
5. If $g^{(n)}(t) \equiv 0$, then $K^{(n)}$ is a local minimizing element.

   If $g^{(n)}(t) \neq 0$, update the parameter $K^{(n)}$ using the following relation
   $$K^{(n+1)}(t) = K^{(n)}(t) + \varepsilon\, g^{(n)}(t);\ t \geq 0,$$
   where $\varepsilon\ (>0)$ is chosen so that $K^{(n+1)} \in \mathcal{K}$ and that $J(K^{(n+1)}) \leq J(K^{(n)})$.
6. If $\left|J(K^{(n+1)}) - J(K^{(n)})\right| \leq \delta$, for some sufficiently small $\delta\ (>0)$, then stop;

   otherwise set n=n+1, and $K^{(n)} \to K^{(n+1)}$ and go to step 2.

Remark 5.1

In the above algorithm we have added another constraint to the control parameter K. This constraint gauarantees that the chosen K is such that $|u(t)| \leq \beta$ for some positive constant β (see equation (2.3)). The example given below shows this fact.

Example

Consider the following (bilinear) stochastic differential equation on the state space R

$$\begin{aligned} dx(t) &= a_1 x(t)dt + b_1 du(t) + c_1 x(t)dw_1(t)\,; \qquad t \geq 0,\ x(0) = x_0, \\ dy(t) &= \left(a_2 x(t) + b_2 y(t)\right)dt + c_2 y(t)dw_2(t)\,; \qquad t \geq 0,\ y(0) = 0, \end{aligned} \tag{5.1}$$

where x and y are the state and output (scalar) variables, w_1 and w_2 are independent standard Wiener processes, a_1, a_2, b_1, b_2, c_1, and c_2 are given constants and x_o denotes the initial state which is assumed to be constant. We wish to design a controller having the following structure

$$du(t) = K_1 y(t)dt + K_2 dy(t);\ t \ge 0, \tag{5.2}$$

where K_1 and K_2 are unknown constants which are chosen so that $|u(t)| \le \beta$ for some constant β, and that

$$J(K) \equiv \lambda\, E\int_0^T (y(t))^2 dt \equiv \min. \tag{5.3}$$

with λ being any arbitrary positive constant. Substituting (5.2) into (5.1) and letting $\xi = (x, y)'$, equation (5.1) can then be written as

$$\begin{aligned} d\xi(t) &= \mathcal{A}(K)\,\xi(t)dt + C\xi(t)dw_1(t) + D(K)dw_2(t), \\ \xi(0) &= \xi_0;\ t \in I, \end{aligned} \tag{5.4}$$

where

$$\mathcal{A}(K) \equiv \begin{pmatrix} a_1 + b_1 a_2 K_2 & b_1(K_1 + b_2 K_2) \\ a_2 & b_2 \end{pmatrix} \tag{5.5}$$

$$C \equiv \begin{pmatrix} c_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad D(K) \equiv \begin{pmatrix} 0 & b_1 c_2 K_2 \\ 0 & c_2 \end{pmatrix}. \tag{5.6}$$

Now the problem is to find a control parameter $K \in \mathcal{K}$ so that the performance index J is minimized subject to the dynamic constraint (3.6), with $\mathcal{A}$, C, and D are as given above and that

$$J(K) \equiv \int_0^T tr\left(\overline{Q}\Lambda(t)\right)dt, \tag{5.7}$$

where

$$\overline{Q} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}.$$

The solution of this (identification) problem is given by the necessary conditions (4.3)-(4.5) (see Theorem 4.2) with

$$\widetilde{\mathcal{A}}(K - K^o) \equiv \begin{pmatrix} b_1 a_2 (K_2 - K_2^o) & b_1[(K_1 - K_1^o) + b_2(K_2 - K_2^o)] \\ 0 & 0 \end{pmatrix}, \tag{5.8}$$

and

$$\widetilde{D}(K-K^{o}) \equiv \begin{pmatrix} 0 & b_1 c_2 (K_2 - K_2^{o}) \\ 0 & 0 \end{pmatrix}. \tag{5.9}$$

Using the inequality (4.5) one can easily verify that the gradient vector $g \equiv (g_1, g_2)'$ is given by

$$\begin{aligned} g_1(t) &= b_1 \Gamma_{11}(t)\Lambda_{12}(t) + b_1\, \Lambda_{22}(t)\Gamma_{12}(t) \\ g_2(t) &= \Gamma_{11}(t)\, b_1\left(\Lambda_{11}(t) a_2 + \Lambda_{12}(t)\, b_2 + \Lambda_{12}(t)\, b_2^2\, c_2^2\, K_2\right) \\ &\quad + \Gamma_{12}(t)\, b_1\left(\Lambda_{12}(t)\, a_2 + \Lambda_{22}(t)\, b_2 + \Lambda_{22}(t) c_2^2\right). \end{aligned} \tag{5.10}$$

Note that the gradients g_1 and g_2 given above have been used in step 5 of the algoritm for updating the control parameters K_1 and K_2.

For numerical simulations, we have taken $a_1 = b_1 = c_2 = 0.2$, $a_2 = b_2 = 0.5$, and $c_2 = 1$. Using the above algorithm, we have determined the optimal (control) parameters K_1 and K_2, the control process u (t); $t \geq 0$, and the corresponding state and output processes. Table (I) shows the values of K_1 and K_2 as a function of β, whereas Fig. (5) shows the control u for different β. From these results it is clear that for each β, the choice of control parameters K_1 and K_2 satisfies the fact that $|u(t)| \leq \beta$. Figs.(2-4) show the processes x, y, and Λ_{ij}; $i, j = 1,2$, for different β's. From these figs. one observes that the proposed controller is capable of regulating the state and output processes about the origin. Further, it is also clear that as we relax the control constraints (i.e. increasing β) regulation becomes faster and viceversa. It should be noted also that there is a range for β beyond which the system may become unstable or the controller fails to regulate the system. In this example, this range is found to be $15 < \beta < 50$.

<u>Table (I)</u>

β	K_1	K_2
15	-1.4	-1.93
22	-0.26×10^3	-0.19×10^3
25	-0.50×10^3	-0.35×10^3
30	-0.60×10^3	-0.27×10^3
35	-0.65×10^3	-0.22×10^3
45	-0.73×10^3	-0.15×10^3

6. CONCLUSIONS

In this paper we have considered the optimal control problem for a class of partially observed bilinear stochastic systems with control constraints. Assuming linear control structure, driven by the output process, we have converted the original stochastic control problem into an equivalent deterministic identification problem in which the controller parameters are the unknowns. Further, using variational arguments and the Gateaux differentiability of the process Λ on the parameter set, we have obtained the corresponding necessary conditions for optimal identification.Based on these necessary conditions, we have presented an iterative scheme for computing the optimal parameters (and hence optimal control) along with some numerical simulations to illustrate the effectiveness of the proposed control scheme. The results showed that the proposed controller is capable of regulating the state and output processes about the origin effectively. Further, as it has been indicated in figs. (1-4), regulation of the state and output process have improved as we relaxed control constraints (i.e., increasing β).

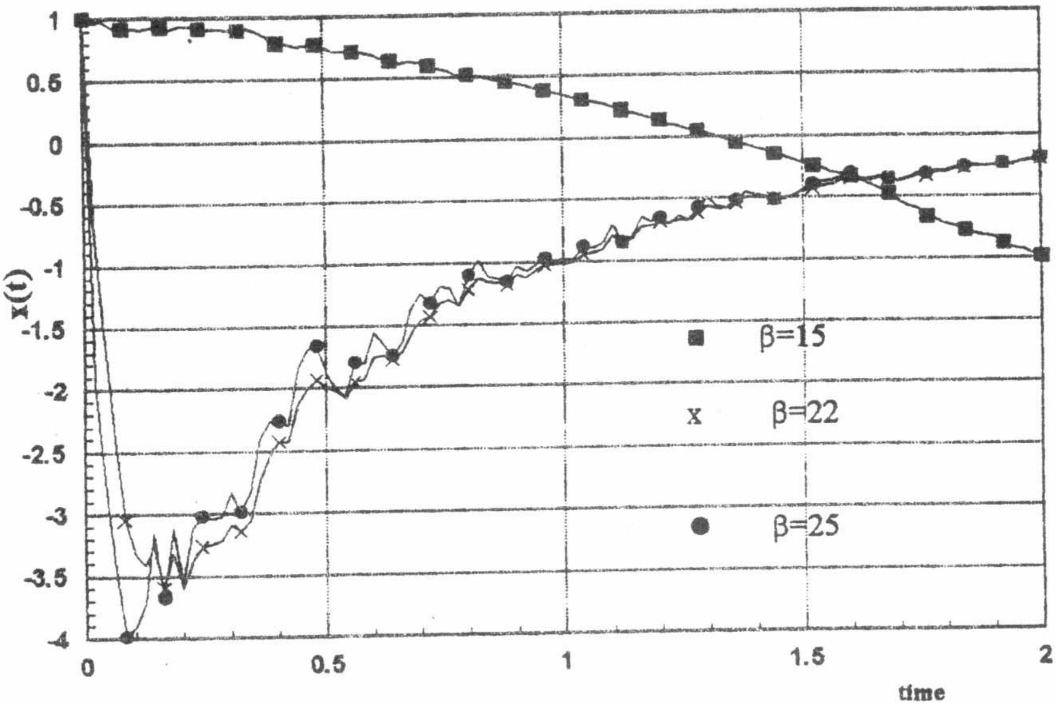


Fig.1-(a) state process for different β

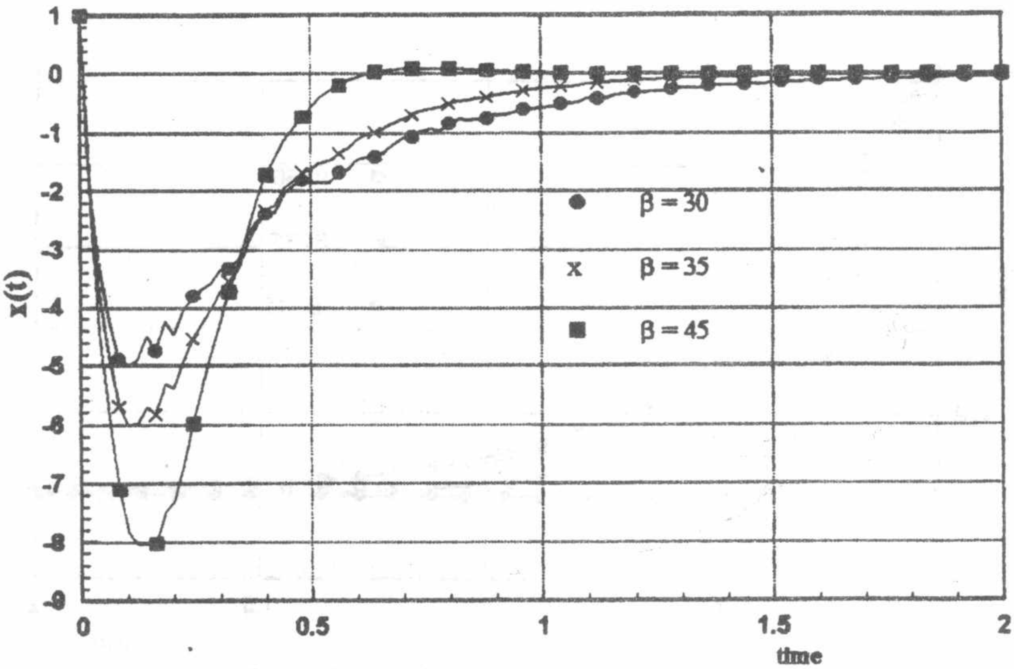


Fig.1-(b) State process for different β

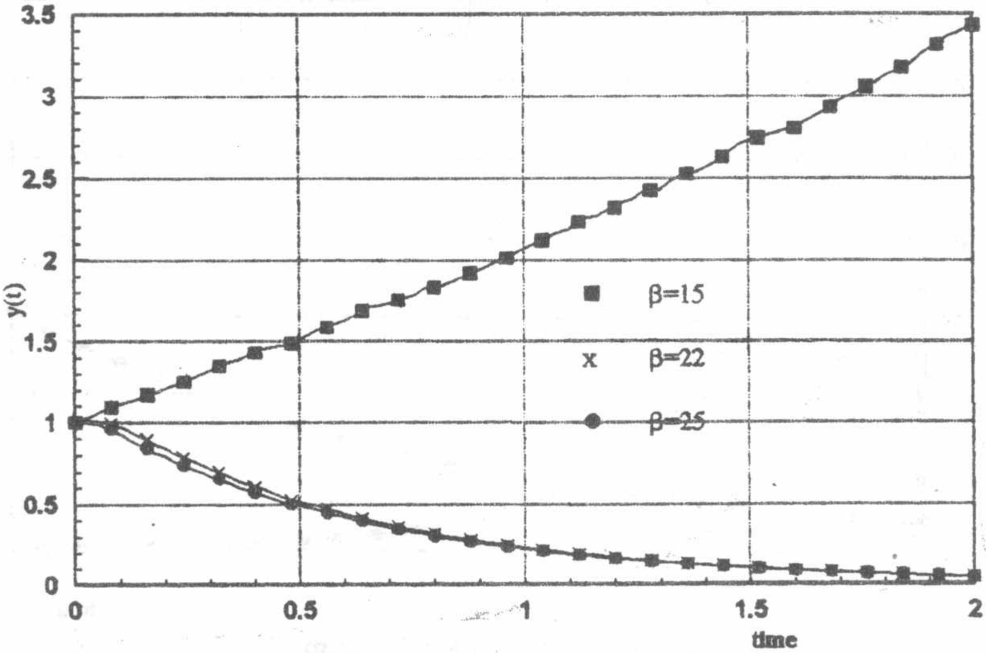


Fig.2-(a) observation process for different β

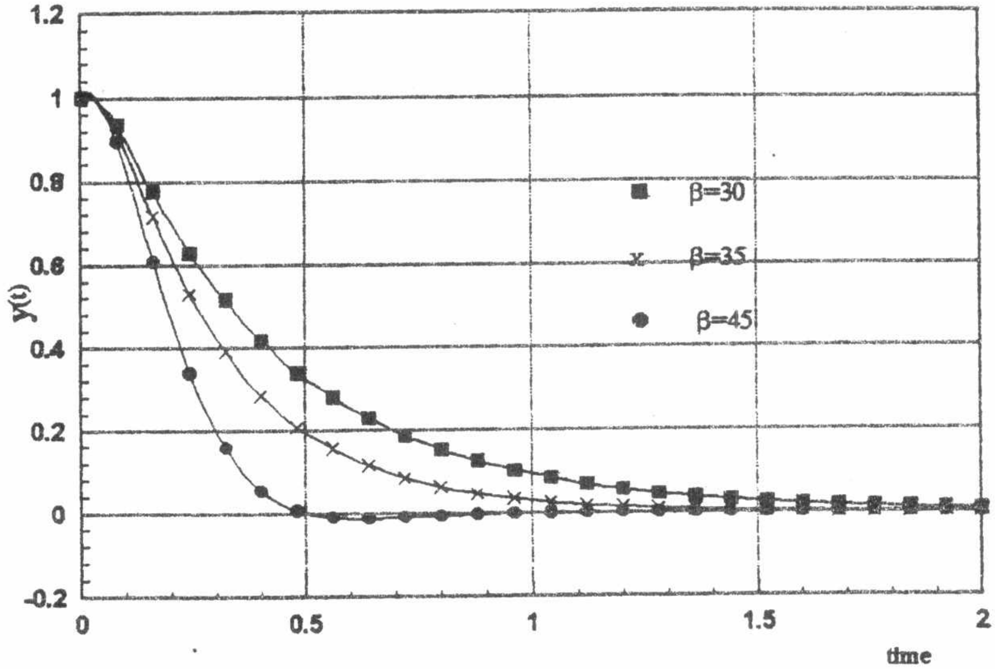
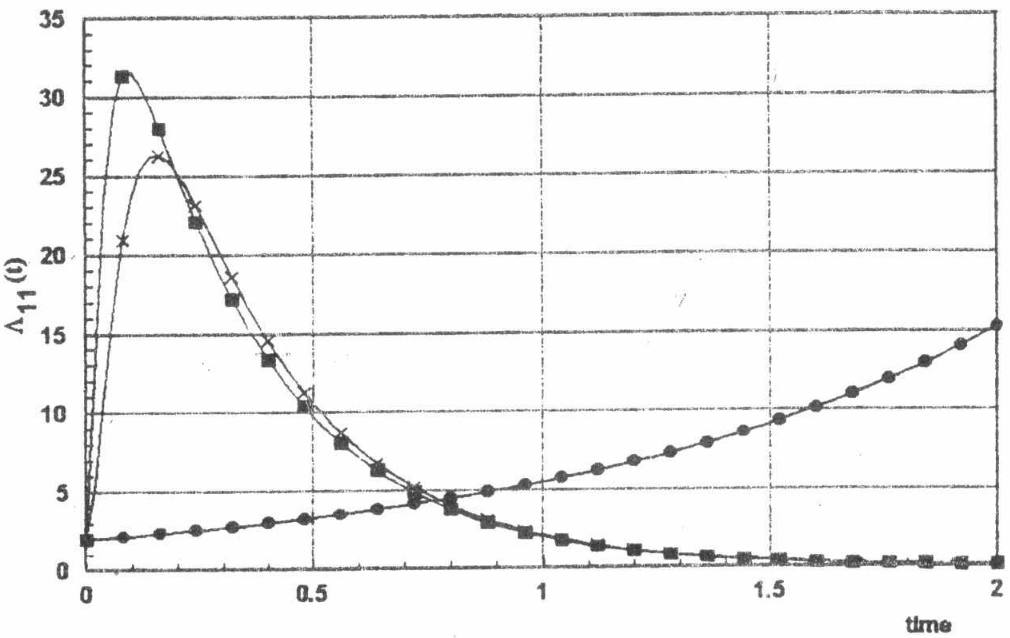


Fig.2-(b) Observation process for different β



F ig.3-(a) The process $\Lambda_{11}(t)$ for different β

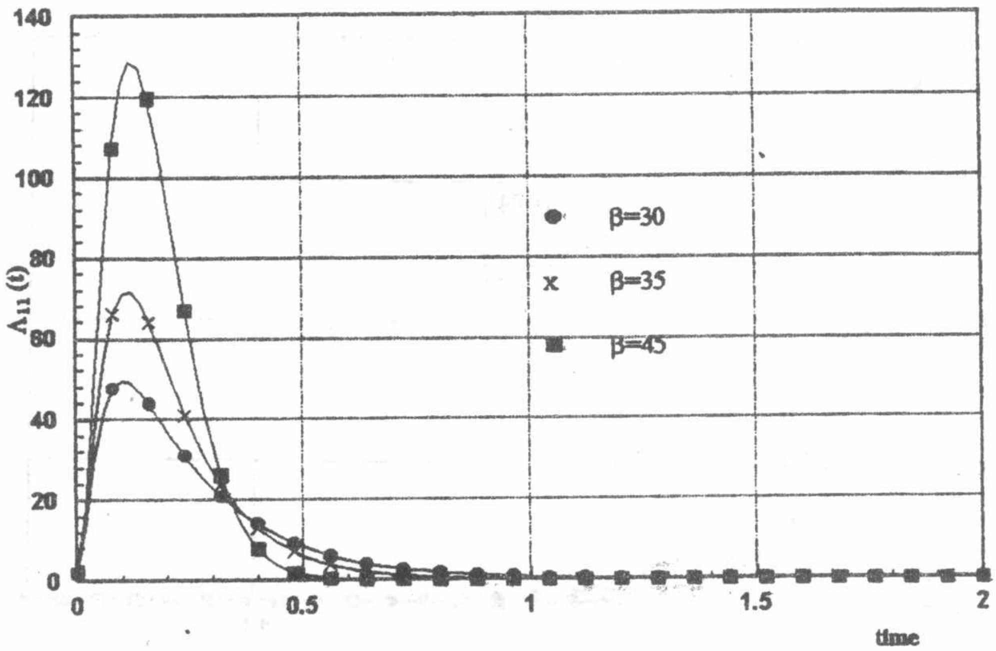


Fig.3-(b) The process $\Lambda_{11}(t)$ for different β

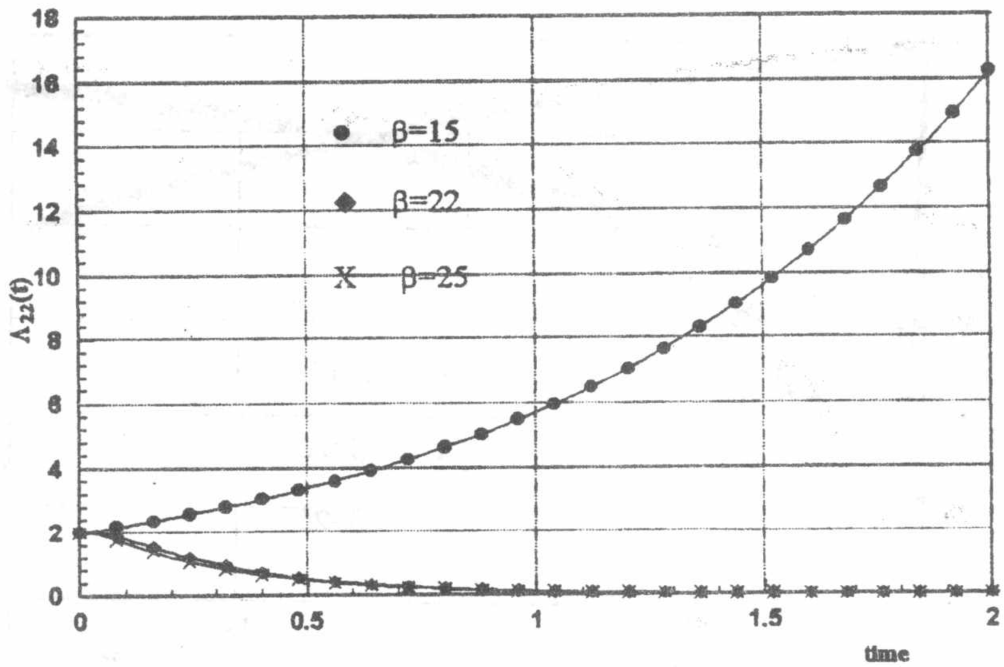


Fig.4-(a) The process Λ_{22} (t) for different β

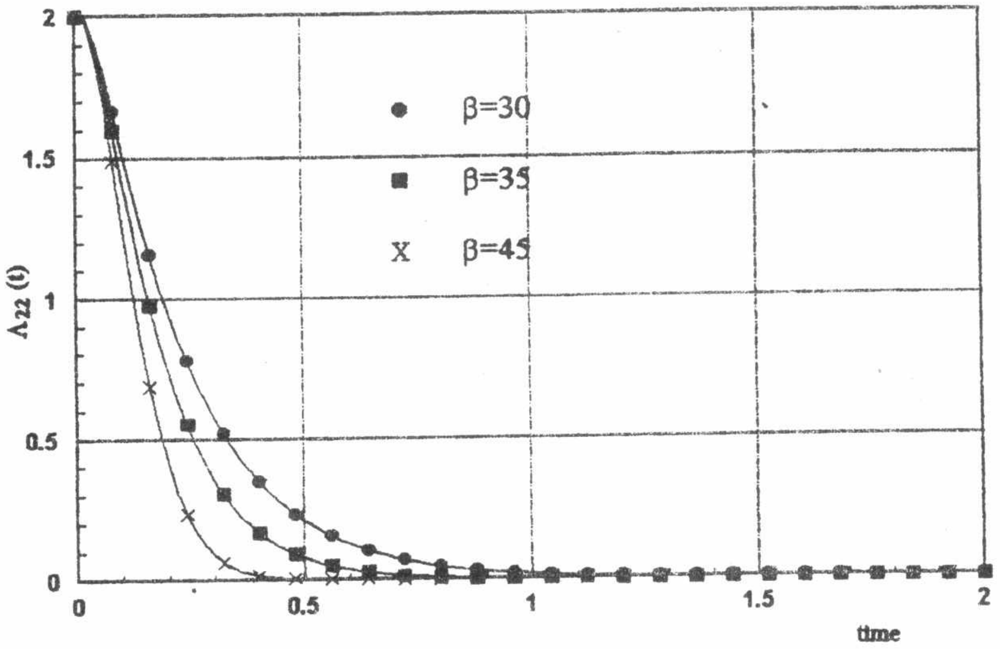


Fig.4-(b) The process Λ_{22} (t) for different β

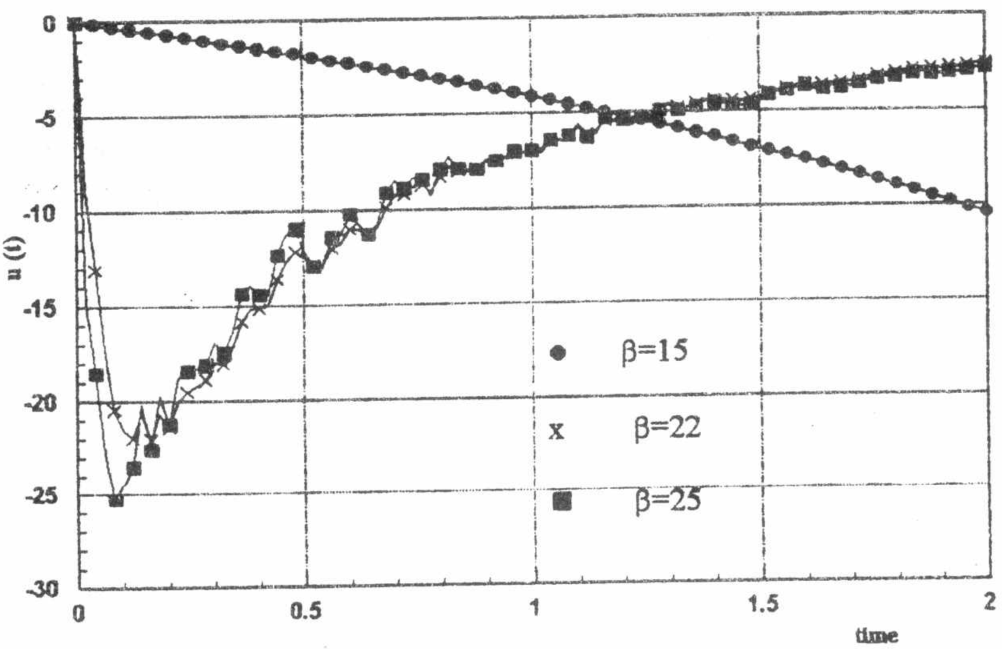


Fig.5-(a) The control process for different β

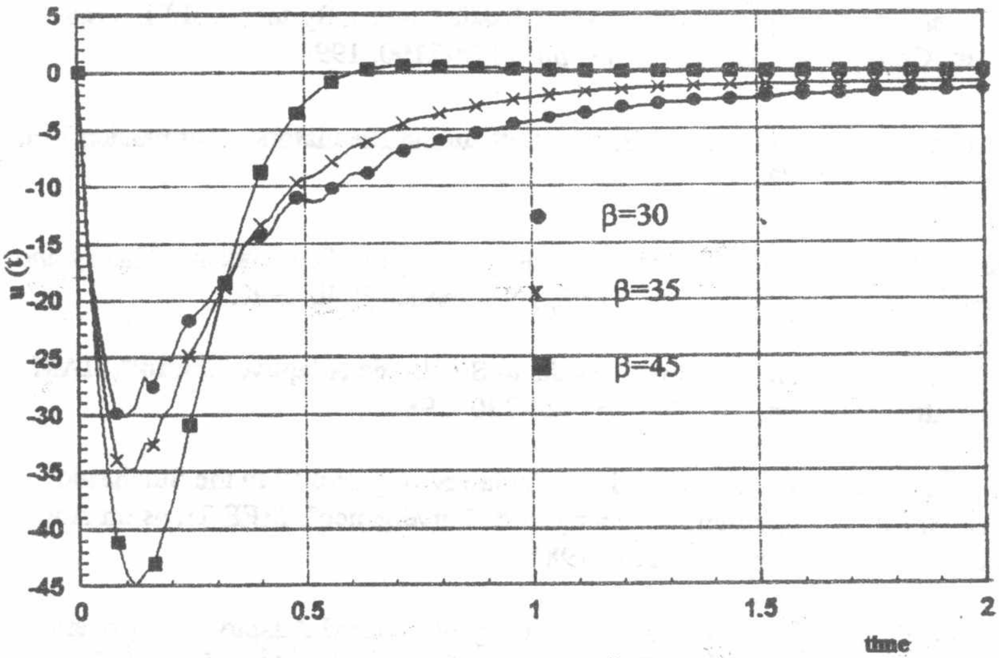


Fig.5-(b) The control process for different β

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