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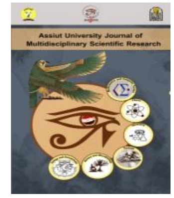

Approximation of analytic functions of exponential derived and integral bases in Fréchet spaces

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ABSTRACT

The purpose of this paper is to establish some theorems on the representation of analytic functions by exponential derived bases ($EDBs$) and exponential integral bases ($EIBs$) in Fréchet spaces. Theorems are proven to show such that representation is possible in closed disks, open disks, open regions surrounding closed disks, at the origin and for all entire functions. Also, some results concerning the growth order and type of EDBs and EIBs are determined. Moreover, the $T\rho$-property of $EDBs$ and $EIBs$ are discussed. Finally, some applications to the $EDBs$ and $EIBs$ of Bernoulli, Euler, Bessel, and Chebyshev polynomials have been studied.

1 Introduction

The theory of basic sets (bases) of polynomials (BPs) has a significant role in mathematics and its applications, e.g., approximation theory, mathematical physics, Geometry, and partial differential equations. The interest of the present work focused on the expansion of analytic functions into BPs. Given a sequence of a base of polynomials $\{P_n(z)\}$. The expansion of an analytic function $f(z)$ as a basic series $\sum_n a_n\ P_n(z)$ began with the papers by Whittaker and Cannon [14, 15, 30, 31] about 90 years ago. Basic series generalize Taylor series, where $P_n(z)$ can be Legendre, Laguerre, Chebyshev, Hermite, Bessel, Bernoulli and Euler polynomials (see [1, 4, 5, 10, 11, 21]). This theory found a lot of applications, mainly in the theory of functions depending on one or several complex variables as well as in the approximation of solutions of differential equations or matrix functions.

The topic of derivative BPs in one complex variable has been studied early (see [26, 27, 28]), the searchers considered the disks in the complex plane C. For several complex variables (see [16, 17, 19, 24, 25]), the representation domains are polycyclinderical, hyperspherical and hyperelliptical regions. Recently, in [12, 35] the authors investigated this problem in Clifford setting which is called hypercomplex derivative bases of special monogenic polynomials, where the representation in closed balls. In [9] Adepoju developed the concept of BPs of a single complex variable in the Banach space, which depends on some basic concepts in functional analysis. Also, the authors in [16, 17] studied the BPs in higher dimensions in Banach space. Hassan et al [22] introduced a study of the idea of BPs based primarily on Clifford analysis and functional analysis. Their study constructed a criterion, of general type, for effectiveness (convergence properties) of BPs in Fréchet modules. They gave some applications of the convergence properties of BPs in approximation theory concerned with the approximation of special monogenic functions by an infinite series in a sequence of special monogenic polynomials in closed and open ball.

In the current work, we define two new bases are called EDBs and EIBs. We investigate the effectiveness and the growth order and type of EDBs and EIBs in Fréchet spaces in several domains:
closed disks $\overline{D}(R)$, open disks $D(R)$, open regions surrounding closed disks $D_{+}(R)$, at the origin and for all entire functions. Furthermore, we will give some applications on the EDBs and EIBs of Bernoulli, Euler, Bessel, and Chebyshev polynomials.

2 Preliminaries

In this section, we recall several definitions, notations and results which will be essential in this study (see [9, 13, 16, 17, 22, 28]).

Definition 2.1. *A seminorm $\|.\|$ is a function from a vector space X to the real numbers R satisfying*

(a) $\|f\| \geq 0 \; \forall f \in X,$
(b) $\|f+g\| \leq \|f\| + \|g\| \; \forall f, g \in X,$
(c) $\|a f\| = |a| \, \|f\| \; \forall f \in X \; and \; a \in C,$
(d) $\|f\| = 0 \; \Rightarrow f = 0.$

Seminorms are essential to define Fréchet spaces as follows:

Definition 2.2. *An F-space E over* C *satisfies the following three properties*

(i) *E is a Hausdorff space,*

(ii) *E is topology may be induced by a countable family of seminorms* $P = (\|.\|_k)_k \geq 0: k < l \Rightarrow \| g \|_k \leq \| g \|_l; (g \in E)$. *This means that* $V \subset E$ *is open if and only if* $\forall g \in V$ *: there exists* $\epsilon > 0, N \geq 0$ *such that* $\{f \in E : \| g - f \|_k \leq \epsilon\} \subset V, \forall k \leq N.$

(iii) *E is complete with respect to the family of semi-norms.*

Definition 2.3. *A sequence* $(g_k)_k \geq 0$ *in an F-space E converges to f in E if and only if, for all* $\|.\|_k \in P$*, we have* $\lim_{n\to\infty} \| g_n - f \|_k = 0.$

Remark 2.1. *Suppose $H[S]$ denote the space containing analytic functions in a region S where S stands for $D(R), \overline{D}(R)$ or $D_+(R)$ as indicated. Then the space $H[D(R)]$ is an F-space using the family of semi-norms defined as*

$$\|g\|_r = sup_{\overline{D}(r)}|g(z)|, \forall r < R,\ g \in H[D(R)].$$

The space $H[\overline{D}(R)]$ with the seminorm

$$\|g\|_R = sup_{\overline{D}(R)}|g(z)|,\ \ g \in H[\overline{D}(R)].$$

is an F-space. The family of semi-norms

$$\|g\|_r = sup_{\overline{D}(r)}|g(z)|, \forall R < r,\ g \in H[D_+(R)].$$

define $\mathrm{H}[D_+(R)]$ *as an F-space. Now, let* $\mathrm{H}[\infty]$ *be the space of entire functions on the complex plane. By defining the seminorms family in* $\mathrm{H}[\infty]$ *as follows*

$$\|g\|_n = sup_{\overline{D}(n)}|g(z)|,\ \ g \in H[\infty], n < \infty.$$

then $\mathrm{H}[\infty]$ *is an F-space. Let* $\mathrm{H}[0^+]$ *be the space of analytic functions at the origin. The family of semi-norms*

$$\|g\|_\epsilon = sup_{\overline{D}(\epsilon)}|g(z)|, \epsilon > 0\ \forall g \in H[0^+]$$

makes $\mathrm{H}[0^+]$ *into an F-space where $\overline{D}(\epsilon)$ is a some disk surrounding* 0.

Definition 2.4. *A sequence $\{P_n(z)\}$ of an F-space E is said to form base if the z^n admits a unique representation of the form*

$$z^n = \sum_{k=0}^{\infty} \pi_{n,k} P_k(z). \tag{2.1}$$

The matrix $\Pi = (\pi_{n,k})$ is called the matrix of operators of the base $\{P_n(z)\}$. The base $P_n(z)$ will always be written as

$$P_n(z) = \sum_{k=0}^{\infty} p_{n,k} z^k. \tag{2.2}$$

The matrix $P = (p_{n,k})$ is called the matrix of coefficients of the base $\{P_n(z)\}$. Thus according to ([22]) the set $\{P_n(z)\}$ will be base if and only if

$$P\Pi = \Pi P = I. \tag{2.3}$$

Let $g(z) = \sum_{n=0}^{\infty} a_n(g)\, z^n$ be any element of an F-space E, substituting for z^n from (2.1) we obtain the basic series

$$g(z) \sim \sum_{n=0}^{\infty} \Pi_n(g)\, P_n\,, \tag{2.4}$$

where

$$\Pi_n(g) = \sum_{k=0}^{\infty} a_k(g)\, \pi_{k,n}. \tag{2.5}$$

Definition 2.5. *A base $\{P_n(z)\}$ is effective for an F-space E if the basic series (2.4) converges uniformly to every element $g \in E$.*

From Definition 2.5, we take the F-space E to be the space $H_{[\bar{D}(R)]}$. The base $\{P_n(z)\}$ will be effective for $H_{[\bar{D}(R)]}$ if the basic series converges uniformly to every analytic function $g \in H_{[\bar{D}(R)]}$ which is analytic in $\bar{D}(R)$. Similar definitions are used for the following spaces $H_{[D(R)]}, H_{[D_+(R)]}, H_{[\infty]}$ *and* $H_{[0+]}$. Theorems concerning the effectiveness of bases are due to [5, 9, 22, 28]).

Write:

$$\|P_n\|_R = \sup_{\bar{D}(R)} |P_n(z)|, \tag{2.6}$$

$$\omega_{P_n}(R) = \sum_k |\pi_{n,k}|\, \|P_k\|_R\,, \tag{2.7}$$

$$\lambda_P(R) = \lim sup_{n\to\infty} \{\omega_{P_n}(R)\}^{\frac{1}{n}}. \tag{2.8}$$

Theorem 2.1. *A necessary and sufficient condition for a base $\{P_n(z)\}$ to be effective for $H_{[\bar{D}(R)]}, H_{[D(R)]}, H_{[D_+(R)]}, H_{[\infty]}$ or $H_{[0]}$ is that $\lambda_P(R) = R, \lambda_P(r) < R\ \forall r < R, \lambda_P(R^+) = R, \lambda_P(R) < \infty\ \forall R < \infty$ or $\lambda_P(0^+) = 0, respectively$.*

Cauchy's inequality for the base in (2.2) is given by (see [22])

$$|P_{n,i}| \le \frac{\|P_n\|_R}{R^i}\,, \tag{2.9}$$

Where

$$\|P_n\|_R = sup_{\bar{D}(R)} |P_n(z)|.$$

Definition 2.6. *When the base $\{P_n(z)\}$ is polynomials then representation (2.1) is finite. If $N(n)$, the number of non-zero terms in (2.1) is such that*

$$\{N(n)\}^{\frac{1}{n}} \to 1, as\ n \to \infty \tag{2.10}$$

the base $\{P_n(z)\}$ is called Cannon base of polynomials (see [22]).

When $\{N(n)\}^{\frac{1}{n}} \to\ a\ >\ 1$, as $n \to \infty$, the set $\{P_n(z)\}$ is said to be general base.

Definition 2.7. *A base $\{P_n(z)\}$ of polynomials in which the polynomial $P_n(z)$ is of degree n is called simple base.*

Definition 2.8. *The order and type of a base $\{P_n(z)\}$ introduced in [30, 31]) by*

$$\rho_P = \lim_{R\to\infty} \limsup_{n\to\infty} \frac{\log \omega_{P_n}(R)}{n\ \log n}. \tag{2.11}$$

and

$$\tau_P = \lim_{R\to\infty} \frac{e}{\rho} \limsup_{n\to\infty} \frac{\{\omega_{P_n}(R)\}^{\frac{1}{n\,\rho}}}{n}. \tag{2.12}$$

The importance of order and type lies in that if the BPs $\{P_n(z)\}$ has finite order ω and finite type τ, it represent every entire function of order less than $\frac{1}{\omega}$ and type less than $\frac{1}{\tau}$ in any finite disk (c.f. [20, 21, 30]). Related results to order and type of the BPs can be found in [2, 33, 34].

The definition of the Tρ-property of bases of polynomials of one complex variable in disks was first proposed by Eweida [18]. In addition, the study of the Tρ-property of bases of polynomials of several complex variables in complete Reinhardt domains (polycylinderical regions) was introduced by Kishka et al. [23]. Moreover, the definition of the Tρ-property of bases of polynomials in Clifford analysis in balls was introduced by Abul-Ez and Constales [7].

We recall the definition of the Tρ-property as provided by Eweida [18] as follows:

Definition 2.9. If 0 < ρ < ∞, then a base is said to have property T_ρ in a closed disk $\overline{D}(R)$ open disk $D(R)$ or at the origin, if it represents all entire functions of order less than ρ in $\overline{D}(R)$, D(R) or at the origin.

Let

$$\omega_P(R) = \limsup_{n\to\infty} \frac{\log \omega_{P_n}(R)}{n\ \log n}.$$

Concerning the necessary and sufficient condition for the base of polynomials $\{P_n(z)\}$ to have the property T_ρ in a closed disk $\overline{D}(R)$, open disk $D(R)$ or at the origin (see [18]), we have the following result:

Theorem 2.2. *Let $\{P_n(z)\}$ be a base of polynomials and suppose that the function $f(z)$ is an entire function of order less than ρ. Then the necessary and sufficient condition for the base $\{P_n(z)\}$ to have property T_ρ in the closed disk $\overline{D}(R)$, open disk $D(R)$, or at the origin $\omega(P,R) \le \frac{1}{\rho}$, $\omega(P,r) \le \frac{1}{\rho}$, $\forall\, r < R$ or $\omega(P\,,0^+) \le \frac{1}{\rho}$.*

For more information about the study of ***BP*** *in Complex and Clifford analysis, we refer to*

([6, 8, 24, 29, 34]).

3 Exponential derived and integral bases

Definition 3.1. *The exponential derived $exp(D)$ and the exponential integral $exp(I)$ acting on the monomial z^n are defined by:*

$$exp(D)z^n = e^n z^n, \tag{3.1}$$

$$exp(I)z^n = e^{\frac{1}{n+1}} z^n. \tag{3.2}$$

Where

$$D = z\frac{d}{dz}\,, D^n = D^{n-1}D \text{ and } I = \frac{1}{z}\int_0^z dz,\ I^n = I^{n-1}I$$

Definition 3.2. *Let $\{P_n(z)\}$ be a base. By applying $exp(D)$ and $exp(I)$ into (2.2), we get*

$$\exp(D)\,P_n(z) = \sum_k p_{n,k} e^k z^k, \tag{3.3}$$

$$\exp(I)\,P_n(z) = \sum_k p_{n,k} e^{\frac{1}{k+1}} z^k. \tag{3.4}$$

The set $\{\exp(D)\,P_n(z)\} = \{E_n^D(z)\}$ is called exponential derived base of polynomials (EDBPs) and the set $\{\exp(I)\,P_n(z)\} = \{E_n^I(z)\}$ is called exponential integral base of polynomials (EIBPs).

In this paper, we thoroughly discuss the following important questions:

1. If $\{P_n(z)\}$ is a base, and we apply the exponential derived $exp(D)$ or exponential integral $exp(I)$ does the resulting sets $\{E_n^D(z)\}$ *or* $\{E_n^I(z)\}$ define a base?
2. If the base $\{P_n(z)\}$ is effective for the spaces $H_{[\overline{D}(R)]}$, $H_{[D(R)]}$, $H_{[D_+(R)]}$, $H_{[\infty]}$ *or* $H_{[0^+]}$, is the base $\{E_n^D(z)\}$ *or* $\{E_n^I(z)\}$ also effective in the same spaces?.

3. How related is the growth rate of the base $\{P_n(z)\}$ of and the growth rate of $\{E_n^D(z)\}$ or $\{E_n^I(z)\}$?

4.If the base $\{P_n(z)\}$ has T_ρ-property, does the base$\{E_n^D(z)\}$ or $\{E_n^I(z)\}$ posses the same property?.

4 Effectiveness of the EDBPs and EIBPs

Before we start characterizing the effectiveness properties, we need to prove that the constructed set EDBs or EIBs is indeed a base.

Theorem 4.1. *Let* $\{P_n(z)\}$ *be a base. Then the set* $\{E_n^D(z)\}$ *is base.*

Proof. We construct the matrix of coefficient E^D, then by apply the $exp(D)$ into (2.2), it follows that

$$exp(D)P_n(z) = \textstyle\sum_k P_{n,k}\, e^k z^k.$$

Thus, the matrix of coefficients E^D of this base is given by:

$$E^D = (E_{n,k}^D) = (e^k\, P_{n,k}).$$

Now, the matrix of operator Π^D follows from the representation:

$$z^n = \frac{1}{e^n}\sum_k \pi_{n,k} E_n^D(z)$$

which means that

$$\Pi^D = (\pi_{n,k}^D) = \left(\tfrac{1}{e^n}\, \pi_{n,k}\right).$$

Consequently,

$$E^D\Pi^D = \left(\sum_k E_{n,k}^D \pi_{k,h}^D\right) = \left(\sum_k p_{n,k}\pi_{k,h}\right) = (\delta_{n,h}) = I$$

Moreover,

$$\Pi^D E^D = \left(\sum_k \pi_{n,k}^D E_{k,h}^D\right) = \left(\frac{e^h}{e^n}\delta_{n,h}\right) = I.$$

According to the base property (2.3), it follows that the set $\{E_n^D(z)\}$ is indeed a base. Also, we may proceed very similar as above to prove that the set $\{E_n^I(z)\}$ is a base and the theorem is therefore established. □

Theorem 4.2. *Let* $\{P_n(z)\}$ *be a BPs satisfying the condition:*

$$\lim_{n\to\infty}\frac{D_n}{n}=1, \tag{4.1}$$

Where D_n is the degree of the polynomial of highest degree in the representation (2.1). If the BP $\{P_n(z)\}$ is effective for $H[\overline{D}(R)]$, then the EDBs $\{E_n^D\ (z)\}$, also, is effective for $H[\overline{D}(R)]$.

Proof. If $P_n(z)$ is a base, $\|P_n\|_R=sup_{\overline{D}(R)}|P_n(z)|$ and $\|E_n^D\|_R=sup_{\overline{D}(R)}|E_n^D(z)|$, then

$$\begin{aligned}\|E_n^D\|_R &= sup_{\overline{D}(R)}|E_n^D(z)|\\ &= sup_{\overline{D}(R)}\left|\textstyle\sum_j p_{n,j}e^j z^j\right|\\ &\le \|P_n\|_R\textstyle\sum_j e^j\\ &= \|P_n\|_R e^{d_n}(d_n+1)\end{aligned} \tag{4.2}$$

where d_k is the degree of the polynomial $P_k(z)$, $d_k \le D_n$ by the definition. Owing to (2.7) and (4.2), we obtain

$$\begin{aligned}\omega_{E_n^D}(R) &= \textstyle\sum_k\left|\pi_{n,k}^D\right|\ \|E_k^D\|_R\\ &\le \frac{1}{e^n}\textstyle\sum_k\left|\pi_{n,k}\right|\ \|P_k\|_R\ e^{d_k}(d_k+1)\\ &\le \frac{1}{e^n}e^{d_n}(d_n+1)\omega_{P_n}(R)\\ &\le e^{D_n-n}(D_n+1)\omega_{P_n}(R).\end{aligned} \tag{4.3}$$

Using (2.8) and (4.3) and applying condition (4.1), it follows that $\omega_{E^D}(\mathrm{R}) \le \omega_{\mathrm{P}}(\mathrm{R}) \le \mathrm{R}$. But $\omega_{E^D}(R) \ge R$, then

$$\omega_{E^D}(R) = R. \tag{4.4}$$

Hence, applying (4.4) and using Theorem 2.1, we conclude that the effectiveness of BPs $\{P_n(z)\}$ for $H[\overline{D}(R)]$ implies the effectiveness of the EDBPs $\{E_n^D\ (z)\}$ for $H[\overline{D}(R)]$. □

The absence of condition (4.1) implies that Theorem 4.2 is not always true. We illustrate this fact with the following example.

Example 4.1. *Consider the BPs* $\{P_n(z)\}$ *defined by*

$$P_n(z)=\begin{cases}z^n, & n \text{ is even}\\ z^n+z^b, & b=2n, n \text{ is odd}.\end{cases}$$

In this base $z^n = P_n(z)$ when n is even. Hence $\omega_{P_n}(R) = R^n$. Thus taking R = 1, $\omega_{P_n}(1) = 1$. and $\lim sup_{n\to\infty}\{\omega_{P_{2n}}(1)\}^{\frac{1}{2n}} = 1$.

Furthermore, $z^n = P_n(z) - P_b(z)$, (n is odd), then $\omega_{P_{2n}}(R) = R^n + 2R^b$.

Taking $R = 1$, $\omega_{P_n}(1) = 3$, *we obtain*

$$\lim\sup_{n\to\infty}\left\{\omega_{P_{2n+1}}(1)\right\}^{\frac{1}{2n+1}} = 1.$$

Consequently $\lambda_P = \lim sup_{n\to\infty}\{\omega_{P_n}(1)\}^{\frac{1}{n}} = 1$, *which implies that the base* $\{P_n(z)\}$ *is effective for* $H_{[\bar{D}(1)]}$.

Now, construct the EDBs $\{E_n^D(z)\}$ *as follows:*

$$E_n^D(z) = \begin{cases} e^n z^n, & n \text{ is even} \\ e^n z^n + e^b z^b, & b = 2n, n \text{ is odd}. \end{cases}$$

Since $z^n = (\frac{1}{e^n})\, E_n^D(z)$, *when* n *is even, then* $\omega_{E_n^D}(R) = R^n$, *taking* $R = 1$, $\omega_{E_n^D}(1) = 1$.

Hence,

$$\lim\sup_{n\to\infty}\{\omega_{E_n^D}(R)\}^{\frac{1}{n}} = 1.$$

Moreover, when n is odd, we have $z^n = (\frac{1}{e^n})[E_n^D(z) - E_b^D n(z)]$. *Thus it follows that,*

$$\omega_{E_n^D}(R) = R^n + 2e^{b-n}R^n .$$

Consider $R = 1$, *then we obtain*

$$\omega_{E_n^D}(1) = 1 + 2e^{b-n}.$$

$$\lambda_{E^D}(1) = \lim\sup_{n\to\infty}\left\{\omega_{E_{2n+1}^D}(1)\right\}^{\frac{1}{2n+1}} = e > 1.$$

In this case, it follows that the EDBs $\{E_n^D(z)\}$ *is not effective for* $H_{[\bar{D}(1)]}$, *although the original BPs* $\{P_n(z)\}$ *is effective for* $H_{[\bar{D}(1)]}$.

Before we conclude this section, we show the validity and applicability of the obtained results on some special bases of polynomials. Consider the simple bases

$\{P_n(z)\}$ of the Bessel polynomials and $\{Q_n(z)\}$ of the general Bessel polynomials; given respectively by

$$\mathbb{P}_0(z) = 1, \quad \mathbb{P}_n(z) = \sum_{k=0}^{n} \frac{(n+k)!}{k!(n-k)!} (\frac{z}{2})^k; \quad (n \geq 1),$$

$$\mathbb{Q}_0(z) = 1, \quad \mathbb{Q}_n(z) = 1 + \sum_{k=0}^{n} \frac{n!(n+b-1)(n+b)...(n+k+b-2)}{k!(n-k)!} (\frac{z}{a})^k;$$

where a and $b \neq 0$ are given numbers. Recently the authors [1, 4] proved that both the bases $\{P_n(z)\}$ and $\{Q_n(z)\}$ are effective for $H_{[\bar{D}(R)]}$.

Consider the base of Chebychv polynomials $\{T_n(z)\}$ given by

$$\mathbb{T}_0(z) = 1, \quad \mathbb{T}_n(z) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(n)!}{(2k)!(n-2k)!} \; z^{n-2k} \; (z^2-1)^k; \quad (n \geq 1),$$

In a recent paper [5] the authors proved that the Chebyshev polynomial $\{T_n(z)\}$ is effective for $H_{[\bar{D}(R)]}$. As immediate consequences of Theorem 4.2, we have the following corollaries:

Corollary 4.1. *The EDB of Bessel polynomial* $\{E^D P_n(z)\}$ *is effective for* $H_{[\bar{D}(R)]}$.

Corollary 4.2. *The EDB of general Bessel polynomial* $\{E^D Q_n(z)\}$ *is effective for* $H_{[\bar{D}(R)]}$.

Corollary 4.3. *The EDB of Chebyshev polynomial* $\{E^D T_n(z)\}$*is effective for* $H_{[\bar{D}(1)]}$.

Now, we can proceed very similar as in 4.2 to prove the following results

Theorem 4.3. *If the base* $\{P_n(z)\}$ *is effective for* $H[D(R)]$ *and satisfying the condition (4.1), then so is the EDBs* $\{E_n^D(z)\}$.

Theorem 4.4. *If the base* $\{P_n(z)\}$ *is effectiveness for* $H[0]$ *or* $H[\infty]$ *and satisfying the condition (4.1), then EDBs* $\{E_n^D(z)\}$ *is effective in the corresponding space.*

Theorem 4.5. *If the base* $\{P_n(z)\}$ *is effectiveness for* $H[D_+(R)]$ *and satisfying the condition (4.1), then so is the EDBs* $\{E_n^D(z)\}$.

It is worthy to ensure that Theorems 4.1, 4.2, 4.3, 4.4 and 4.5 will be still true when we replace the base $\{E_n^D(z)\}$ by the base $\{E_n^I(z)\}$.

5 Order, Type and Tρ-Property of the EDBPs

This section creates a relation between the orders and types of the original base $\{P_n(z)\}$ and the
EDBPs $\{E_n^D(z)\}$. For this purpose, let ρ_P, τ_P and ρ_{E^D}, τ_{E^D} denote the orders and types of the bases $\{P_n(z)\}$ and $\{E_n^D(z)\}$, respectively.

Theorem 5.1. *If the BPs* $\{P_n(z)\}$ *is of order* ρ_P *and type* τ_P *and satisfying the condition:*

$$D_n = O[n], \tag{5.1}$$

then the base $\{E_n^D(z)\}$ *will be of order* $\rho_{E^D} \leq \rho_P$ *and type* $\tau_{E^D} \leq \tau_P$ *whenever* $\rho_{E^D} = \rho_P$ *. Moreover, the two upper bounds are attainable.*

Proof. In the beginning, we shall prove that the order and type of the EDBPs is at most ρ_P and τ_P. Since

$$\omega_{E_n^D}(R) \leq e^{D_n - n}(D_n + 1)\, \omega_{P_n(R)}.$$

Then

$$\lim_{R\to\infty}\lim_{n\to\infty}\sup \frac{\log \omega_{E_n^D}(R)}{n \log n} \leq \lim_{R\to\infty}\lim_{n\to\infty}\sup \frac{\log e^{D_n - n}(D_n + 1) + \log \omega_{P_n}(R)}{n \log n}$$

Using the definition of order, then the order of the EDBPs is at most ρ_P.
Now, suppose that $\rho_{E^D} = \rho_P$. It follows that

$$\lim_{R\to\infty}\frac{e}{\rho_{E^D}}\lim_{n\to\infty} sup \frac{\{\omega_{E_n^D}(R)^{\overline{\frac{1}{n(\rho_{E^D})}}}\}}{n} \leq \lim_{R\to\infty}\frac{e}{\rho_P}\lim_{n\to\infty} sup \frac{\{\omega_{p_n}(R)\}^{\overline{\frac{1}{n(\rho_P)}}}\}}{n}$$

Therefore, the type of the EDBPs is at most τ_P . □

Now, we show that the two bases $\{P_n(z)\}$ and $\{E_n^D(z)\}$ may be of the same order and type by the following example:

Example 5.1. *Let* $\{P_n(z)\}$ *be BPs given by* $P_n(z) = n^n + z^n$, $P_0(z) = 1$.

Then, we have

$$\omega_{P_n}(R) = n^n \left[2 + \left(\frac{R}{n}\right)^n\right],$$

and the base $\{P_n(z)\}$ *is of order* $\rho_P = 1$ *and type* $\tau_P = e$. *The base* $\{E_n^D(z)\}$ *is given as follows:*

$$E_n^D(z) = n^n + e^n z^n, \;\; P_0(z) = 1$$

Hence

$$\omega_{E_n^D}(R) = \left(\frac{n}{e}\right)^n \left[2 + \left(e\,\frac{R}{n}\right)^n\right],$$

and hence the base $\{E_n^D(z)\}$ *is of order* $\rho_{E^D} = 1$ *and type* $\tau_{E^D} = e$. *This proves that both bases* $\{P_n(z)\}$ *and* $\{E_n^D(z)\}$ *have the same order and type.*

Recently, the authors [21] proved that the Bernoulli polynomials $\{B_n(z)\}$ is of order 1 and type $\frac{1}{2\pi}$ and the Euler polynomials $\{E_n(z)\}$ is of order 1 and type $\frac{1}{\pi}$.

From Theorem 5.1, we get the following corollaries:

Corollary 5.1. *The EDBPs of Bernoulli polynomials* $\{E^D B_n(z)\}$ *is of order* 1 *and type* $\frac{1}{2\pi}$.

Corollary 5.2. *The EDBPs of Euler polynomials* $\{E^D E_n(z)\}$ *is of order* 1 *and type* $\frac{1}{\pi}$.

In the following result, we determine the Tρ-property of the base $\{E_n^D(z)\}$.

Theorem 5.2. *Let the BP* $\{P_n(z)\}$ *have* Tρ*-property in* $\overline{D}(R), R > 0$ *and satisfy the condition:*

$$\lim_{n\to\infty} \frac{\log D_n}{n \log n} = 0. \tag{5.2}$$

Then the associated base $\{E_n^D(z)\}$ *have the same property.*

Proof. Consider the function $\omega_{E^D}(R)$ given by

$$\omega_{E^D}(R) = \limsup_{n\to\infty} \frac{\log \omega_{E_n^D}(R)}{n \log n}, \tag{5.3}$$

where $\omega_{E_n^D}(R)$ is the Cannon sum of the EDBPs $\{E_n^D(z)\}$. Then by using (4.3), (5.2) and (5.3), we obtain

$$\omega_{E^D}(R) \le \limsup_{n\to\infty} \frac{\log e^{D_n - n}(D_n+1) + \log \omega_{P_n}(R)}{n \log n} \le \omega_P(R). \tag{5.4}$$

Besides, suppose that the base $\{P_n(z)\}$ have property Tρ in $\overline{D}(R), R > 0$. Then according to Theorem 2.2 and inequality (5.4), it follows:

$$\omega_{E^D}(R) \leq \omega_P(R) \leq \frac{1}{\rho}.$$

That is to say that, the base $\{E_n^D(z)\}$ have property Tρ in $\overline{D}(R), R > 0$. Thus Theorem 5.2 is proved.

□

To show that the EDBPs $\{E_n^D(z)\}$, can not have property Tρ in $\overline{D}(R), R > 0$, though the original base have this property unless the later satisfies condition (5.2), we give the following example:

Example 5.2. *Consider the BPs* $\{P_n(z)\}$ *defined by given by*

$$P_n(z) = \begin{cases} z^n, & n \text{ is even} \\ z^n + \dfrac{z^{t(n)}}{2^{n^n}}, & n \text{ is odd} \end{cases}$$

where $t(n)$ *is the nearest even integer to* $n^n + n \log n$.

Therefore, when n is odd

$$z^n = P_n(z) - \frac{P_{t(n)}(z)}{2^{n^n}}.$$

Hence,

$$\omega_{P_n}(R) = R^n + 2\frac{R^{t(n)}}{2^{n^n}}.$$

Putting $R = 2$, we get

$$\omega_{Pn}(2) = 2^n + 2^{nlogn+1},$$

so that

$$\omega_P(2) = \limsup_{n\to\infty} \frac{\log \omega_{P_n}(2)}{n \log n} \leq \log 2.$$

Thus, the base $P_n(z)$ have property $T_{\frac{1}{\log 2}}$ in $\overline{D}(2)$.

In the other hand, the EDBPs $\{E_n^D(z)\}$, is

$$E_n^D(z) = \begin{cases} e^n z^n, & n \text{ is even} \\ e^n z^n + e^{t(n)} \dfrac{z^{t(n)}}{2^{n^n}}, & n \text{ is odd} \end{cases}$$

and

$$\omega_{E^D}(2) = \limsup_{n\to\infty} \frac{\log \omega_{E_n^D}(2)}{n \log n} \leq 1 + \log 2.$$

that is to say the EDBPs $\{E_n^D(z)\}$, have not $T_{\frac{1}{\log 2}}$ -property in $\overline{D}(2)$ while the original base have the same property. It is clear that the base $\{P_n(z)\}$ does not satisfy condition (5.2), as required.

In [21] the Bernoulli polynomials $\{B_n(z)\}$ and the Euler polynomials $\{E_n(z)\}$ have property T_1.

Corollary 5.3. *The EDB of Bernoulli polynomials* $\{E^D B_n(z)\}$ *have property* T_1.

Corollary 5.4. *The EDB of Euler polynomials* $\{E^D E_n(z)\}$ *have property* T_1.

Now, we can proceed very similar as in 5.2 to prove the following result:

Theorem 5.3. *If the base of polynomials* $\{P_n(z)\}$ *has the property* $T\rho$ *in an open disk* $D(R), R > 0$ *or at the origin, then the base* $\{E_n^D(z)\}$ *associated with it has the same property.*

Also, it is worthy to ensure that Theorems 5.1, 5.2, 5.3 will be still true when we replace the base $\{E_n^D(z)\}$ by the base $\{E_n^I(z)\}$.

6 Conclusion

We construct exponential derived and integral bases. The effectiveness properties, order and type and the $T\rho$-Property, have been characterized for the derived EDBs in various regions in Fréchet spaces. The current work suggests exploring other possible generalizations using other derivative. Furthermore, this study paves the way to develop the EDBs in the case of several complex variables or higher dimensional spaces such as the Clifford analysis setting. In previous studies, [3, 9, 20, 24, 28, 29, 32, 33], the convergence properties in different regions of associated BSP (such as inverse set, product set, transpose set, transposed inverse set, square root set, similar set, Hadamard product set) were studied. It is of great interest to examine the convergent properties for the EDBs of these sets in the corresponding regions. In the future, it is likely to study the convergent properties of new sets of polynomials in different regions (e.g., Laguerre, Lagender, Hermit and Gontcharoff polynomials) where the EDBs these sets can be studied in the same regions.

Data Availability There are no data associate with this research.

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