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CLOSED FORM SOLUTION FOR THE NUMERATOR POLYNOMIAL OF SAMPLED DATA SYSTEMS

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ABSTRACT

A problem of continuing interest in control is the specification of sampling period that results in a system response with specified time domain characterestics such as minimum phase behaviour, overshoot and settling time. This paper presents a closed form solution for the numerator polynomial of sampled data systems. The coefficients of the numerator polynomial are given in terms of the residues and poles of the continuous time system, and the sampling period. Also, numerical algorithm is proposed for determining reasonable sampling period for minimum phase behaviour.

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1. INTRODUCTION

Poles and zeros are fundamental properties of linear time invariant systems. The poles reflect the internal couplings in the system and thus its autonomous behavior. The zeros reflect the way the internal variables are coupled to the inputs and outputs. It is well known that unstable zeros limit the performance that can be achieved when controlling a system. Feedforward compensation would require an unstable inverse model of the system. Many techniques are based on the cancellation of process zeros. Such methods will not work when the process has unstable zeros. Several of the adaptive algorithms that are currently investigated belong to this category. When a continuous time system is sampled the poles s_i are transformed as

$$z_i = e^{s_i T} \tag{1}$$

where T is the sampling period. The transformation (1) maps the left half plane onto the unit circle. This means that the stability is preserved. There is unfortunately no simple transformation which shows how the zeros of a continuous time system are transformed by sampling. The type of hold circuit used critically influences the position of the zeros. It can be actually shown that the zeros of an Nth-order strictly proper system can be placed arbitrarily, if the control signal has a constant value over each Nth part of the sampling interval. Most digital control systems, however, use a zero-order hold, and we limit ourselves to that. It is then not true that a continuous time system with zeros in the left half plane will transform to a sampled system with zeros inside the unit circle or vice versa. Design methods for sampled systems which are based on cancellation of process zeros can thus work well for certain sampling periods and fail for others.

Many research efforts have been spent in the area of choice of sampling period for minimum phase behaviour. It was shown that a continuous time system with pole excess larger than 2 will always give a pulse transfer function with zeros outside the unit circle provided that the sampling period is sufficiently short. Also, there are continuous time transfer functions which have unstable zeros which give sampled systems with stable zeros [1]. A relationship between real poles and real zeros of SISO sampled data systems, which is independent of the sampling period is given in [2]. The relation is stated in terms of a parity properity involving the number of real zeros between any two real poles. This properity allows to investigate conditions for the preservation of stable or unstable zeros under sampling.

The purpose of this paper is to introduce a numerical approach for choice of sampling period to achieve minimum phase system. The paper is organized as follows :section 2 derives the pulse transfer function of linear continuous time system. A closed form expression for the numerator polynomial of the pulse transfer function is given in section 3. In section 4, the algorithm of choice of sampling period is introduced. Numerical examples are found in section 5. Final conclusions are presented in section 6.

2. PULSE TRANSFER FUNCTION

Consider a discrete time system composed of a zero-order hold, a plant, and a sampler in series as shown in Fig.(1).

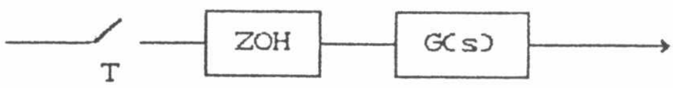

Fig.(1) Discrete Time System.

G(s) denotes the transfer function of a linear continuous time controlled plant. This configuration is used in almost all digital control systems. The controlled plant is an Nth-order strictly proper linear continuous time whose transfer function is given by

$$G(s) = B(s)/A(s) \qquad (2)$$

where A(s) and B(s) are coprime polynomials, and A(s) is an Nth-degree monic polynomial. For simplicity, we suppose that all the poles of G(s) are simple, but the discussion is valied even if G(s) has multiple poles. Using partial fraction expansion G(s) can be written as

$$G(s) = \sum_{i=1}^{Nr} G_{r,i}(s) + \sum_{i=1}^{Nc} G_{c,i}(s) \tag{3}$$

where Nr is the number of distincit real poles, Nc is the number of complex pole pairs, and

$$G_{r,i}(s) = \frac{b_i}{s + a_i} \tag{4a}$$

$$G_{c,i}(s) = \frac{p_i\ s + q_i}{(s + c_i)^2 + d_i^2} \tag{4b}$$

a_i, b_i, c_i, d_i, p_i, and q_i are real numbers that satisfy $a_i \neq a_j$ and $(c_i, d_i) \neq (c_j, d_j)$, and $d_j \neq 0$.

The pulse transfer function of the sampled data system with sampling period T for G(s) is represented by

$$G(z) = (1 - z^{-1}) \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{e^{sT}}{z - e^{sT}} \frac{G(s)}{s} ds \tag{5}$$

where σ is a real number such that all the poles $\frac{G(s)}{s}$ have real parts less than σ. G(z) can be expressed as

$$G(z) = \beta(z)/\alpha(z) \tag{6}$$

where $\alpha(z)$ is an Nth-degree monic polynomial and $\beta(z)$ is a polynomial. We now calculate the pulse transfer function for G(s) given by (3) for two cases :

(a) A simple real pole case

$$G_r(s) = \frac{b}{s + a} \quad \rightarrow \quad G_r(z) = \frac{\beta}{z - \alpha} \tag{7}$$

where

$$\alpha = e^{-aT} \tag{8a}$$

$$\beta = b\ (1 - \alpha)/a \tag{8b}$$

(b) A complex pole pair case

$$G_c(s) = \frac{p\,s + q}{(s + c)^2 + d^2} \quad \Rightarrow \quad G_c(z) = \frac{\zeta\, z + \eta}{(z - \phi)^2 + \psi^2} \tag{9}$$

where ϕ, ψ, ζ, and η are determined by

$$\phi = e^{-cT} \cos(dT) \tag{10a}$$

$$\psi = e^{-cT} \sin(dT) \tag{10b}$$

$$\zeta = \frac{-qd(\phi - 1) + [p(c^2 + d^2) + qc\,]\,\psi}{d(c^2 + d^2)} \tag{10c}$$

$$\eta = \frac{-qd(\phi - e^{-2cT}) - [p(c^2 + d^2) + qc\,]\,\psi}{d(c^2 + d^2)} \tag{10d}$$

Summarizing the above results, the transfer function of a sampled data system for G(s) given by (3) can be expressed as

$$G(z) = \sum_{i=1}^{Nr} G_{r,i}(z) + \sum_{i=1}^{Nc} G_{c,i}(z) \tag{11}$$

where $$G_{r,i}(z) = \frac{\beta_i}{z - \alpha_i} \tag{12a}$$

$$G_{c,i}(z) = \frac{\zeta_i z + \eta_i}{(z - \phi_i)^2 + \psi_i^2} \tag{12b}$$

(α_i, β_i) and $(\phi_i, \psi_i, \zeta_i, \eta_i)$ are defined by (8) and (10).

3. NUMERATOR POLYNOMIAL

The numerator polynomial $\beta(z)$ of the pulse transfer function G(z) can be written as

$$\beta(z) = \sum_{i=1}^{N} \gamma_i\, z^{N-i} \tag{13}$$

$$= \gamma_1\, z^{N-1} + \gamma_2\, z^{N-2} + \ldots + \gamma_{N-1}\, z + \gamma_N$$

where γ_1, γ_2, ... , and γ_N are the coefficients of numerator polynomial given by

$$S * \beta_v = \gamma_v \tag{14}$$

where S is NxN matrix which is function of α_i, ϕ_i, and ψ_i. γ_v and β_v are given by

$$\gamma_v = [\ \gamma_1 \quad \gamma_2 \quad \dots \quad \gamma_N\ ]^T$$

$$\beta_v = [\ \beta_1 \quad \beta_2 \quad \dots \quad \beta_{Nr} \quad \zeta_1 \quad \zeta_2 \quad \dots \quad \zeta_{Nc} \quad \eta_1 \quad \eta_2 \quad \dots \quad \eta_{Nc}\ ]^T$$

Examples of several transfer functions and their $\beta(z)$ are given below, then the results are generalized. Low order systems are used for presentation, because high order requires large space..

Example 1 : Case of real poles

$$G(s) = \sum_{i=1}^{4} \frac{b_i}{s + a_i} \quad \Rightarrow \quad G(z) = \sum_{i=1}^{4} \frac{\beta_i}{z - \alpha_i}$$

$$\beta_v = [\ \beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4\ ]^T$$

$$\gamma_v = [\ \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4\ ]^T$$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -(\alpha_2+\alpha_3+\alpha_4) & -(\alpha_1+\alpha_3+\alpha_4) & -(\alpha_1+\alpha_2+\alpha_4) & -(\alpha_1+\alpha_2+\alpha_3) \\ (\alpha_2\alpha_3+\alpha_3\alpha_4+\alpha_2\alpha_4) & (\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_3\alpha_4) & (\alpha_1\alpha_2+\alpha_1\alpha_4+\alpha_2\alpha_4) & (\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) \\ -\alpha_2\alpha_3\alpha_4 & -\alpha_1\alpha_3\alpha_4 & -\alpha_1\alpha_2\alpha_4 & -\alpha_1\alpha_2\alpha_3 \end{bmatrix}$$

It is noticed that the elements of the ith column of S matrix, are the coefficients of a polynomial of order Nr-1 whose roots are α_k, $k = 1, \dots, Nr$, and $k \neq i$.

Example 2 : Case of Complex poles

$$G_c(s) = \sum_{i=1}^{2} \frac{p_i s + q_i}{(s + c_i)^2 + d_i^2} \Rightarrow G_c(z) = \sum_{i=1}^{2} \frac{\zeta_i z + \eta_i}{(z - \phi_i)^2 + \psi_i^2}$$

$$\beta_v = [\ \zeta_1 \quad \zeta_2 \quad \eta_1 \quad \eta_2\ ]^T$$

$$\gamma_v = [\ \gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \gamma_4\ ]^T$$

$$S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -2\phi_2 & -2\phi_1 & 1 & 1 \\ \phi_2^2 + \psi_2^2 & \phi_1^2 + \psi_1^2 & -2\phi_2 & -2\phi_1 \\ 0 & 0 & \phi_2^2 + \psi_2^2 & \phi_1^2 + \psi_1^2 \end{bmatrix}$$

It is noticed that the elements of the ith column of S matrix, i = 1, ... , Nc, are the coefficents of a polynomial of order 2(Nc-1), whose roots are $-\phi_k \pm j\ \psi_k$, k = 1, ... , Nc, k ≠ i. The last element of the column is zero. The next Nc columns are the same, except that the first element is zero.

Example 3 : Case of real and complex pole pair

$$G(s) = \frac{b}{s + a} + \frac{p\ s + q}{(s + c)^2 + d^2}$$

$$G(z) = \frac{\beta}{z - \alpha} + \frac{\zeta\ z + \eta}{(z - \phi)^2 + \psi^2}$$

$$S = \begin{bmatrix} 1 & 1 & 0 \\ -2\phi & -\alpha & 1 \\ \phi^2+\psi^2 & 0 & -\alpha \end{bmatrix}$$

$$\beta_v = [\ \beta \quad \zeta \quad \eta\ ]^T$$

$$\gamma_v = [\ \gamma_1 \quad \gamma_2 \quad \gamma_3\ ]^T$$

It is noticed that the first Nr columns of S matrix are similar to the case of real poles. The Next 2*Nc columns are similar to the case of complex poles.

4. REASONABLE SAMPLING PERIOD

Since the numerator polynomial $\beta(z)$ can be expressed as function of sampling peiod as shown in the previous section. One can choose sampling time to gurantee a minimum phase system. The following algorithm describes the details of finding reasonable sampling period

(a) Given the continuous time transfer function G(s), use partial fraction expansion to rewrite it in the form given in (3), i.e, determine a_i, b_i and/or p_i, q_i, c_i, d_i.

(b) For T = 0 to Tmax (a predefined maximum value) step ΔT (a predefined incremental value).

(c) Calculate α_i, β_i and/or ζ_i, η_i, ϕ_i, ψ_i using equations (8,10).

(d) Find coefficents of numerator polynomial, equation (14)

(e) Solve for the roots of the numerator polynomial $\beta(z)$.

(f) Let $T = T + \Delta T$ until $T > Tmax$, and goto step (c).

5. NUMERICAL EXAMPLES

The following examples are applications of the algorithm given in the previous section.

Example 4 : The transfer function

$$G(s) = \frac{6(1 - s)}{(s + 2)(s + 3)}$$

has an unstable zero at s = 1, the pulse transfer function, has a zero at -1 for T = 1.2485 sec., and for larger T, the zero is always in the unit circle.

Example 5 : The transfer function

$$G(s) = \frac{9}{s^2 + 3s + 9}$$

The zero of the pulse transfer function is always in the unit circle for T > 0.0012 sec.

Example 6 : The transfer function

$$G(s) = \frac{s^2 + 2s + 0.75}{s^5 + 27.5 s^4 + 261.5 s^3 + 1039 s^2 + 1668 s + 864}$$

For minimum phase behaviour T must be greater than 0.2209 sec.

6. CONCLUSIONS

Closed form solution for the coefficients of the numerator polynomial of sampled systems is given as function of sampling period. Numerical algorithm for evaluating reasonable sampling period for minimum phase behaviour have been proposed. The basic theme behind this new algorithm is to find the intersection points of the numerator polynomial and the unit circle on the z-plane. This algorithm can be used to study the effects of sampling period on the transient behaviour of sampled systems. Several examples are illustrated.

7. REFERENCES

1. Astrom, K. J., Hangander, P., and Sternby, J.,"Zeros of Sampled Systems", Automatica, Vol.20-1, pp. 31-38, 1984.
2. Hara, S., Katori, H.,and Kondo, R.,"The Relationship Between Real Poles and Real Zeros in SISO Sampled Data Systems", IEEE Trans. Automat. Contr., Vol. AC-34, No. 6, JUNE 1989.
3. Fu, Y., and Dumont, G.A., "Choice of Sampling to Ensure Minimum Phase Behaviour", IEEE Trans. Automat. Contr., Vol. 34, No.5, May 1989.
4. Moore, K.L., and Bhattacharyya, S.P., "A Technique for Choosing Zero Locations for Minimum Overshoot", IEEE Trans. Automat. Contr., Vol. 35, No.5, May 1990.
5. Astrom, K.J., and Wittenmark, B., "Computer controlled systems", Prentice Hall, 1984.
6. Phillips, C.L., and Nagle, H.N., "Digital Control Systems Analysis and Design", Prentice Hall, 1984.