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COMPUTATION OF ROBOT MANIPULATORS JACOBIAN : A QUATERNION APPROACH

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ABSTRACT

This work deals with the Jacobian computation for robot manipulators using a unified quaternion parameterization of position and orientation of members.
Proposed generalized relations expressing the elements of manipulators Jacobian are derived in terms of Boolean parameters. These parameters introduce the contribution of both revolute and prismatic joints. The proposed relations are applied on a case study of a PUMA 560 robot manipulator for which analytical expressions of the Jacobian are obtained and tested by simulation.

1. INTRODUCTION

The major task of an industrial robot manipulator is to position and orient an end effector (EEF) during an approach phase of motion over a predefined time-based trajectory. To perform this task, a robot manipulator should have a mechanical structure of variable configuration with several powered joints between its members. In order to position and orient the EEF arbitrarily w.r.t a base fixed frame, the number of joints n or the degree of freedom of the manipulator must be greater than or equal to six. The first step in the formulation of the control problem of manipulators is to establish a relationship between the vector of EEF spatial coordinates $\underline{X}$ and the vector $\underline{q}$ of the joint coordinates or the generalized coordinates in the Lagrangian sense.
The trajectory control of manipulators consists of forcing the vector $\underline{X}(t)$ to track a vector $\underline{X}_d(t)$ representing the desired evolution of the position and orientation of the EEF. A differential Inverse Kinematics Algorithm calculates the error $\Delta\underline{q}$ of the joint variables corresponding to an error vector $\Delta\underline{X} = \underline{X}_d(t) - \underline{X}(t)$ to be compensated for. This process passes by the

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computation of the manipulator Jacobian Matrix $J(\underline{q})$. The computation of the elements of $\underline{X}(t)$ and those of $J(\underline{q})$ depends on the kinematical parameterization of the manipulator. The computation of $J(\underline{q})$ for manipulators using the Homogeneous Transformation Matrix parameterization is given in [1,2]. The use of Quaternions for the parameterization of manipulators has been recently investigated [4-9]. They provide a unified representation of both position and orientation of members. In this paper, the computation of manipulators Jacobian in terms of quaternion parameterization is investigated.

The organization of the paper is as follows : In section 2, the fundamental relations defining the unified quaternion position and orientation parameterization are shortly described. These relations are judged as essential for the derivation of the Jacobian matrix presented in section 3. The rest of the paper is reserved to the presentation of an application where a Quaternion based Jacobian of the PUMA 560 manipulator is obtained. A solution of the differential kinematic equations based on the Jacobian inversion is used to verify the validity of the obtained Jacobian by numerical simulation.

2. A UNIFIED QUATERNION PARAMETERIZATION OF POSITION & ORIENTATION

Consider a frame ${}^{b}F$ in a general space motion w.r.t. a frame ${}^{n}F$. Based on Chasle's theorem [3], the motion of ${}^{b}F$ w.r.t. ${}^{n}F$ is a superposition of a translation following any point (say O_b), the origin of ${}^{b}F$, and a rotation about that point.
This rotation may be defined by a Quaternion of Finite Rotation $\hat{e}_{n,b}$ as [8,9],

$$\hat{e}_{n,b} = \cos(\theta/2) + \sin(\theta/2)\vec{u} \tag{1}$$

where θ and $\vec{u}$ are respectively the principal angle and the principal axis associated with Euler's theorem of finite rotation [3]. The quaternion of finite rotation $\hat{e}_{n,b}$ is a unit quaternion having the same representation on both frames ${}^{b}F$ and ${}^{n}F$, i.e.,

$$\hat{e}_{n,b} * \hat{e}^{c}_{n,b} = 1 \tag{2}$$

$${}^{n}\underline{e}_{n,b} = {}^{b}\underline{e}_{n,b} = [e_0 \quad e_1 \quad e_2 \quad e_3]^t \tag{3}$$

The four elements of the quaternion representation ${}^{n}\underline{e}_{n,b}$ are the Euler parameters of finite rotation :

$$\begin{aligned} e_0 &= \cos(\theta/2) \quad ; \quad & e_1 &= u_1 \sin(\theta/2) \\ e_2 &= u_2 \sin(\theta/2) \quad ; \quad & e_3 &= u_3 \sin(\theta/2) \end{aligned} \tag{4}$$

The angular velocity of ${}^{b}F$ w.r.t. ${}^{n}F$ may be expressed as [5];

$$ {}^{n}\underline{w}_{n,b} = 2\,{}^{n}\dot{\underline{e}}_{n,b} * {}^{n}\underline{e}^{c}_{n,b} \tag{5}$$

The position of ${}^{b}F$ w.r.t. ${}^{n}F$ is defined by the vector quaternion $\hat{P}_{n,b}$ from the origin O_n of ${}^{n}F$ to the origin O_b of ${}^{b}F$. $\hat{P}_{n,b}$ and its representation are expressed as :

$$\begin{aligned} \hat{P}_{n,b} &= 0 + \overrightarrow{O_n O_b} \\ {}^{n}\underline{P}_{n,b} &= {}^{n}[0 \quad x \quad y \quad z]^{t} \quad ; \end{aligned} \tag{6}$$

where x,y and z are the coordinates of O_b in ${}^{n}F$.

Vector Transformations - Successive Rotations :

Consider a spatial quaternion vector $\hat{V}$. It may be shown [9] that the representations of $\hat{V}$ w.r.t. ${}^{b}F$ and ${}^{n}F$ are related by

$$ {}^{n}\underline{V} = {}^{n}\underline{e}_{n,b} * {}^{b}\underline{V} * {}^{n}\underline{e}^{c}_{n,b} \tag{7}$$

Eq.(7) is the quaternion equivalent of Vector Transformations. Consider three frames ${}^{1}F$, ${}^{m}F$ and ${}^{n}F$ in successive rotations; it may be shown [9] that when the quaternions of finite rotations are represented w.r.t. local frames, the representation of the overall rotation corresponds to the post multiplication of the quaternion for the fore rotation by that for the following rotation i.e.,

$$ {}^{1}\underline{e}_{1,n} = {}^{1}\underline{e}_{1,m} * {}^{m}\underline{e}_{m,n} \tag{8}$$

Manipulator Kinematics

A robot manipulator is considered here as an open kinematic chain of rigid links. The first link is fixed to the base frame and the last link is attached to an end effector. The links are numbered starting from the base link which is given the number 0. The last link is given the number n which is the degree of freedom of the manipulator,

A right handed orthogonal frame ${}^{i}F$ is assigned to each link i according to Denavit & Hartenberg convention [2]. ${}^{i}F$ may be obtained starting from ${}^{i-1}F$ through four steps of successive rotations and translations as follows : a) Rotation around the z_{i-1} at an angle θ_i; b) Translation in the direction of x_i by the member length a_i; c) Translation in the direction of z_{i-1} by d_i; d) Rotation about x_i by the member twist α_i.

Based on Eq.(8), the orientation of ${}^{i}F$ w.r.t. ${}^{i-1}F$ is given by,

$$^{i-1}\underline{e}_{i-1,i} = {}^{i-1}\underline{e}_{i-1,a}(\theta_i) * {}^{a}\underline{e}_{a,i}(\alpha_i) \tag{9}$$

where

$$^{i-1}\underline{e}_{i-1,a}(\theta_i) = [\cos\theta_i/2 \quad 0 \quad 0 \quad \sin\theta_i/2]^t \quad ;$$

$$^{a}\underline{e}_{a,i}(\alpha_i) = [\cos\alpha_i/2 \quad \sin\alpha_i/2 \quad 0 \quad 0]^t \quad ;$$

$$^{i-1}\underline{e}_{i-1,i} = \begin{bmatrix} \cos\theta_i/2 & \cos\alpha_i/2 \\ \cos\theta_i/2 & \sin\alpha_i/2 \\ \sin\theta_i/2 & \sin\alpha_i/2 \\ \sin\theta_i/2 & \cos\alpha_i/2 \end{bmatrix} \tag{10}$$

The position of ^{i}F w.r.t. ^{i-1}F may be specified by

$$^{i-1}\underline{P}_{i-1,i} = [0 \quad a_i \cos\theta_i \quad a_i \sin\theta_i \quad d_i]^t \tag{11}$$

For a revolute joint, the joint variable $q_i = \theta_i$ while for a prismatic joint $q_i = d_i$.

Based on Eq.(8) the orientation of link i w.r.t. link j(j<i) is,

$$^{j}\underline{e}_{j,i} = {}^{j}\underline{e}_{j,j+1} * {}^{j+1}\underline{e}_{j+1,j+2} * \ldots * {}^{i-1}\underline{e}_{i-1,i} \tag{12}$$

The position of link i w.r.t. link j may, however, be defined by the position vector $P_{j,i}$ from O_j to O_i represented in ^{j}F as

$$^{j}\underline{P}_{j,i} = \sum_{k=j+1}^{i} {}^{j}\underline{e}_{j,k-1} * {}^{k-1}\underline{P}_{k-1,k} * {}^{j}\underline{e}^{c}_{j,k-1} \tag{13}$$

For a manipulator having n links, the position and orientation of the EEF w.r.t. the base frame are given respectively by $\hat{P}_{0,n}$ and $\hat{e}_{0,n}$. Their representations in the base frame are expressed using Eqs.(12) and (13) as :

$$^{0}\underline{P}_{0,n} = \sum_{i=1}^{n} {}^{0}\underline{e}_{0,i-1} * {}^{i-1}\underline{P}_{i-1,i} * {}^{0}\underline{e}^{c}_{0,i-1} = [0 \quad X \quad Y \quad Z]^t \tag{14}$$

$$^{0}\underline{e}_{0,n} = {}^{0}\underline{e}_{0,1} * {}^{1}\underline{e}_{1,2} * \ldots * {}^{n-1}\underline{e}^{c}_{n-1,n} = [E_0 \quad E_1 \quad E_2 \quad E_3]^t \tag{15}$$

The elements of $^{0}\underline{P}_{0,n}$ and $^{0}\underline{e}_{0,n}$ are grouped in a vector $\underline{X}$:

$$\underline{X} = [\underline{X}_1 \quad \underline{X}_2]^t \tag{16}$$

where $\underline{X}_1 = M\ {}^{0}\underline{p}_{0,n} = [X \quad Y \quad Z]^t$ (17)

$\underline{X}_2 = {}^{0}\underline{e}_{0,n} = [E_0 \quad E_1 \quad E_2 \quad E_3]^t$ (18)

$M = [0 \quad I_3]$; and I_3 is a 3×3 unit matrix.

The vector $\underline{X}(7\times1)$ hence defines uniquely the position and orientation of the EEF w.r.t. the base frame. It should however be noted that the last four elements of $\underline{X}$ are not independent. They are inter-related by the normality condition (Eq.(2)). The right hand side of Eqs.(17) and (18) are function of the manip lator configuration defined by the vector $\underline{q}$.

manipulator using quaternion parameterization and may be rewritten as ;

$$\underline{X} = \underline{X}(\underline{q}) \quad (19)$$

3. QUATERNION-BASED JACOBIAN COMPUTATION

The differentiation of Eq.(19) w.r.t. time yields

$$\dot{\underline{X}} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}(\underline{X})\ \dot{q}_i \quad (20\text{-a})$$

or :

$$\begin{bmatrix} \dot{\underline{X}}_1 \\ \dot{\underline{X}}_2 \end{bmatrix} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \begin{bmatrix} \underline{X}_1 \\ \underline{X}_2 \end{bmatrix} \dot{q}_i = \sum_{i=1}^{n} \begin{bmatrix} \underline{p}_i \\ \underline{h}_i \end{bmatrix} \dot{q}_i \quad (20\text{-b})$$

Eq.(20-b) may be rewritten in the following matrix form

$$\begin{bmatrix} \dot{\underline{X}}_1 \\ \dot{\underline{X}}_2 \end{bmatrix} = \begin{bmatrix} \underline{p}_1 & \underline{p}_2 & \cdots\cdots\cdots & \underline{p}_n \\ \underline{h}_1 & \underline{h}_2 & \cdots\cdots\cdots & \underline{h}_n \end{bmatrix} \dot{\underline{q}} = J_e\ \dot{\underline{q}} \quad (20\text{-c})$$

where J_e is a 7×n jacobian matrix which relates the motion rate of the EEF $\dot{\underline{X}}$ to the joint speeds $\dot{\underline{q}}$.

The upper 3×1 portion of the <u>ith</u> column of J_e is associated with position of the EEF and is given by :

$$\underline{p}_i = \frac{\partial}{\partial q_i}\left({}^{0}\underline{p}_{0,n}\right) = \frac{\partial}{\partial q_i}\left({}^{0}\underline{p}_{0,i-1} + {}^{0}\underline{p}_{i-1,n}\right) \quad (21\text{-a})$$

The analysis of Eqs.(10-13) shows that $\frac{\partial}{\partial q_i}\left({}^{0}\underline{p}_{0,i-1}\right) = 0$. Eq.(21-a), hence, reduces to;

$$\underline{p}_i = \frac{\partial}{\partial q_i}\left({}^{0}\underline{p}_{i-1,n}\right) = \frac{\partial}{\partial q_i}\left({}^{0}\underline{e}_{0,i-1} * {}^{i-1}\underline{p}_{i-1,n} * {}^{0}\underline{e}^{c}_{0,i-1}\right) \quad (21\text{-b})$$

The analysis of Eqs.(10-13) shows also that :

$$\frac{\partial}{\partial q_i}\left({}^{0}\underline{e}_{0,i-1}\right) = \frac{\partial}{\partial q_i}\left({}^{0}\underline{e}^{c}_{0,i-1}\right) = \frac{\partial}{\partial q_i}\left({}^{i}\underline{p}_{i,n}\right) = 0 \tag{22}$$

However;

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{p}_{i-1,n}\right) = \frac{\partial}{\partial q_i}\left({}^{i-1}\underline{p}_{i-1,i}\right) + \frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i}\right) \tag{23}$$

<u>For a prismatic joint</u> and based on Eq.(11),

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{p}_{i-1,i}\right) = [0 \quad 0 \quad 0 \quad 1]^{t} = {}^{i-1}\underline{k}_{i-1} \tag{24}$$

Using Eq.(22) it may be shown that

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i}\right) = 0 \tag{25}$$

Substituting from Eqs.(22-25) into (21-b), $\underline{p}_i$ for a prismatic joint is expressed as :

$$\underline{p}_i = {}^{0}\underline{e}_{0,i-1} * {}^{i-1}k_{i-1} * {}^{0}\underline{e}^{c}_{0,i-1} \tag{26}$$

<u>For a revolute joint</u> and based on Eq.(11),

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{p}_{i-1,i}\right) = [0 \quad -a_i \sin q_i \quad a_i \cos q_i \quad 0]^{t} \tag{27}$$

Considering on the other hand the quaternion product it may be shown that :

$${}^{i-1}k_{i-1} * {}^{i-1}\underline{p}_{i-1,i} = [d_i \quad -a_i \sin q_i \quad a_i \cos q_i \quad 0]^{t} \tag{28}$$

and based on Eqs.(9,10) it may also be shown that :

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}_{i-1,i}\right) = \frac{1}{2}\left({}^{i-1}k_{i-1} * {}^{i-1}\underline{e}_{i-1,i}\right); \tag{29-a}$$

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}^{c}_{i-1,i}\right) = \frac{1}{2}\left({}^{i-1}\underline{e}^{c}_{i-1,i} * {}^{i-1}k_{i-1}\right) \tag{29-b}$$

Based on Eqs.(22) and (29) the last term in the R.H.S. of Eq.(23) may be rewritten as :

$$\frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i}\right) = \frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}_{i-1,i}\right) * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i} + {}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * \frac{\partial}{\partial q_i}\left({}^{i-1}\underline{e}^{c}_{i-1,i}\right) \tag{30-a}$$

$$\frac{\partial}{\partial q_i}\left[{}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i}\right] =$$

$$= \frac{1}{2}\left[{}^{i-1}k_{i-1} * {}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i}\right] +$$

$$\frac{1}{2}\left[{}^{i-1}\underline{e}_{i-1,i} * {}^{i}\underline{p}_{i,n} * {}^{i-1}\underline{e}^{c}_{i-1,i} * {}^{i-1}k_{i-1}\right] \tag{30-b}$$

$$= \frac{1}{2}\left[{}^{i-1}k_{i-1} * {}^{i}\underline{p}_{i,n}\right] + \frac{1}{2}\left[{}^{i}\underline{p}_{i,n} * {}^{i-1}k_{i-1}\right] \tag{30-c}$$

Substituting from eqs.(22,23,27,28 and 30-c) into (21-b), $\underline{p}_i$ for a revolute joint is expressed as;

$$\underline{p}_i = M\left[{}^{0}\underline{e}_{0,i-1} * {}^{i-1}k_{i-1} * {}^{i-1}\underline{p}_{i-1,n} * {}^{0}\underline{e}^{c}_{0,i-1}\right] \tag{31}$$

Based on Eqs.(26) and (31), a generalized expression for the position portion of the Jacobian matrix is given by,

$$\underline{p}_i = M\left[{}^{0}\underline{e}_{0,i-1} * \left\{(1-\sigma_i)\ {}^{i-1}k_{i-1} + \sigma_i\ {}^{i-1}k_{i-1} * {}^{i-1}\underline{p}_{i-1,n}\right\} * {}^{0}\underline{e}^{c}_{0,i-1}\right] \tag{32}$$

Where σ_i is a Boolean parameter, $\sigma_i = 0$ for a prismatic joint and $\sigma_i = 1$ for a revolute joint.

The lower 4×1 portion of the ith column of J_e is associated with the orientation of the EEF and is given by,

$$\underline{h}_i = \frac{\partial}{\partial q_i}\left[{}^{0}\underline{e}_{0,n}\right] = {}^{0}\underline{e}_{0,i-1} * \frac{\partial}{\partial q_i}\left[{}^{i-1}\underline{e}_{i-1,i}\right] * {}^{i}\underline{e}_{i,n} \tag{33-a}$$

Using the results of Eq.(29), Eq.(33-a) reduces to

$$\underline{h}_i = \frac{1}{2}\left[{}^{0}\underline{e}_{0,i-1} * {}^{i-1}k_{i-1} * {}^{i-1}\underline{e}_{i-1,n}\right] \tag{33-b}$$

for joint i being a revolute joint and;

$$\underline{h}_i = 0 \tag{33-c}$$

for joint i being a prismatic joint.

A generalized expression for $\underline{h}_i$ may hence be written as :

$$\underline{h}_i = \frac{1}{2}\sigma_i\left[{}^{0}\underline{e}_{0,i-1} * {}^{i-1}k_{i-1} * {}^{i-1}\underline{e}_{i-1,n}\right] \tag{34}$$

Equations (32) and (34) express in general the elements of J_e which is defined by Eq.(20-c). It should be noted that the left hand side of Eq.(20-c) is a 7×1 vector $\dot{\underline{X}}$ consisting of two parts: 1) the upper 3×1 vector $\dot{\underline{X}}_1 = v$ representing the components of the velocity of the EEF origin; 2) the lower 4x1 vector $\dot{\underline{x}}_2$ representing the time rate of change of the Euler parameters of the EEF. Hereafter, another formulation for Eq.(20-c) is presented where $\dot{\underline{x}}_2$ is replaced by $\dot{\underset{\sim}{x}}_2 = {}^0\underline{w}_{0,n}$, the angular velocity of the EEF. Eq.(5) may be rewritten as :

$$
\begin{aligned}
{}^0\underline{w}_{0,n} &= 2 \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial q_i} \left({}^0\underline{e}_{0,n} \right) \dot{q}_i \right\} * {}^0\underline{e}^c_{0,n} \\
&= 2 \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial q_i} \left({}^0\underline{e}_{0,n} \right) * {}^0\underline{e}^c_{0,n} \right\} \dot{q}_i = \sum_{i=1}^{n} \underline{H}_i \, \dot{q}_i
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
\underline{H}_i &= 2 \, \underline{h}_i * {}^0\underline{e}^c_{0,n} \\
&= \sigma_i \left({}^0\underline{e}_{0,i-1} * {}^{i-1}k_{i-1} * {}^{i-1}\underline{e}_{i-1,n} \right) * {}^0\underline{e}^c_{0,n} \\
&= \sigma_i \left({}^0\underline{e}_{0,i-1} * {}^{i-1}k_{i-1} * {}^0\underline{e}^c_{0,i-1} \right)
\end{aligned}
\tag{36}
$$

Eq.(20-c) may, hence, be rewritten as

$$
\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underset{\sim}{x}}_2 \end{bmatrix} = \begin{bmatrix} \underline{p}_1 & \underline{p}_2 & \cdots\cdots\cdots & \underline{p}_n \\ \underline{H}_1 & \underline{H}_2 & \cdots\cdots\cdots & \underline{H}_n \end{bmatrix} \dot{\underline{q}}
\tag{37}
$$

or

$$
\dot{\underset{\sim}{x}} = J \, \dot{\underline{q}}
\tag{38}
$$

where

$$
\dot{\underset{\sim}{x}} = \left\{ \dot{\underline{x}}_1, \; \dot{\underset{\sim}{x}}_2 \right\}^t ; \qquad J = \begin{bmatrix} \underline{p}_1 & \underline{p}_2 & \cdots\cdots\cdots & \underline{p}_n \\ \underline{H}_1 & \underline{H}_2 & \cdots\cdots\cdots & \underline{H}_n \end{bmatrix}
\tag{39}
$$

where J is a (6xn) Jacobian matrix. The definition of J based on Eq.(39) is identical to that given in [1]. The only difference is that the elements of J here as calculated by Eqs.(31 and 36) are based on the quaternion parameterization of members.

4. APPLICATION : QUATERNION BASED JACOBIAN COMPUTATION OF THE PUMA 560 MANIPULATOR

The Denavit-Hartenberg link parameters of the Puma 560 Manipulator are given in Table 1. Based on these parameters and using Eqs.(10,11) the orientation and position of the successive frames are given by :

$$\left.\begin{array}{l} {}^{0}\underline{e}_{0,1}=1/\sqrt{2}\begin{bmatrix} C_1 & -C_1 & -S_1 & S_1 \end{bmatrix}^t; \quad {}^{1}\underline{e}_{1,2}=\begin{bmatrix} C_2 & 0 & 0 & S_2 \end{bmatrix}^t \\ {}^{2}\underline{e}_{2,3}=1/\sqrt{2}\begin{bmatrix} C_3 & C_3 & S_3 & S_3 \end{bmatrix}^t; \quad {}^{3}\underline{e}_{3,4}=1/\sqrt{2}\begin{bmatrix} C_4 & -C_4 & -S_4 & S_4 \end{bmatrix}^t \\ {}^{4}\underline{e}_{4,5}=1/\sqrt{2}\begin{bmatrix} C_5 & C_5 & S_5 & S_5 \end{bmatrix}^t; \quad {}^{5}\underline{e}_{5,6}=\begin{bmatrix} C_6 & 0 & 0 & S_6 \end{bmatrix}^t \end{array}\right\} (40)$$

$$\left.\begin{array}{l} {}^{0}\underline{P}_{0,1}={}^{4}\underline{P}_{4,5}={}^{5}\underline{P}_{5,6}=[0 \;\; 0 \;\; 0 \;\; 0]^t; \quad {}^{1}\underline{P}_{1,2}=[0 \;\; a_2\underline{C}_2 \;\; a_2\underline{S}_2 \;\; d_2]^t \\ {}^{2}\underline{P}_{2,3}=[0 \;\; 0 \;\; 0 \;\; d_3]^t; \quad {}^{3}\underline{P}_{3,4}=[0 \;\; 0 \;\; 0 \;\; d_4]^t \end{array}\right\} (41)$$

where $C_i = \cos(\theta_i/2)$; $S_i = \sin(\theta_i/2)$; $\underline{C}_i = \cos(\theta_i)$; $\underline{S}_i = \sin(\theta_i)$

Using the values of the relative position and orientation of the Puma 560 defined by Eqs.(40,41), the columns of the Jacobian matrix are calculated based on Eqs.(39) as follows :

$$\left.\begin{array}{l} J_1 = \begin{bmatrix} -Y & X & 0 & 0 & 0 & 1 \end{bmatrix}^t \\ J_2 = \begin{bmatrix} Z\underline{C}_1 & Z\underline{S}_1 & -(X\underline{C}_1 + Y\underline{S}_1) & -\underline{S}_1 & \underline{C}_1 & 0 \end{bmatrix}^t \\ J_3 = \begin{bmatrix} C_{23}d_4\underline{C}_1 & C_{23}d_4\underline{S}_1 & -S_{23}d_4 & -\underline{S}_1 & \underline{C}_1 & 0 \end{bmatrix}^t \\ J_4 = \begin{bmatrix} 0 & 0 & 0 & S_{23}\underline{C}_1 & S_{23}\underline{S}_1 & C_{23} \end{bmatrix}^t \\ J_5 = \begin{bmatrix} 0 & 0 & 0 & -(\underline{C}_4\underline{S}_1+\underline{S}_4S_{23}\underline{C}_1) & -(\underline{S}_4C_{23}\underline{S}_1+\underline{C}_4\underline{C}_1) & \underline{S}_4S_{23} \end{bmatrix}^t \\ J_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \underline{S}_5(\underline{C}_4C_{23}\underline{C}_1-\underline{S}_4\underline{S}_1) + \underline{C}_5S_{23}\underline{C}_1 \\ \underline{S}_5(\underline{C}_4C_{23}\underline{S}_1+\underline{S}_4\underline{C}_1) + \underline{C}_5S_{23}\underline{S}_1 \\ \underline{C}_5C_{23} - \underline{S}_5\underline{C}_4S_{23} \end{bmatrix} \end{array}\right\} (42)$$

Where

$$X = \underline{C}_2\underline{C}_1a_2 + d_4S_{23}\underline{C}_1 - d_{23}\underline{S}_1$$

$$Y = \underline{C}_2\underline{S}_1a_2 + d_4S_{23}\underline{S}_1 + d_{23}\underline{C}_1$$

$$Z = d_4C_{23} - a_2\underline{S}_2$$

$$d_{23} = d_2 + d_3, \quad C_{23} = \cos(q_2 + q_3) \quad S_{23} = \sin(q_2 + q_3)$$

Validation by Numerical Simulation :

Fig.(1) shows the scheme of numerical simulation proposed to verify the validity of the analytical expressions obtained for the Jacobian matrix (Eq.42). The motion rate $\dot{\tilde{X}}(t)$ corresponding to a desired EEF time based trajectory $\underline{X}_d(t)$ are calculated and used as input to a Differential Kinematics Equation Solver (DKES). This DKES is based on the inversion of the obtained Jacobian. The output $\underline{X}_a(t)$ of the DKES is compared to $\underline{X}_d(t)$ to calculate the EEF attitude error Δ_e and position error Δ_p [9] associated with the solution every sampling period.

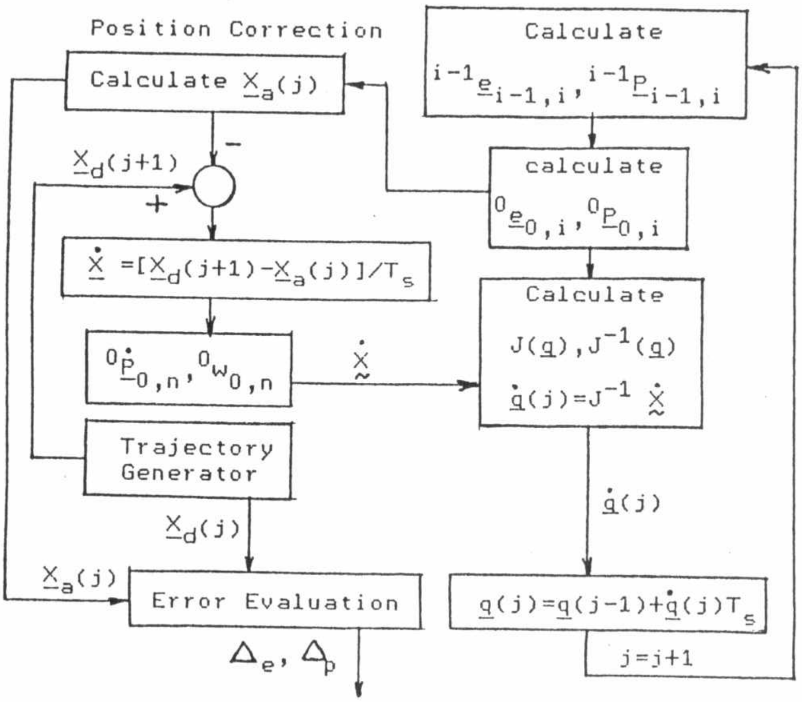

Fig.1. Scheme of Numerical Simulation

Table 1: Link and joint parameters

i	d_i mm	a_i mm	α_i °
1	0	0	-90
2	223	432	0
3	-73.9	0	90
4	433	0	-90
5	0	0	90
6	0	0	0

Results and Discussions :

The simulation have been carried out for several trajectories. The typical results presented hereafter correspond to a closed circular path of the EEF origin with a constant speed v. The location of the EEF over this path is uniquely specified by a polar angular coordinate ϕ.

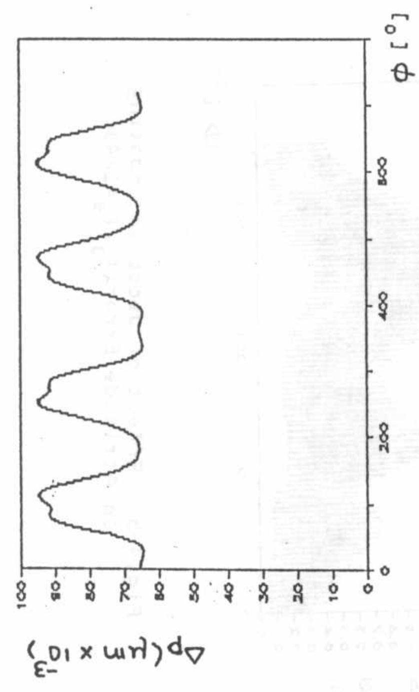

Fig.2-a Evolution of Position Error (v_o=0.2m/s f_j/f_s=1)

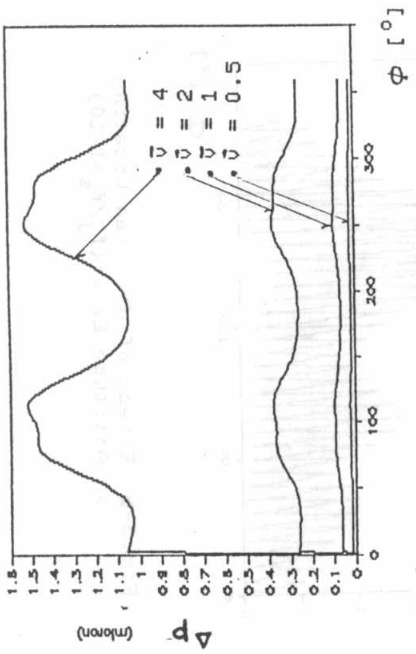

Fig.3-a Effect of EEF Speed on Position Error(f_j/f_s = 1)

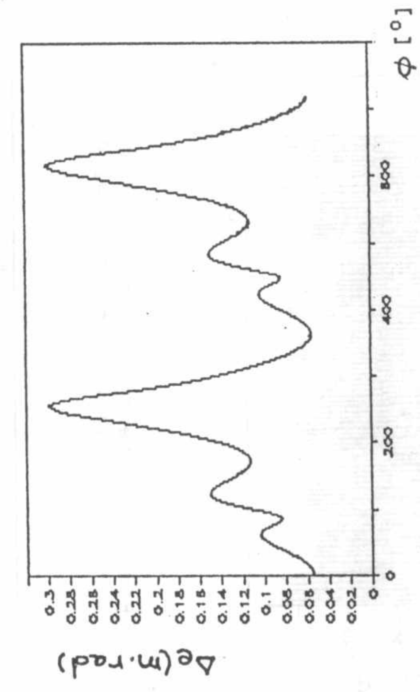

Fig.2-b Evolution of Attitude Error (v_o=0.2m/s f_j/f_s=1)

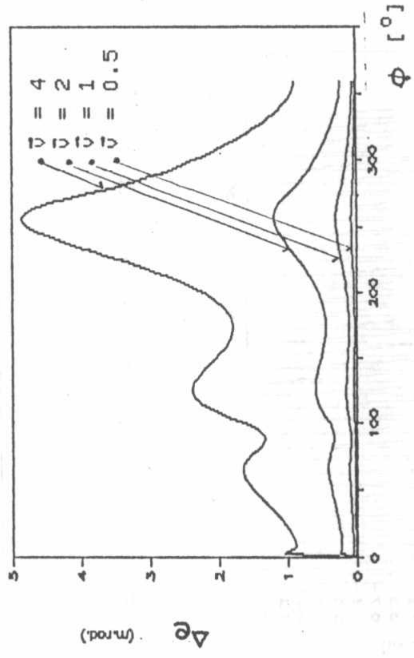

Fig.3-b Effect of EEF Speed on Attitude Error(f_j/f_s=1)

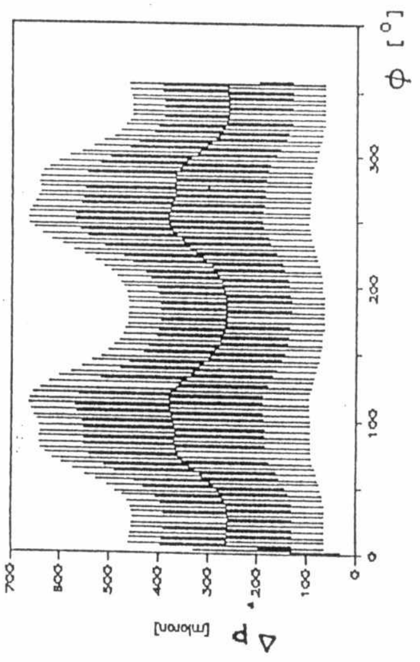

Fig.4-a Effect of Jacobian Updating on Position Error(f_j/f_s=1/4)

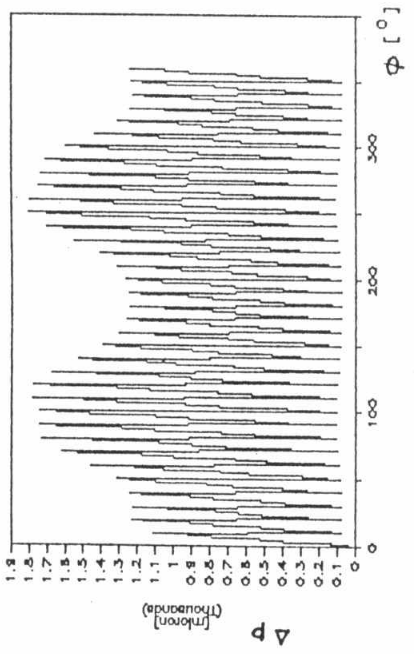

Fig.5-a Effect of Jacobian Updating on Postion Error(f_j/f_s=1/10)

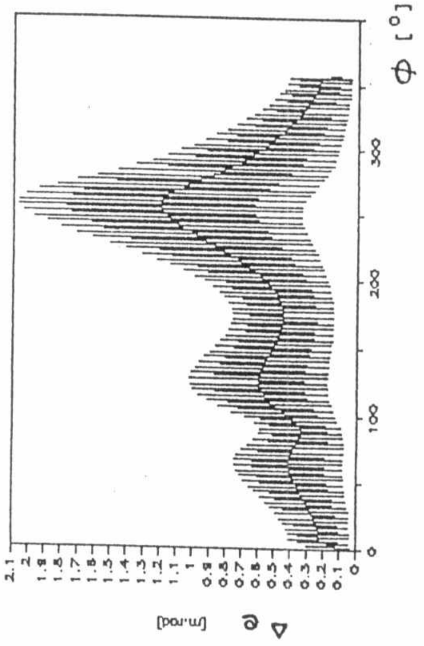

Fig.4-b Effect of Jacobian Updating on Attitude Error(f_j/f_s=1/4)

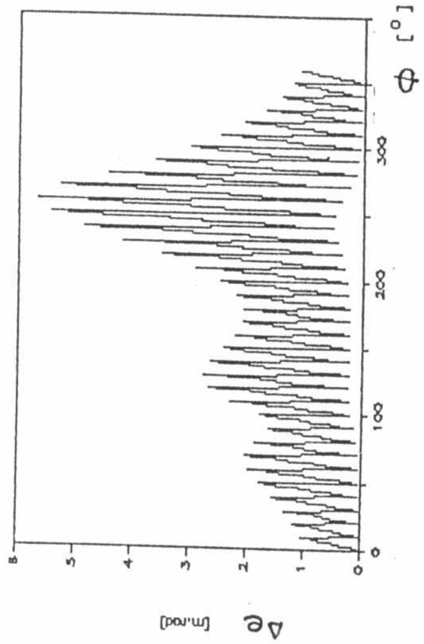

Fig.5-b Effect of Jacobian Updating on Attitude Error(f_j/f_s=1/10)

Figs.(2.a,b) show respectively the evolution of position and attitude errors for two successive turnes of the EEF over the circular path with a reference speed $v = v_o = 0.2$ m/s. The Jacobian here is updated every computation step. It can be seen that both errors are bounded (this is due to the effect of the position correction loop used). The of magnitude of errors are $\Delta_e \leq 0.3$ m.rad. and $\Delta_p \leq 0.1\ \mu$m.
Figs.(3.a,b) represent the evolution of errors for different values of speed v as indicated by the dimensionless parameter $\overline{v} = v/v_o$. Both errors are again bounded, however, the order of magnitude of errors increases as $\overline{v}$ increases.
The main disadvantage of using a DKES based on the Jacobian inversion is the computation effort associated with the matrix inversion. To overcome this problem, one may reduce the Jacobian updating frequency f_j w.r.t. the solution frequency f_s. The effect of f_j on the solution accuracy may be seen from the comparison of Figs.(4-a,b) and Figs.(5-a,b). Both are calculated for $\overline{v} = 0.5$ and for f_j/f_s respectively equal to 1/4 and 1/10. It can be recognized that as f_j decreases Δ_p and Δ_θ increase. A compromise between the computation effort and the solution accuracy may be achieved by a proper selection of f_j.

CONCLUSION

Quaternions establish a unified representation and a symmetric treatment of manipulators members position and orientation. Proposed generalized relations expressing the elements of manipulators Jacobian based on quaternion parameter- izations are derived. The study of these relations shows that they have the same physical meaning as those derived on basis of Homogeneous Transformation parameterization [1]. Using the derived relations the Jacobian matrix for the Puma 560 manipulator is obtained and tested by numerical simulation. The test results are quite satisfactory. Proper selection of the frequency of Jacobian updating enables one to reduce the computation burden associated with the Jacobian inversion keeping the level of the solution accuracy.

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ABBREVIATION AND NOTATION

EEF End Effector

DKES Differential Kinematic Equation Solver

* Denotes a quaternion product

Head Symbols : ˆ : Quaternion

→ : Vector

Bottom notation – : 4×1 column which correspond to the representation of a quaternion in a given frame.

Left superscript denotes the basis or the frame w.r.t which a quaternion representation is considered.

Right superscript c denotes the conjugate of a quaternion or its representation.

Right Subscript under a quaternion of rotation denotes the direction of rotation.