Steady –State Solution of the Queue $E_r/M/2/k/N$ With Heterogeneous Repairmen

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Abstract- The machine interference queue with Erlangian interarrival and exponential service time is solved for the steady-state case and heterogeneous repairmen .The queue discipline considered here is first in first out(FIFO). The recurrence relations that give all the probabilities in terms of $P_{k,r}$ and some measures of effectiveness are derived. Also, some special cases are deduced.

Keywords: Heterogeneous repairmen; machine interference

1- Introduction

Kleinrock [4] studied the queue : $M/M/C/k/N$, Gross and Harris [3] discussed the queue :M/M/C/ k / N with Spars , Medhi [6], Bunday [1] and White et al [9] treated the queue : M/M/C/ k / k . Shawky [7] treated the queue : $M/M/2/k/N$ with balking, reneging and two heterogeneous repairmen. The present work treats the queue: $E_r/M/2/k/N$ with heterogeneous repairmen. The recurrence relations that give all the probabilities in terms of $P_{k,r}$ and some measures of effectiveness are derived. Finally, some special cases are obtained.

2- Model Description

Consider the two heterogeneous servers (repairmen) interarrival Erlangian queue having r-stages each with interarrival rate $r\lambda$ and where its service time distribution is an exponential with service rates μ_1, and μ_2 ($\mu_1 > \mu_2$). We assume that we have a finite source (population) of N customers and the system has a finite waiting room such that the total number of customers(machines)in the system is no more than $k(k < N)$. The queue discipline considered here is a modification of both Singh ($\pi_1 = 1, \pi_2 = 0$)[8] and Krishnamoorthi [5], and it shows that:

(i) If both repairmen are free, the head customer of the queue goes to the first repairman with probability π_1 or to the second repairman with probability π_2 thus $\pi_1 + \pi_2 = 1$.

(ii) If only one repairman is free, the head machine goes directly to it

(iii) If the two repairmen are busy, the machines wait in their order until any repairman becomes available.

3- The Steady- State equations and their solution

Define the equilibrium probabilities:

$p_{0,0,s}$= prob. {there is no machine in the system and s^{th} arrival stage occupied the next arriving machine},

$p_{1,0,s}$= prob. {there is one machine in repairman $I$and s^{th} arrival stage occupied the next arriving machine},

$p_{0,1,s}$= prob. {there is one machine in repairman $II$and s^{th} arrival stage occupied the next arriving machine},

$p_{n,s}$= prob. {there are n machines in the system and s^{th} arrival stage occupied the next arriving machine} , n=2.,3,..k; s =1,2,… r .

Also, $p_{0,s} = p_{0,0,s}$, $p_{1,s} = p_{1,0,s} + p_{0,1,s}$ and $p_{2,s} = p_{1,1,s}$.

Consequently, the steady-state probability difference equations are:

$$\left.\begin{aligned} Nr\lambda\, p_{0,0,1} &= \mu_1 p_{1,0,1} + \mu_2 p_{0,1,1} \\ Nr\lambda\, p_{0,0,s} &= Nr\lambda\, p_{0,0.s-1} + \mu_1 p_{1,0,s} + \mu_2 p_{0,1,s} \quad, s = 2(1)r \end{aligned}\right\}, \qquad (1)$$

$$\left.\begin{aligned} \{(N-1\}r\lambda + \mu_1\} p_{1,0,1} &= Nr\lambda\pi_1 p_{0,0,r} + \mu_2 p_{1,1,1} \\ \{(N-1\}r\lambda + \mu_1\} p_{1,0,s} &= (N-1)r\lambda, p_{1,0,s-1} + \mu_2 p_{1,1,s} \quad, s = 2(1)r \\ \{(N-1\}r\lambda + \mu_2\} p_{0,1,1} &= Nrr\lambda\pi_2, p_{0,0,r} + \mu_1 p_{1,1,1} \\ \{(N-1\}r\lambda + \mu_2\} p_{0,1,s} &= (N-1)r\lambda, p_{1,0,s-1} + \mu_1 p_{1,1,s} \quad, s = 2(1)r \end{aligned}\right\}, \qquad (2)$$

$$\left.\begin{aligned} \{(N-n\}r\lambda + \mu\} p_{n,,1} &= (N-n+1)r\lambda\, p_{n-1,r} + \mu p_{n+1,1} \\ \{(N-n\}r\lambda + \mu\} p_{n,,s} &= (N-n)r\lambda p_{n,s-1} + \mu\, p_{n+1,s}, \qquad s = 2(1)r \end{aligned}\right\}, n = 2(1)\, k\text{-}1, \quad (3)$$

$$\left.\begin{aligned} \{(N-k\}r\lambda + \mu\} p_{k,1} &= (N-k+1)r\lambda\,, p_{k-1,r} + (N-k)r\lambda\, p_{k,r} \\ \{(N-k\}r\lambda + \mu\} p_{k,,s} &= (N-k)r\lambda\,, p_{k,s-1}, \qquad s = 2(1)r \end{aligned}\right\}, \qquad (4)$$

where : $\mu = \mu_1 + \mu_2$.

Summing up equation (1) over s we get:

$$Nr\lambda\, p_{0,0,r} = \mu_1 \sum_s p_{1,0,s} + \mu_2 \sum_s p_{0,1,s}. \qquad (5)$$

Summing up equation (2) over s, and using (5) we obtain :

$$(N-1)r\lambda\, p_{1,r} = \mu \sum_s p_{2,s} \qquad (6)$$

Also , summing up equations (3) over s at n =2 and using (6) we obtain:

$$(N-2)r\lambda\, p_{2,r} = \mu \sum_s p_{3,s} \qquad (7)$$

Similarly, summing up equations (3) over s for n =3(1) k-1 and using (7) we obtain:

$$(N-n)r\lambda\, p_{n,r} = \mu \sum_s p_{n+1,s} \quad , \quad n = 3(1)\, k-1 \qquad (8)$$

From the second equation of (4), we get :

$$p_{k,s} = (1+\gamma_k)^{r-s}\, p_{k,r} \;, \qquad s=1(1)\, r \qquad (9)$$

From equations (3) and (8) we get :

$$p_{n,r} = \gamma_n \sum_s p_{n+1,s} \,,$$

and

$$p_{n,s-1} = (1+\gamma_n)p_{n,s} - \gamma_n p_{n+1,s} \;, \; s=1(1)r\, ;\, n=2(1)\, k\text{-}1,$$

or

$$p_{n,s} = \gamma_n (1+\gamma_n)^{r-s} \left\{ \sum_{i=1}^{r} p_{n+1,i} - \sum_{i=0}^{r-s-1} \left(\frac{1}{1+\gamma_n}\right)^{i+1} p_{n+1,r-i} \right\},$$

$$n = 2(1)k-1;\; s = 1(1)r-1,$$

and

$$p_{n,r} = \gamma_n \sum_{i=1}^{r} p_{n+1,i} \;, n = 2\,(1)k-1 \qquad (10)$$

where: $\gamma_n = \frac{\mu}{(N-n)r\lambda},\; n = 2\,(1)k-1.$.

From equations (1) and (2) we have:

$$p_{0,0,1} = \Phi_1 p_{1,0,1} + \Phi_2 p_{0,1,1}$$

$$p_{0,0,s} = p_{0,0,s-1} + \Phi_1 p_{1,0,s} + \Phi_2 p_{0,1,s}, \qquad s = 1(1)r$$

or

$$p_{0,0,s} = \sum_{i=1}^{s} \left(\Phi_1 p_{1,0,i} + \Phi_2 p_{0,1,i}\right), \qquad s = 1(1)r. \qquad (11)$$

Thus , from equation (2) we get:

$p_{1,0,1} = l_1 \pi_1 p_{0,0,r} + g_{2,1} p_{2,1},$

$P_{1,0,s} = l_1 P_{1,,0,s-1} + g_{2,1} P_{2,s}, \qquad s = 2(1)r$

or

$$p_{1,0,s} = l_1^s \;\; \pi_1 p_{0,0,r} + g_{2,1} \sum_{i=1}^{s} l_1^{s-i}\, p_{2,i}\,, \qquad s = 1(1)r\,. \tag{12}$$

Also, from the equation (2) we get:

$$p_{0,1,s} = l_2^s \pi_2 p_{0,0,r} + g_{1,2} \sum_{i=1}^{s} l_2^{s-i}\, p_{2,i}\;, \qquad s=1(1)r, \tag{13}$$

where , $\Phi_i = \frac{\mu_i}{Nr\lambda}(i = 1,2),\;\; l_i = \frac{1}{1+\Phi_i}(i = 1,2),\;\; g_{1,2} = \frac{\mu_1}{Nr\lambda+\mu_2} \quad and \quad g_{2,1} = \frac{\mu_2}{Nr\lambda+\mu_1}$.

From (12) and (13) we get:

$$p_{1,s} = (l_1^s \pi_1 + l_2^s \pi_2) p_{0,0,r} + \sum_{i=1}^{s} (g_{2,1} l_1^{s-i} + g_{2,1} l_1^{s-i} + g_{1,2} l_2^{s-i}) p_{2,1}\;, \text{ s} = (1)\, r \tag{14}$$

From equations (6) and (14) with s= r we get:

$$p_{0,0,r} = \frac{1}{l_1^r \pi_1 + l_2^r \pi_2} \sum_{i=1}^{r} \eta_i\;\; p_{2,i} \tag{15}$$

where , $\qquad \eta_i = \frac{\mu}{(N-1)r\lambda} + g_{2,1} l_1^{r-i} + g_{1,2} l_2^{r-i}\;, i = 1,2$

Thus:

$$\left.\begin{aligned} p_{1,0,s} &= \frac{l_1^s \pi_1}{l_1^r \pi_1 + l_2^r \pi_2} \sum_{i=1}^{r} \eta_i\, p_{2,i} + g_{2,1} \sum_{i=1}^{s} l_2^{s-i} p_{2,i} \\ p_{0,1,s} &= \frac{l_2^s \pi_2}{l_1^r \pi_1 + l_2^r \pi_2} \sum_{i=1}^{r} \eta_i\, p_{2,i} + g_{1,2} \sum_{i=1}^{s} l_2^{s-i} p_{2,i} \end{aligned}\right\}\;, \text{S}=1(1)\, r \tag{16}$$

Using the equations (9)-(14), we get the recurrence relations as:

$$p_{k,s} = (1+\gamma_k)^{r-s} p_{k,r,} \tag{17}$$

$$p_{n,s} = \gamma_n (1+\gamma_n)^{r-s} \left\{ \sum_{i=1}^{r} p_{n+1,i} - \sum_{i=0}^{r-s-1} \left(\frac{1}{1+\gamma_n}\right)^{i+1} p_{n+1,r-i} \right\}$$

$$, n = 2\,(1)k-1\,; s = 1\,(1)r-1$$

$$p_{n,r} = \gamma_n \;\; \sum_{i=1}^{r} p_{n+1,i}\;\;, n = 2\,(1)k-1\,, \tag{18}$$

$$p_{1,0,s} = \frac{l_1^s \pi_1}{l_1^r \pi_1 + l_2^r \pi_2} \sum_{i=1}^{r} \eta_i\, p_{2,i} + g_{2,1} \sum_{i=1}^{s} l_1^{s-i} p_{2,i}\;\;, \tag{19}$$

$$p_{0,1,s} = \frac{l_2^s \pi_2}{l_1^r \pi_1 + l_2^r \pi_2} \sum_{i=1}^{r} \eta_i\, p_{2,i} + g_{1,2} \sum_{i=1}^{s} l_2^{s-i} p_{2,i}\;\;, \tag{20}$$

and

$$p_{0,0,s} = \sum_{i=1}^{s} \big(\Phi_1 p_{1,0,i} + \Phi_2 p_{0,1,i}\big)\,,\;\; s = 1(1)r. \tag{21}$$

Equations (17) – (21) are the required recurrence relations that give all probabilities in terms of $P_{k,r}$ which itself may now be determined by using the normalizing condition:

$\sum_{n=0}^{k} \sum_{s=1}^{r} p_{n,s} = 1$, hence all the probabilities are completely known in terms of the queue parameter.

Thus, the expected number of machines in the system and in the queue are calculated from the formula

$L = \sum_{n=1}^{k} \sum_{s=1}^{r} n\, p_{n,s}$, $L_q = \sum_{n=1}^{k} \sum_{s=1}^{r} (n-2) p_{n,s}$

and the expected waiting in the queue and in the system

$$w_q = \frac{L_q}{\check{\lambda}} \quad , \quad w = \frac{L}{\check{\lambda}} \qquad \text{where} \quad \check{\lambda} = \frac{\mu}{2}\left(L - L_q\right).$$

4-Numerical Work

The following example illustrates the method discussed above.

Example: In the system: $E_r\ /M/2/k/N$ letting r =2, k =3 and $N = 4$, i.e., the queue : $E_2\ /M/2/3/4$, the results are:

$$P_{3,1} = a P_{3,2}\ , \quad P_{2,1} = b_1 P_{3,2}\ , \quad P_{2,2} = b_2\, P_{3,2}, \quad , P_{1,0,1} = c_1 P_{3,2}\ ,$$

$$P_{1,0,2} = c_2 P_{3,2}\ , \quad P_{0,1,1} = d_1 P_{3,2}\ , \quad P_{0,1,2} = d_2 P_{3,2}\ , \quad P_{0,0,1} = e_1 P_{3,2}\ ,$$

$$P_{0,0,2} = e_2 P_{3,2}\ , \quad P_0 = P_{0,0,1} + P_{0,0,2}.$$

where:

$$a = (1+\gamma_3)\ , \quad b_1 = \gamma_2[(1+\gamma_2)(a+1)-1], \quad b_2 = \gamma_2(a+1)\ ,$$

$$c_1 = \frac{l_1\pi_1}{l_1^2\pi_1 + l_2^2\pi_2}\left(\eta_1 b_1 + \eta_2 b_2\right) + g_{2,1} b_1, \qquad l_1 = \frac{8\lambda}{8\lambda+\mu_1}, \quad \phi_1 = \frac{\mu_1}{8\lambda}$$

$$c_2 = \frac{l_1^2\pi_1}{l_1^2\pi_1 + l_2^2\pi_2}\left(\eta_1 b_1 + \eta_2 b_2\right) + g_{2,1}(l_1 b_1 + b_2)\ , \quad l_2 = \frac{8\lambda}{8\lambda+\mu_2}\ , \phi_2 = \frac{\mu_2}{8\lambda}$$

$$d_1 = \frac{l_2\pi_2}{l_1^2\pi_1 + l_2^2\pi_2}\left(\eta_1 b_1 + \eta_2 b_2\right) + g_{1,2} b_1\ , \qquad g_{1,2} = \frac{\mu_1}{8\lambda+\mu_2}$$

$$d_2 = \frac{l_2^2\pi_2}{l_1^2\pi_1 + l_2^2\pi_2}\left(\eta_1 b_1 + \eta_2 b_2\right) + g_{1,2}(l_2 b_1 + b_2)\ , \quad g_{2,1} = \frac{\mu_2}{8\lambda+\mu_1},$$

$$e_1 = \phi_1 c_1 + \phi_2 d_1\ , \quad e_2 = \phi_1(c_1+c_2) + \phi_2(d_1+d_2), \ \gamma_3 = \frac{\mu}{2\lambda}\ , \quad \gamma_2 = \frac{\mu}{4\lambda}$$

and

$$\eta_i = \frac{\mu}{6\lambda} + g_{2,1}\, l_1^{2-i} + g_{1,2}\, l_2^{2-i}\ , i = 1,2\ .$$

From the normalizing condition: $\sum_{n=0}^{3} \sum_{s=1}^{2} P_{n,s} = 1$, we have

$$P_{3,2}^{-1} = [1 + a + b_1 + b_2 + c_1 + c_2 + d_1 + d_2 + e_1 + e_2\,]\ .$$

Therefore, the expected numbers of customers in the system and in the queue are, respectively.

$L = [\, c_1 + c_2 + d_1 + d_2 + 2b_1 + 2b_2 + 3a + 3\,]P_{3,2}$

$Lq = (a+1)P_{3,2}$,

and the expected waiting time in both the system and the queue are obtained by

$$w = \frac{L}{\tilde{\lambda}} \quad and\; w_q = \frac{L_q}{\tilde{\lambda}}$$

where $\tilde{\lambda} = \frac{\mu}{2}\left(L - L_q\right), \mu = \mu_1 + \mu_2.$

Now, we introduce the two tables for some measures of effectiveness at $\pi_1 = 0.6$ and $\pi_2 = 0.4$ for the different values of λ and μ when one of them is fixed.

Table.1: $\mu = 10$, $(\mu_1 = 6, \mu_2 = 4)$

λ	P_0	L	Lq	w	w_q
3	0.170	1.21	0.0924	0.216	0.0165
4	0.115	1.44	0.162	0.225	0.0253
5	0.0803	1.64	0.236	0.234	0.0337
6	0.0507	1.81	0.306	0.241	0.0408
7	0.0423	1.97	0.372	0.246	0.0465
8	0.032	2.08	0.430	0.252	0.0521

Table 2: $\lambda = 3$, $(\mu_2 = 2, \mu_1 = \mu - 2)$

μ	P_0	L	Lq	w	w_q
5	0.0573	1.81	0.306	0.483	0.0816
6	0.0785	1.64	0.236	0.390	0.0562
7	0.100	1.52	0.185	0.324	0.0394
8	0.121	1.41	0.147	0.279	0.0291
9	0.140	1.31	0.117	0.245	0.0219
10	0.159	1.23	0.0950	0.216	0.0167

From the numerical results, it is clear that $L, L_q, w, and\; w_q$ are increasing as λ is increasing and decreasing as μ is increasing. While P_0 is increasing as μ increasing and decreasing as λ increasing.

5- special Cases

Some queuing systems can be obtained as special cases of this model:

(i) If r=1 we get the model : M/M/2/ k /N, which was studied by Shawky [7] at $\alpha = 0$ and $\beta = 1$

(ii) If . $k = N \to \infty, r$ =1 ,$\pi_1 = 1$, and $\pi_2 = 0$ we obtain the queue M/M/2 which was studied by Singh[8]

(iii) If $k = N = m$, $\mu_1 = \mu_2$ and $\pi_1 = \pi_2 = 1/2$,we get the homogeneous repairmen model :$E_r/M/2/\ m/m$, which was studied by El-Paoumy [2] at C=2

Also, if r =1, the system becomes : M/M/2/ k / k which was discussed by White et al. [8], Medhi [6], Gross and Harris [3] and Bunday [1].

6-Conclusion

In this paper, the machine interference model: $E_r\ /M/2/k/N$ is studied with two heterogeneous repairmen. The recurrence relations that give all the probabilities in terms of $P_{k,r}$ are derived. We illustrated the method by a numerical example and deduced the expected numbers of units in the system and in the queue, also the expected waiting time in the system and in the queue are derived. Some special cases are obtained.

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