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A TERRESTRIAL STRAPDOWN NAVIGATION ALGORITHM

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ABSTRACT

In this paper terrestrial strapdown navigation computation is investigated. A case study of a navigation computation in local level navigation frame is presented. For this case, the basic equations of the navigation algorithm are derived. A simulation scheme for the evaluation of the algorithm is proposed. The evaluation is based on two criteria. One criterion is associated with the position error. The other is associated with attitude error of the vehicle.
The results of simulation show a quite reasonable performance of the navigation algorithm under investigation.

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1. INTRODUCTION

Figure.1 depicts the main computational schemes implemented by software algorithms in typical strapdown Inertial Navigation Systems (INS). The input data to the algorithms is provided by a triad of strapdown gyros and accelerometers.

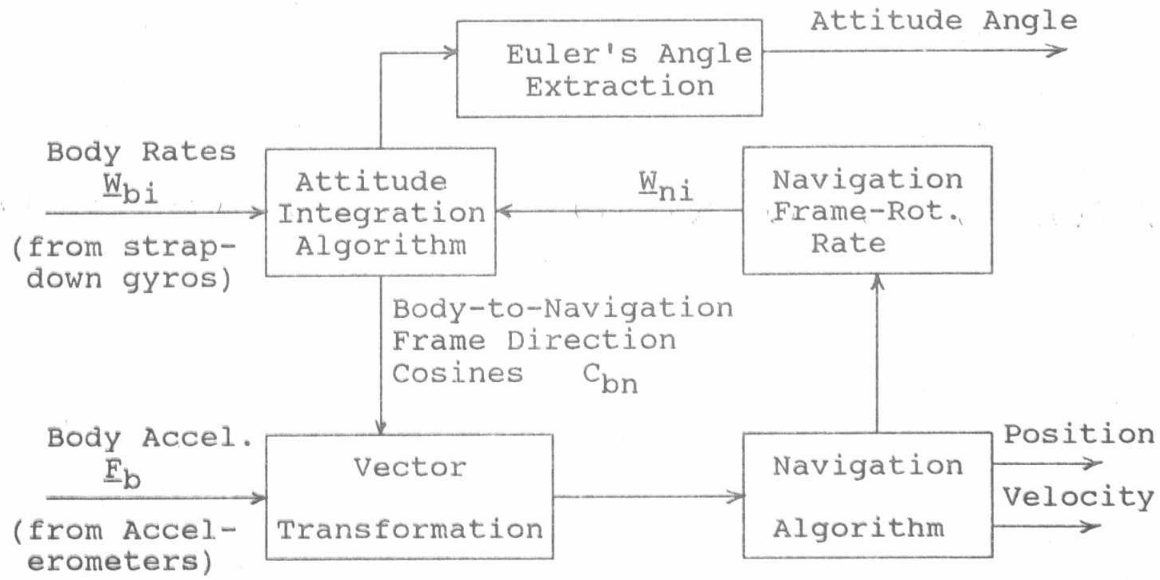


Fig.1 Strapdown Algorithms

The accelerometers provide 3-orthogonal components of the nongravitational acceleration $\underline{F}_b$ of the vehicle w.r.t. the inertial frame along the body axes. The gyros provide the vehicle body frame rotation rate $\underline{W}_{bi}$ w.r.t. the inertial frame. $\underline{W}_{bi}$, plus the data corresponding to the rotation rate of the navigation frame are processed by an attitude integration algorithm to obtain the attitude of the body frame relative to the navigation frame and by consequence the corresponding transformation matrix as a function of time.

The attitude information of body frame relative to navigation frame (transformation matrix) is used to transform the nongravitional acceleration measured in body axes into the navigation frame. The transformed components are complemented by the gravity acceleration and integrated in the navigation frame to calculate the velocity and position of the vehicle.

The inaccuracies in attitude integration represent the major contributers of strapdown position errors [1]. The greatest research effort, is therefore, directed towards the devlopment of software algorithms for attitude computation [1-7]. Relatively little attention has been given, at least in the open literature, to the navigation computation and its optimization. The main objective of this optimization is to reduce the computation burden and to improve its accuracy. Several navigation computation schemes have been proposed [1,5].

In this work a case study of terrestrial strapdown

navigation algorithm in the Local-Level-Navigation Frame (L.L.N.F.) is investigated. The main navigation equations are derived. A simulation scheme for the evaluation of the algorithm is presented. The simulation results show quite satisfactory performance.

NOMENCLATURE

C_{bn}	the direction cosine matrix used to transform vectors from the b-frame into n-frame.
$\frac{d_r}{dt}$	the rate of change w.r.t. r-frame.
ψ, θ, τ	the Yaw, Pitch, and Roll orientation angles.
$\underline{F}$	the specific force vector.
$\underline{G}$	the gravitational vector.
$\underline{g}$	the gravity vector.
Ω	the earth rotation rate vector relative to inertial frame.
lt, lg	the geographic latitude and longitude.
h	the vehicle's altitude.
R_o	the equatorial radius of the earth.
$\underline{R}$	the position vector from the center of the earth to the vehicle's position over earth surface.
$\dot{\underline{R}}_r$	the rate of change of $\underline{R}$ in an arbitrary frame r.
$\ddot{\underline{R}}_i$	the inertially referenced acceleration; or the second derivative of the position vector $\underline{R}$ w.r.t. i-frame.
$\underline{V} \equiv \dot{\underline{R}}_i$	the rate of change of $\underline{R}$ w.r.t. inertial i-frame.
$\underline{U} \equiv \dot{\underline{R}}_e$	the Ground velocity; or rate of change of $\underline{R}$ w.r.t. the earth fixed frame.
$\dot{\underline{U}}_r$	the time derivative of ground velocity $\underline{U}$ in an arbitrary frame r.
$\underline{W}_{br}$	the angular velocity of b-frame w.r.t. r-frame.
$\underline{W}_{br}^{b}$	the angular velocity of b-frame relative to r-frame coordinatized in the b-frame.
$\underline{U}_n$	the ground velocity coordinatized in the n-frame.
$g(\underline{R})_n$	the gravity vector coordinatized in the n-frame.
g_o	the gravitational acceleration at earth's surface.
N, E, D	denote, respectively, the North, East, and Down directions.

The reference frames used are selected to be as shown in fig.(2), taking into consideration that the L.L.N.F has its axes aligned with the North (N), East (E), and Down (D) directions.

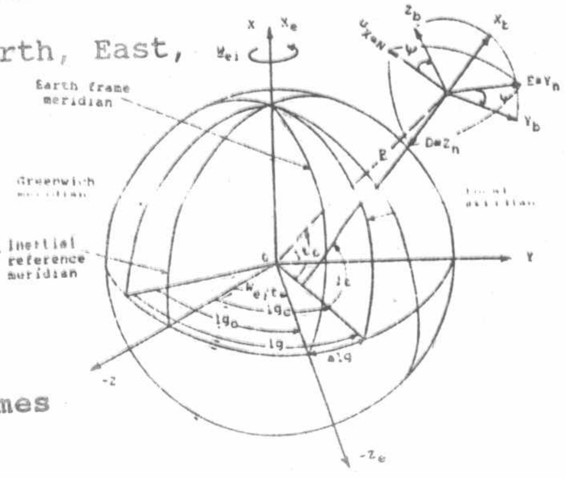


Fig.(2) Ref. Frames

2. STRAPDOWN NAVIGATION BASED ON L.L.N.F. COMPUTATION SCHEME

2.1. Basic equations of inertial navigation

From Newton's second law the output from the accelerometer is proportional to the difference between the inertially referenced acceleration and the net gravitational accelerations at the instrument's location. This difference between inertial and gravitational acceleration is identically equal to the contact force exerted on the accelerometer set by its support structure. Thus, the accelerometer output is written as :

$$\underline{F} = \frac{d_i^2 \underline{R}}{dt^2} - \underline{G} \qquad (1)$$

Where;

$\underline{F}$ represents the sensed acceleration or the Specific force exerted on the instrument set (nonfield contact force per unit mass)

$\frac{d_i^2 \underline{R}}{dt^2}$ is the acceleration of the vehicle w.r.t. the inertial frame.

$\underline{G}$ is the Gravitational acceleration at the system location.

2.2. Computation of ground-velocity in the L.L.N.F.

The rate of change of position vector $\underline{R}$ relative to the inertial frame can be written as

$$\frac{d_i \underline{R}}{dt} = \frac{d_e \underline{R}}{dt} + \underline{W}_{ei} \times \underline{R} \quad \text{or;}$$

$$\underline{V} = \underline{U} + \underline{W}_{ei} \times \underline{R} \qquad (2)$$

Differentiating eqn(2) w.r.t. time in inertial frame yields

$$\frac{d_i \underline{V}}{dt} = \frac{d_i \underline{U}}{dt} + \frac{d_i}{dt} (\underline{W}_{ei} \times \underline{R}) \qquad (3)$$

Since however;

$$\frac{d_i \underline{U}}{dt} = \frac{d_n \underline{U}}{dt} + \underline{W}_{ni} \times \underline{U} \qquad (4)$$

$$\frac{d_i}{dt} (\underline{W}_{ei} \times \underline{R}) = \underline{W}_{ei} \times \underline{U} + \underline{W}_{ei} \times (\underline{W}_{ei} \times \underline{R}) \qquad (5)$$

$$\frac{d_i \underline{V}}{dt} = \frac{d_i^2 \underline{R}}{dt^2} = \ddot{\underline{R}}_i \qquad (6)$$

Then, substituting Eqs(4) and (5) into (3) yields

$$\ddot{\underline{R}}_i = \frac{d_n \underline{U}}{dt} + (\underline{W}_{ni} + \underline{W}_{ei}) \times \underline{U} + \underline{W}_{ei} \times (\underline{W}_{ei} \times \underline{R}) \qquad (7)$$

Where;

$\underline{W}_{ni}$ is the angular velocity of L.L.N.F. relative to Inertial Frame (I.F.). It may be written as

$$\underline{W}_{ni} = \underline{W}_{ne} + \underline{W}_{ei} \qquad (8)$$

$\underline{W}_{ne}$ is the angular velocity of the navigation frame relative to earth. It is known as the transport rate.

Then, eqs(1) and (7) yields

$$\frac{d_n\underline{U}}{dt} = -(2\underline{W}_{ei} + \underline{W}_{ne})x\underline{U} + \underline{F} + \underline{g} \qquad (9)$$

Where;

$\underline{g}$ is the gravity vector defined by eqn(12).

The earth rate $\underline{W}_{ei}$ expressed in the L.L.N.F. can be written as,

$$\underline{W}^n_{ei} = \begin{bmatrix} \Omega\cos(lt) & 0 & -\Omega\sin(lt) \end{bmatrix} \begin{bmatrix} N \\ E \\ D \end{bmatrix} \qquad (10)$$

The transport rate $\underline{W}_{ne}$ may be expressed in the L.L.N.F. as, [5,9,10]

$$W^n_{ne} = \begin{bmatrix} \dfrac{U_E}{R_E + h} & -\dfrac{U_N}{R_N + h} & -\dfrac{U_E\tan(lt)}{R_E + h} \end{bmatrix} \begin{bmatrix} N \\ E \\ D \end{bmatrix} \qquad (11)$$

GRAVITY Model

The relationship between the Gravitational acceleration $\underline{G}_n$ and the Gravity vector $\underline{g}$ has the form [1,5,9,10];

$$\underline{G} = \underline{g}(\underline{R}) + \underline{W}_{ei} \times (\underline{W}_{ei} \times \underline{R}) \qquad (12)$$

and

$$g(\underline{R}) = \begin{bmatrix} 0 \\ 0 \\ g_o\left(1 + \alpha_1\sin^2(lt)\right)\left(1 - \dfrac{2h}{R_o\left(1 - \frac{E^2}{2}\sin^2(lt)\right)}\right) \end{bmatrix} \qquad (13)$$

Where;

g_o is the universal earth gravity at sea level.
h is the vehicle altitude.
lt is the geographic latitude.
R_o is the equatorial radius of the earth.
E is the earth excentricity.
$\alpha_1 = 0.0052884$

Integrating eqn(9) in the L.L.N.F. yields

$$\underline{U} = \int_n(\underline{F} + \underline{g} - (2\underline{W}_{ei} + \underline{W}_{ne})x\underline{U})dt \qquad (14)$$

For the realization of eqn(14) it is necessary to express the individual vectors w.r.t. L.L.N.F. The specific force vector which is measured in the body frame should,

therefore, be transformed first to the L.L.N.F.
For detailed discussion about the problem of vector transformation refer to [1].
Eqn(14) expresses the evolution of the vehicle velocity w.r.t. the earth expressed in the L.L.N.F.

2.3. Computation of the vehicle's position

The transport rate $\underline{W}_{ne}$ may be expressed in the L.L.N.F. in terms of the derivatives of latitude (lt) and longitude (lg) as [1,8,10]

$$W^{n}_{ne} = \begin{bmatrix} lg^{\cdot}\cos(lt) & -lt^{\cdot} & -lg^{\cdot}\sin(lt) \end{bmatrix} \begin{bmatrix} N \\ E \\ D \end{bmatrix} \tag{15}$$

Comparing eqs(15) and (11) yields

$$\begin{aligned} h^{\cdot} &= -U_D \\ lt^{\cdot} &= \frac{U_N}{R_N + h} \\ lg^{\cdot} &= \frac{U_E}{(R_E + h)\cos(lt)} \end{aligned} \tag{16}$$

Where; R_N and R_E are the meridian and transverse radii of curvature of the earth respectively.
The integration of eqs(16) yields the relation for latitude, longitude, and altitude updating in the form

$$\begin{aligned} lt(t+\Delta t) &= lt(t) + lt^{\cdot}\ \Delta t \\ lg(t+\Delta t) &= lg(t) + lg^{\cdot}\ \Delta t \\ h(t+\Delta t) &= h(t) + h^{\cdot}\ \Delta t \end{aligned} \tag{17}$$

The Algorithm of solving the navigational differential equations is as follows :

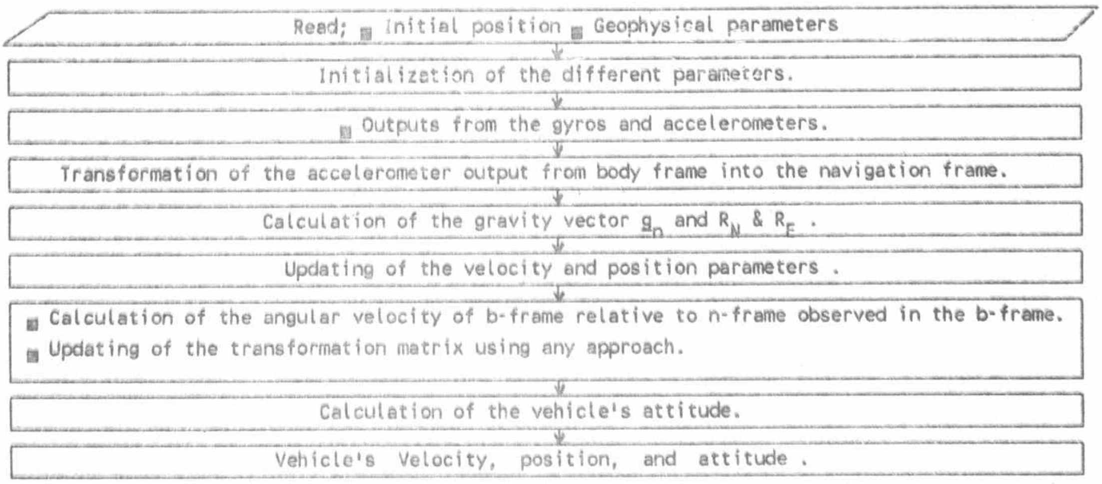


Fig.(3) Strapdown Navigation Algorithm

3. SIMULATION SCHEME FOR THE VALIDATION OF THE NAVIGATION ALGORITHM

The main features of the proposed simulation scheme is shown in Fig.(4). A reference time based vehicle trajectory is selected. For this trajectory the simulated gyros and accelerometers measurements are calculated and given as inputs to the navigation algorithm under investigation. The navigation algorithm processes the sensors measurements and hence a calculated time based trajectory is obtained.

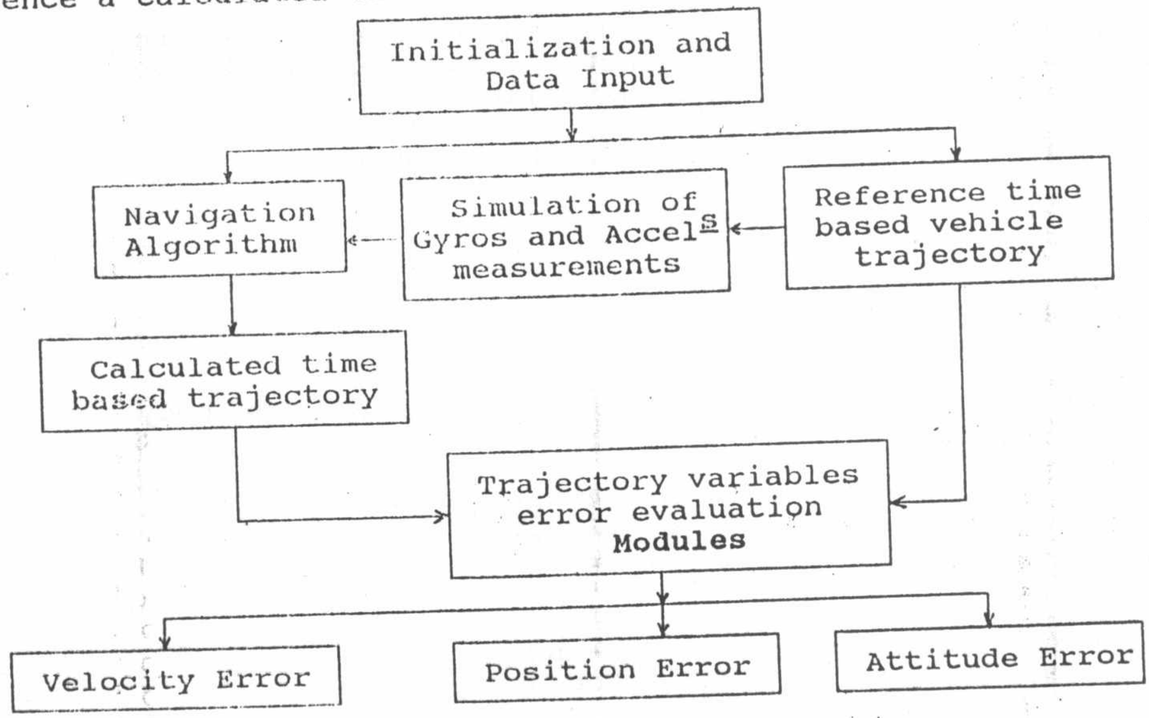


Fig.(4) Navigation Algorithm Validation Scheme

The difference between the reference trajectory and the calculated trajectory are evaluated through the computation of following error criteria :

1- **Position-Error Criteria:**
Position error may be calculated by the study of the latitude, longitude, and altitude errors which represent the difference between the values corresponding to the reference trajectory and those corresponding to the calculated one.

2- **Attitude-Error Criterion:**
An error matrix Δ is obtained as follows [1,8]

$$\Delta = [C^c_{bn} - C^r_{bn}] [C^r_{bn}]^T \qquad (18)$$

Where C_{bn} is the body to navigation frame transformation matrix, and superscripts c & r correspond respectively to calculated and reference trajectories.
The matrix Δ is skew symmetric . It consists of the infinitesimal angular deviations $\Delta\theta_1$, $\Delta\theta_2$, $\Delta\theta_3$ between the body frames associated with the calculated and reference trajectories in the form ;

$$\Delta = \begin{bmatrix} 0 & -\Delta\theta_3 & \Delta\theta_2 \\ \Delta\theta_3 & 0 & -\Delta\theta_1 \\ -\Delta\theta_2 & \Delta\theta_1 & 0 \end{bmatrix} \tag{19}$$

An equivalent attitude error angle is taken as an attitude-error criterion and defined by;

$$\beta = \sqrt{(\Delta\theta_1)^2 + (\Delta\theta_2)^2 + (\Delta\theta_3)^2} \tag{20}$$

The reference time based vehicle trajectory used in simulation hereafter has been selected intentionally to be of simple analytical form defined as follows :

i) A translatory ground velocity with components :

$$\begin{aligned} U_N(t) &= 300 + 100\ t \quad [m/sec] \\ U_E(t) &= 300 + 100\ t \quad [m/sec] \\ U_D(t) &= -300 - 100\ t \quad [m/sec] \end{aligned} \tag{21}$$

ii) Euler angles defining the orientation of the body frame w.r.t. the L.L.N.F. are given by

$$\begin{aligned} \psi &= A_1 \sin(2\pi z_a t) + A_2 \sin(2\pi z_e t) \quad [rad] \\ \theta &= A_1 \sin(2\pi z_a t) + A_2 \sin(2\pi z_t t) \quad [rad] \\ \tau &= A_1 \sin(2\pi z_a t) + A_2 \sin(2\pi z_g t) \quad [rad] \end{aligned} \tag{22}$$

Where,

z_a is a common frequency of Euler angles.
z_e is the frequency of the Yaw-angle (ψ).
z_t is the frequency of the Pitch-angle (θ).
z_g is the frequency of the Roll-angle (τ).
A_1 is the amplitude of the main harmonics.
A_2 is the amplitude of the superimposed harmonics.

Equations (22) show that the Euler's angles vary with time as superposition of two harmonics. The first harmonic corresponds to the low frequency body motion. While the superimposed harmonic corresponds to the higher frequency structural motion.

4. RESULTS AND DISCUSSIONS

The results of simulation are given in Figs.(5 ÷ 8) and Tables(1,2,3). Table(1) shows the maximum values of navigational errors during a time interval of 200 sec for different values of system sampling frequencies and the different indicated simulation conditions.

The effect of system sampling frequency on navigational errors are shown in Table(2) where the maximum values of errors are given for the same conditions of simulation (z_a=0.0033, z_e=0, z_t=0.005, z_g=0 and A_2=.5) and for three different values of sampling frequencies.

N. Errors / I/P Parameters					Velocity Error			Position Error			Attitude Error
Z_E	Z_T	Z_G	A_2	F_s [Hz]	V_N [Km/sec]	V_E [Km/sec]	V_D [Km/sec]	H [Km]	lt [°]	lg [°]	β [°]
.1	.2	.3	,5	1000	$7.2\ 10^{-3}$	$0.8\ 10^{-3}$	$8\ 10^{-3}$	.18	.0043	.0097	0.039
0	.005	0	.5	1000	$5.5\ 10^{-4}$	$3.3\ 10^{-4}$	$3\ 10^{-4}$	.0114	.00024	.00028	0.0023
0	.005	0	.5	128	$4\ 10^{-3}$	$3\ 10^{-3}$	$2\ 10^{-3}$	.09	.0026	.002	0.018
0	.005	0	.5	100	$5.4\ 10^{-3}$	$3.3\ 10^{-3}$	$3\ 10^{-3}$	.114	.0024	.0028	0.023
0	.005	0	.05	128	$3.5\ 10^{-3}$	$3\ 10^{-3}$	$1\ 10^{-3}$	.064	.0015	.0025	0.015
20	20	.3	.001	128	$1.8\ 10^{-2}$	$9\ 10^{-3}$	$9.2\ 10^{-3}$	.860	.017	.007	0,04
20	20	.3	.005	128	$8\ 10^{-2}$	$3.5\ 10^{-2}$	$3.5\ 10^{-2}$	4.5	.06	.025	0.2

Table(1): Maximum values of Navigational Errors (200 sec interval)

Figures (5) and (6) show the evolution of velocity, position, and attitude error for a typical case calculated at different sampling frequencies F_s; fig.(5) for $F_s = 1/t_s =$ 100 Hz, and fig.(6) for $F_s = 1000$ Hz.

The analysis of fig.(5-a,-b,-c) show that velocity, position and attitude errors accumulate with time.

The maximum values during a period of 200 sec are however, given in table(2). The magnitude of the errors is quite small.

The effect of sampling on the evolution of the errors may be seen through the comparison between fig.(5) and fig.(6).

It may be seen that all errors decrease as the sampling frequency increases. A compromise between the computation burden and the algorithm accuracy defines the basis for the selection of the system sampling frequency. The relation between the sampling period and the algorithm error in the used interval ($.001 \le t_s \le .01$) is quasilinear.

Figures (7) and (8) show the evolution of velocity, position and attitude error for another typical case calculated at the same sampling frequency and different amplitudes of the high frequency superimposed harmonics.

The error is affected by the frequency of the harmonic motion and the amplitude of the superimposed harmonics describing the structural vibration i.e. the error decreases as the amplitude or frequency of these harmonics decreases.

Investigating the errors profile shows that :

- The latitude and longitude error profiles have approximately linear form.
- The attitude error is in the form of oscillation superimposed upon approximately quasilinear increasing mean function. These oscillations are bounded with magnitude increasing with increasing the frequency of harmonic motion.

Evolution of Navigation Algorithm Errors

Fig.(7) -a-

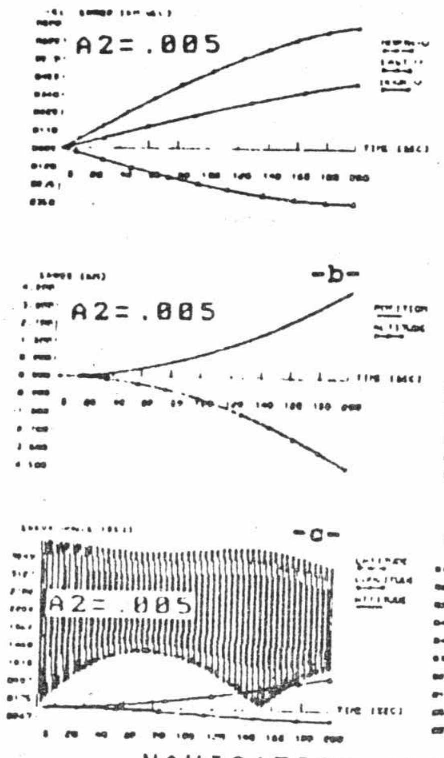
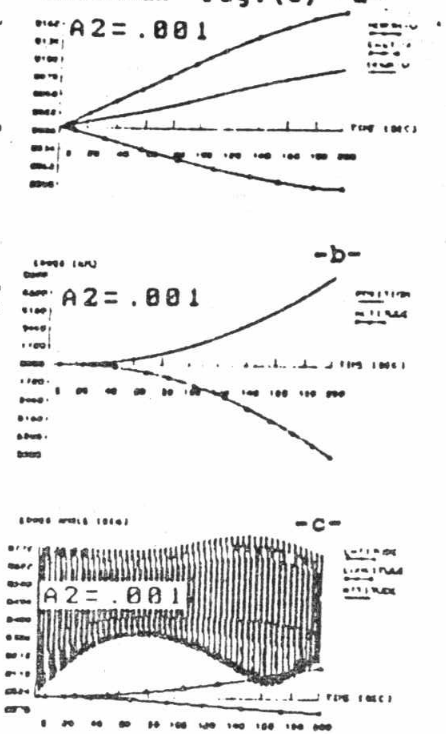


Fig.(8) -a-



NAVIGATION ALGORITHM

ZA= .0033, ZE=20, ZT=20, ZG=.3 C/SEC
Fs= 128 C/SEC

Fig.(5) -a-

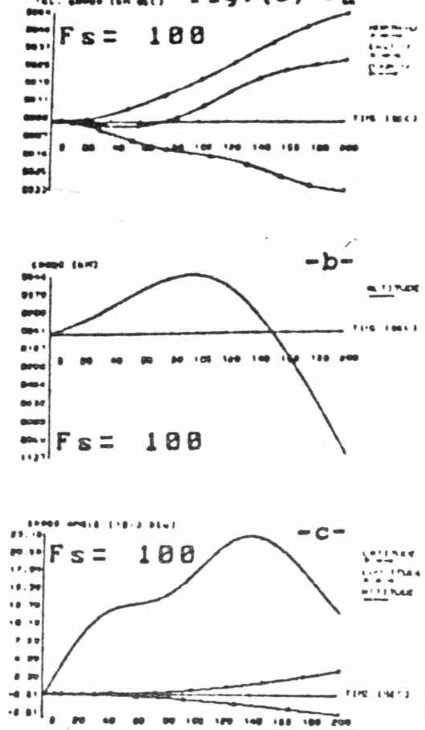
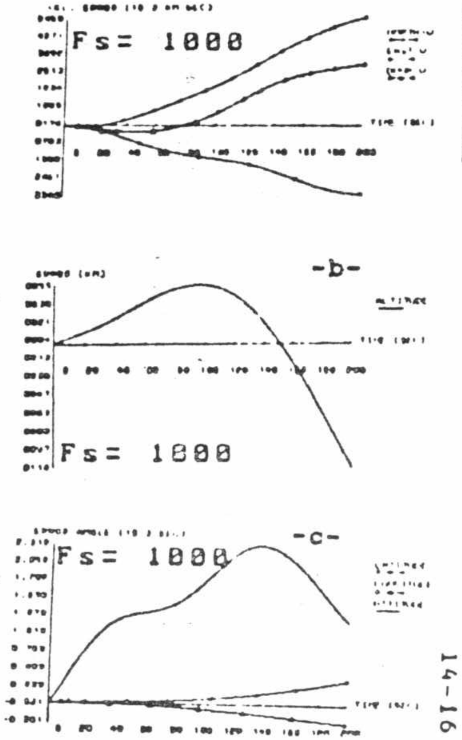


Fig.(6) -a-



NAVIGATION ALGORITHM

ZA=.0033, ZE=0, ZT=.005, ZG=0 , A2=.5

MAXIMUM ERROR x 10-3

	VN	VE	VD	H	It	Iq	φ
A2= .001	18	9	9.2	.86	17	7	40
A2= .005	80	35	35	4.5	60	25	200

Table(3): Effect of higher frequency component of attitude variation on navigational errors

MAXIMUM ERROR x 10-3

	VN	VE	VD	H	It	Iq	φ
Fs= 1000	.55	.33	.3	.01	.24	.28	2.1
Fs= 128	4	3	2	.09	2.6	2	18
Fs= 100	5.4	3.3	3	.1	2.4	[illegible].8	[illegible]

Table(2): Effect of system sampling frequency on navigational errors

5. CONCLUSIONS

A terrestrial strapdown navigation algorithm is investigated. The results of testing the algorithm by numerical simulation are presented. When the lower frequency harmonics of attitude variation are applied, the accuracy of the algorithm is quite reasonable. When the higher frequency components of attitude variation are applied, the algorithm error increases. A development in the algorithm to eliminate the errors caused by high frequency components is the subject of the current authers' investigations.

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