Modified Poisson Regression Models of Count Data and Parameter Estimation

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Abstract

This paper discusses the use of regression models of count data. It presents the estimation of the parameters for Poisson regression and zero-truncated Poisson regression models using the maximum likelihood method. We are interested in studying the performance of the estimators of modified Poisson regression models of count data. An assessment of the maximum likelihood estimates of the parameters is presented through a numerical study for different sample sizes for each model. Two empirical applications for non-truncated and truncated count data are presented, the first studies the effect of wave damage to cargo-ship and the second application investigates the length of hospital stay in days.

Keywords: Count data models; Poisson regression; generalized linear models; zero-truncated Poisson.

1. Introduction

Count data is data obtained from counting of the number of occurrences of an event of interest. The outcome observations in the count data can take only the non-negative integer values {0, 1, 2, 3 …} with or without explicit upper limit. Further, counts often display positive skew such that the frequency for low counts is considerably higher than the frequencies as the count levels increase. A univariate statistical model of event counts usually specifies a probability distribution for the number of occurrences of the event known up to the value of some parameters where the counts of events are assumed to be independently and identically distributed.

However, it is common to encounter data where the counts are affected by the values taken by one or more factor or variable. For such situations we opt to analyze the data using regression models. In count data regression, the main focus is on the effect of covariates on the frequency of an event, measured by non-negative integer values or counts. This is usually accomplished by a regression model for event count; see Cameron and Trivedi (1998).

Count data regression modelling have become important tools in empirical studies of economic behavior and their applicability continues to grow in various areas of economic. In the specialized literature numerous examples of economic studies utilizing count data methodology is mentioned covering topics including health economics, financial economics, industrial organization; see Hellström (2002).

Although these modelling techniques have a rather recent origin; their statistical analysis has a rich history. The Poisson distribution can form the basis for some analyses of count data and in this case "Poisson regression" model is the most important count data (see Cameron and Trivedi (1998), Winkelmann (2008). It has a serious constraint of equidispersion (*i.e* mean and variance are equal).

The restriction of equidispersion is not usually realized, for example cases of zero-truncated data. In practical situations, where zero count is a potential possible value, but is missing in the data set, it has been called it zero truncated data. The missing of zero count happens due to the sampling scheme, in which the zero count is impossible to be observed; see Creel and Loomis (1990); Shaw (1988). For this kind of data, the estimation methods that do not take truncation into account will generally lead to inconsistent estimates. The zero truncated models are more appropriate than traditional regression models for this kind of data. The Zero Truncated Poisson (ZTP) regression model was suggested by Shaw (1988) to model positive count data; see Wang *et al* (2011) and Cameron and Trivedi (1998).

After a review of related work this article studies the performance of the maximum likelihood estimation of the parameters for Poisson and zero-truncated Poisson regression models.

This paper is organized as follow. The next section reviews the generalized linear model for count data. Section (3) gives the general frame work for the Poisson regression model of count for non-truncated sample. Section (4) introduces different applications for truncated count data and presents the zero-truncated Poisson regression model. Section (5) presents the maximum likelihood estimator of the parameters for both Poisson and truncated Poisson regression model. Section (6) introduces an assessment of the MLE of the parameter through a simulation study. Section (7) presents two empirical applications for non-truncated and truncated count data.

2. Generalized Linear Models For Count Data

The introduction of the "*generalized linear models*" played a role in the development of count data regression models. Generalized linear models constitute a class of models that generalize the linear models used for regression and analysis of variance. They were first suggested by Nelder and Wedderburn (1972). An extensive treatment of them is given by McCullagh and Nelder (1989). Generalized linear models include logistic regression as a special case. Another special case, Poisson regression, provides the same analysis for count data as log-linear models. Poisson regression and logistic regression are generalized linear models for one-parameter exponential families. The other two distributions, normal theory linear models and gamma distribution regression, involve the two-parameter family of distributions that was used in the basic theory, see Christensen (1997) and Cameron and Trivedi (1998), Dobson (2002).

All generalized linear models (GLMs) have three components: the first one, the random component, identifies the response variable Y and assumes a probability distribution for it. The second component, the *systematic component,* specifies the explanatory variables for the model. The third component, the "*link function*"*,* specifies a function of the expected value (mean) of Y, which the generalized linear model relates to the explanatory variables through a prediction equation having linear form; see Agresti (2007).

The, *link function*, specifies a function $g(\cdot)$ that relates λ to the linear predictor as

$$g(\lambda) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k. \qquad (1)$$

The link function $g(\cdot)$ connects the random and systematic components. The simplest "*link function*" is $g(\lambda) = \lambda$. This models the mean directly and is called the *identity link*. It specifies a linear model for the mean response,

$$\lambda = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k. \qquad (2)$$

This is the form of ordinary regression models for continuous responses.

The simplest GLMs for count data assume a *Poisson distribution* for the random component. GLMs for the Poisson mean can use the identity link, but it is more common to model the log of the mean. Like the linear predictor $\alpha + \beta x$, the log of the mean can take any real-number value. A *Poisson log linear model* is a GLM that assumes a Poisson distribution for Y and uses the log link function; see Agresti (2007).

For a single explanatory variable x, the Poisson log linear model has form

$$log(\mu) = \alpha + \beta x. \tag{3}$$

The mean satisfies the exponential relationship

$$\mu = exp\,(\alpha + \beta x) \tag{4}$$

3. Poisson regression models

The Poisson regression model (PRM) is the widely used model for analyzing count data. Sometimes it is defined as a Poisson log-linear model. The basic Poisson regression model relates the probability function of a dependent variable y_i (also referred to as regressand and endogenous) to a vector of independent variables x_i (also referred to as regressors and exogenous). This is done by allowing the intensity parameter λ to depend on covariates; see Cameron and Trivedi (1998), Winkelmann (2008), Agresti (2007) and Hilbe (2011).

The standard univariate Poisson regression model makes the following three assumptions:

Assumption 1

$$f(y_i|\lambda_i) = \frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!}, \qquad y_i = 0,1,2,\dots \;, \lambda > 0 \tag{5}$$

where $f(y|\lambda)$ is the conditional probability function of y given λ.

Assumption 2

$$\lambda_i = exp(\boldsymbol{x}_i{}'\boldsymbol{\beta}), \qquad i = 1,2,\dots,n \tag{6}$$

where $\boldsymbol{\beta}$ *is a* $(k \times 1)$ vector of parameters, $\boldsymbol{x}_i$ is a $(k \times 1)$ vector of regressors, including a constant and n is the number of observations in the sample.

Assumption 3

Observation pairs $(y_i, \boldsymbol{x}_i')$, $i = 0,\dots,n$ are independently distributed.

Assumption 1 and 2 can be combined to obtain the following conditional probability function:

$$f(y_i|x_i;\beta) = \frac{exp(-exp(x_i'\beta))(exp(x_i'\beta))^{y_i}}{y_i!}, \quad y_i = 0,1,2,\ldots \tag{7}$$

Thus,

$$E(y_i|x_i;\beta) = Var(y_i|x_i;\beta) = \lambda_i = exp(x_i'\beta), \tag{8}$$

See Agresti (2007).

The Poisson regression model (PRM) imposes the restriction that the conditional variance equals the conditional mean. In practice this restriction is often not realized, which reveals the over/under-dispersion phenomena. Over-dispersion arises when the conditional variance is greater than the conditional mean, whereas in under-dispersion the conditional variance is smaller than the conditional mean. Other models have been suggested to deal with this problem of the dispersions. On one hand, the "Negative binomial regression" is suggested as to model to handle count data with overdispersion. On the other hand, the "generalized Poisson regression" model is claimed to be suitable for accommodating both types of dispersions (see Consul and Famoye (1992), Famoye (1993) and Wang and Famoye (1997)). More general count data regression models relax the restriction of the Poisson regression model by introducing a dispersion parameter which may complicate computation; see Lambert and Roeder (1995), Mullahy (1997) and Liu and Cela (2008).

4. Models for Truncated Counts

In many applications, the analyst does not observe the entire distribution of counts. In particular the zeros often are not observed. For example, zero truncated (ZT) samples occur when observations enter the sample only after the first count occurs. So far the truncated count data models have been concerned with the data generating process. A different issue is that of sampling modalities. For instance, choice-based samples do not represent the entire population but only those individuals who have experienced at least (at most) a certain number of events. Such samples are called truncated from below (from above), or left truncated (right truncated). The most common form of truncation from below is truncation at zero; see Winkelmann and Zimmermann (1995).

Winkelmann (2008) explains the use of a two-part process to model truncated count data. The first part consists of an untruncated latent distribution for X^*. The second part consists of a binary indicator variable c. The observed distribution for X is truncated if $c = 0$, and untruncated if $c = 1$.

The generic model for truncation is then

$$X = \begin{cases} X^* & if\ c = 1 \\ unobserved & if\ c = 0 \end{cases} \quad (9)$$

Further, assume that

$$c = \begin{cases} 1 & if\ X^* \in A \\ 0 & if\ X^* \notin A \end{cases} \quad (10)$$

That is, c is uniquely determined through the latent count variable X^*.

The two most commonly encountered situations are:

1. A is the set of positive integers (“truncation at zero”)
2. A is the set $\{0, \ldots, a\}$ where a is some positive integers, (“truncation from above”)

For instance, assume that c is defined as (10) and X^* is Poisson distributed with parameter λ.

For $A = \{1,2, \ldots\}$, then $p(c = 1) = 1 - exp(-\lambda)$.

And for $A = \{0,1, \ldots, a\}$, then $p(c = 1) = F(a)$, where F is the cumulative distribution function of X^*. In general,

$$p(X = y) = \frac{p(X^* = y \mid c = 1)}{p(c = 1)}, \quad (11)$$

4.1 Zero-Truncated Poisson Regression Models

The Zero Truncated Poisson (ZTP) regression model is used to model positive count data, where zero cannot occur due to the nature of study and its design. For example, the number of bus trips made per week in surveys taken on buses, the number of shopping trips made by individuals sampled at a mall, and the number of unemployment spells among a pool of unemployed. In all these cases we do not observe zero counts, so the data are said to be zero truncated, or more generally left truncated. Right truncation results from loss of observations greater than some specified value, see Wang *et al* (2011) and Cameron and Trivedi (1998).

Let y_i be a discrete count random variable with probability density function $g(y_i,\varphi) = Pr(Y_i = y_i)$. The subscript i $(i = 1,2,\ldots,n)$ refers to observation y_i which takes values $(0,1,2,\ldots)$ and φ is a parameter vector. The distribution that is truncated at zero is obtained if realizations of y_i at zero are omitted.

The truncated-at-zero count distribution will be denoted by:

$$f(y_i,\varphi|y_i > 0) = Pr(Y_i = y_i|Y_i > 0) = \frac{Pr(Y_i = y_i)}{p(Y_i > 0)}. \qquad (12)$$

This is a special case of left truncated or truncated-from-below distribution. It is well known that the mean of the truncated at zero, and generally of the left truncated, random variable exceeds the corresponding mean of the untruncated distribution. It is useful to express this as:

$$E(y_i|Y_i > 0) = E(y_i) + \delta_i, \qquad (13)$$

where δ_i depends on the parameters of the model. Gurmu (1991) called δ_i the *adjustment factor*. The adjustment factor is the difference between the means of the truncated and untruncated parent distributions.

The zero-truncated Poisson distribution is given by:

$$f(y_i,\lambda|Y_i \geq 0) = \frac{e^{-\lambda_i}\lambda_i^{y_i}}{y_i!\,[1-exp(-\lambda_i)]}, \qquad y_i = 1,2,3,\ldots \qquad (14)$$

Consider the speciation $\lambda_i = exp(\boldsymbol{x}_i'\boldsymbol{\beta})$, the probability density function of positive Poisson (*PP*) regression model is given by:

$$f(y_i,\boldsymbol{\beta}|\boldsymbol{x}_i,y_i > 0) = \frac{exp(-exp(x_i'\beta)) * (exp(x_i'\beta))^{y_i}}{(1-exp(-exp(x_i'\beta))) * y_i!}, \quad y_i = 1,2,3,\ldots \qquad (15)$$

where, $\boldsymbol{\beta}$ is a $(k \times 1)$ vector of parameters, $\boldsymbol{x'}_i$ is a vector of covariate values for subject i, that $\boldsymbol{x'}_i = (1, x_{i1}, \ldots, x_{i(k-1)})$.

The rth descending factorial moment $E(y^{(r)}|x, Y > 0) = E(y\,(y-1)\ldots(y-r+1))$ of the PP can be expressed in terms of the mean of the regular Poisson (λ_i) and the adjustment factor δ_i. It can be shown that the rth order factorial moments of the PP are,

$$E\left(y_i^{(r)} \middle| \boldsymbol{x}_i, Y_i > 0\right) = \lambda_i^r + \lambda_i^{r-1} \cdot \delta_i. \qquad (16)$$

The mean, the variance, and the adjustment factor for the positive Poisson model are as follows:

$$E(y_i|\boldsymbol{x}_i, Y_i > 0) = \mu_i = exp(\boldsymbol{x}_i'\boldsymbol{\beta}) + \delta_i = \frac{exp(x_i'\beta)}{1-exp(-exp(x_i'\beta))}, \qquad (17)$$

$$var(y_i|\boldsymbol{x}_i, Y_i > 0) = \sigma_i^2 = \left(exp(\boldsymbol{x}_i'\boldsymbol{\beta}) - \delta_i(\mu_i - 1)\right) = \mu_i(1 - \delta_i) \qquad (18)$$

and

$$\delta_i = \frac{exp(\boldsymbol{x}_i'\boldsymbol{\beta})}{(\boldsymbol{x}_i'\boldsymbol{\beta}) - 1}, \qquad (19)$$

It is noteworthy that for the positive Poisson (pp) model δ_i is written,

$$\delta_i = E(y_i|x_i).\, g(0, \lambda_i)/(1 - g(0, \lambda_i). \qquad (20)$$

See Gurmu (1991) and Winkelmann and Zimmermann (1995) for more details.

Since λ_i, the mean of untruncated distribution, is greater than zero, then $(0 < \exp(-\lambda_i) < 1)$, and the truncated mean is shifted to the right. Moreover, the truncated at zero model displays underdispersion relative to the untruncated Poisson model since $0 < \left(1 - \frac{exp(x_i'\beta)}{(x_i'\beta)-1}\right) < 1$.

Count data are sometimes collected by on-site[1] sampling of users. Pollock *et al.* (1994) present a number of on-site sampling methods for angler surveys. Here interest is in the population of fishing trips and estimation of total catch and total angler effort. Such samples involve truncation because only those who use the facility at least once are included in the survey. An alternative way to account for truncation at zero has been proposed by Shaw (1988) with an on-site samples Poisson regression. It is obtained not by conditioning, but by shifting the sample space.

5. Maximum Likelihood Estimation of Count Data Regression Models

[1] On-site sampling means intercepting respondents at public places like shopping centers, airports, recreation sites, etc., followed up by interviews and/or address collection.

5.1 Likelihood Function and Maximization of Poisson Regression Model

Let Y be a random variable that is known to follow a certain probability distribution which may not be completely specified; the mathematical form of this distribution may be assumed to be known but it depends on one or more unknown parameter $\boldsymbol{\beta}$. A common practice is to observe a random sample from Y and try to estimate the proper value of $\boldsymbol{\beta}$ to complete our knowledge about the probability distribution of Y. The maximum likelihood method is used for estimating the vector of parameters $(\boldsymbol{\beta} = \beta_1, \beta_2, \beta_3)$ for this model.

For the regression model of count data the probability density function (pdf) of the random variable Y conditioned on a set of covariate $\boldsymbol{x}$ and a set of parameters $\boldsymbol{\beta}$ is given by eq.7. The likelihood function of Poisson regression is written as:

$$L = f(y_1, \dots, y_n | \boldsymbol{x}_i; \boldsymbol{\beta}) = \prod_{i=1}^{n} f(y_i | \boldsymbol{x}_i; \boldsymbol{\beta}). \tag{21}$$

The log-likelihood function for the Poisson regression model takes the form:

$$\ell(\boldsymbol{\beta}; Y, X) = \sum_{i=1}^{n} [-exp(\boldsymbol{x}_i'\boldsymbol{\beta}) + y_i \boldsymbol{x}_i'\boldsymbol{\beta} - log(y_i!)]. \tag{22}$$

The maximizing value for $\boldsymbol{\beta}$, denoted as $\widehat{\boldsymbol{\beta}}$, is found by computing the first derivatives of the log-likelihood function and setting them equal to zero. In the Poisson regression model, there are k such derivatives, with respect to β_1, β_2 and β_3. The (column) vector that collects these k first derivatives is alternatively denoted as *gradient vector*, or as *score vector*. The latter term is used in the following. We write,

$$S_n(\boldsymbol{\beta}; y, x) = \frac{\partial \ell(\boldsymbol{\beta}; y, x)}{\partial \beta} = \sum_{i=1}^{n} [y_i - exp(\boldsymbol{x}_i'\boldsymbol{\beta})] \boldsymbol{x}_i, \tag{23}$$

we use the subscript "n" as a reminder that the score depends on the sample size.

The maximum likelihood estimates $\widehat{\boldsymbol{\beta}}$ is the value of $\boldsymbol{\beta}$ that solves the first order conditions for a maximum,

$$S_n(\hat{\boldsymbol{\beta}}; y, \boldsymbol{x}) = 0. \tag{24}$$

The Hessian matrix of the Poisson log-likelihood function is given by:

$$H_n(\boldsymbol{\beta}; y, \boldsymbol{x}) = \frac{\partial^2 \ell(\boldsymbol{\beta}; y, \boldsymbol{x})}{\partial \boldsymbol{\beta}\, \partial \boldsymbol{\beta}'} = -\sum_{i=1}^{n} [exp(x_i' \boldsymbol{\beta})]\, \boldsymbol{x}_i \boldsymbol{x}_i' . \tag{25}$$

The analytical solation of eq.24 is difficult to establish; the non-linearity of the equations is one reason and the problem is further complicated by the multiplicity of the equations. For solving this kind of equations one usually relies on numerical techniques. A common choice that works well for most smooth function is the Newton-Raphson method. It is the most common numerical technique used for solving equations. There exist many software programs and packages that offer procedure for solving equations numerically. Among these packages there is one which is most popular among statisticians, that is the R package. Acutely, R offers a plethora of computational techniques for solving equations and data analysis. Some of these techniques are designed especially for obtaining maximum likelihood estimates; see Winkelmann (2008), Greene (2002) and Beckett *et al*. (2013).

The asymptotic variance-covariance matrix of the ML estimates for the three parameters β_1, β_2 and β_3 is defined as follow:

$$\hat{V} = \left[\hat{I}_{j,j'}(\boldsymbol{\beta})\right]^{-1}, \tag{26}$$

where, the observed information matrix $\hat{I}_{j,j'}(\boldsymbol{\beta})$ is defined as:

$$\hat{I}_{j,j'}(\boldsymbol{\beta}) = -\left[\frac{\partial^2 \ell(\boldsymbol{\beta}; y, \boldsymbol{x})}{\partial \boldsymbol{\beta}\, \partial \boldsymbol{\beta}'}\right]_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}} \forall\, j, j' = 1, 2, 3. \tag{27}$$

The elements of the observed information matrix are obtained by differentiating eq.23 with respect to (β_j, $j = 1, 2, 3$):

$$\frac{\partial^2 \ell}{\partial^2 \beta_j} = -\sum_{i=1}^{n} (x_{ji})^2 exp\left(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3\right), \quad \forall\, j = 1, 2, 3. \tag{28}$$

$$\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_{j'}} = -\sum_{i=1}^{n} x_{ji} x_{j'i}\, exp\left(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3\right), \quad j, j' = 1, 2, 3\ \forall\, j \neq j'. \tag{29}$$

By the statistical (mathematical) computing package R to yield the MLE's of the parameters the results obtained for complete sample data are displayed in Tables (1.1) to (1.10).

5.2Likelihood Function and Maximization of the Zero-Truncated Poisson Regression Model

A truncated distribution is a conditional distribution that results from restricting the domain of some other probability distribution. For the Poisson distribution, the zero-truncated Poisson regression is given by eq.14. The estimation of the zero-truncated Poisson model involves a simple modification to the likelihood equation for the Poisson regression model (PRM) as given as:

$$L(\lambda; y, x) = \prod_{i=1}^{n} f(y_i, \lambda | y_i \geq 0) = \prod_{i=1}^{n} \frac{e^{-\lambda_i} \lambda_i^{y_i}}{y_i! [1 - exp(-\lambda_i)]}, \qquad (30)$$

The natural logarithm of the likelihood function is given by,

$$\ell(\lambda; y, x) = \sum_{i=1}^{n} \left[y_i log\lambda_i - \lambda_i - log\left(1 - exp(-\lambda_i)\right) - lo \right. \qquad (31)$$

where n is the number of observations in the sample, and using the specifying $\lambda_i = exp(\boldsymbol{x}_i' \boldsymbol{\beta})$, eq. (31) can be written as:

$$\ell(\boldsymbol{\beta}; y, \boldsymbol{x}) = \sum_{i=1}^{n} \begin{bmatrix} y_i(\boldsymbol{x}_i\boldsymbol{\beta}) - exp(\boldsymbol{x}_i' \boldsymbol{\beta}) - \\ log\left(1 - exp\left(-exp(\boldsymbol{x}_i' \boldsymbol{\beta})\right)\right) - log\, y_i! \end{bmatrix}, \qquad (32)$$

The *score* vector that collects the k first derivatives is written as:

$$\begin{aligned} S_n(\boldsymbol{\beta}; y, \boldsymbol{x}) &= \frac{\partial \ell(\boldsymbol{\beta}; y, x)}{\partial \boldsymbol{\beta}} \\ &= \sum_{i=1}^{n} x_i \left(y_i - exp(\boldsymbol{x}_i' \boldsymbol{\beta}) - \frac{exp(x_i' \boldsymbol{\beta})}{exp\left(exp(x_i' \boldsymbol{\beta})\right) - 1}\right), \end{aligned} \qquad (33)$$

we use the subscript "n" as a reminder that the score depends on the sample size. The maximum likelihood estimate (MLE) $\bar{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ is the solution of the following equation:

$$S_n(\bar{\boldsymbol{\beta}}; y, \boldsymbol{x}) = 0, \qquad (34)$$

Once again we use **R** package to solve the likelihood equations; See Hellström (2002), Grogger and Carson (1991), Long (1997), Winkelmann (2008) and Gurmu (1991).

The Hessian matrix for the positive Poisson regression model is given,

$$H_n(\boldsymbol{\beta}; y, \boldsymbol{x}) = \left[\frac{\partial^2 \ell(\boldsymbol{\beta}; y, x)}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'}\right] = \sum_{i=1}^{n} \lambda_i \left[1 - \frac{\lambda_i}{exp(\lambda_i) - 1}\right] \boldsymbol{x}_i \boldsymbol{x}_i'. \qquad (35)$$

Similarly, we use eq. (26) and (27) to obtain the observed information matrix. The elements of the observed information matrix are obtained by differentiating eq.33 with respect to $(\beta_j, j = 1, 2, 3)$,

$$\frac{\partial^2 \ell}{\partial^2 \beta_j} = \sum_{i=1}^{n} \left[\begin{array}{c} \left(-exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)(x_{ji})^2\right) * \\ [1 + \frac{exp^{exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)} exp^{(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)}}{\left(exp^{exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)} - 1\right)^2} - \\ \frac{1}{\left(exp^{exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)} - 1\right)}] \end{array} \right], \quad j = 1, 2, 3. \qquad (36)$$

$$\frac{\partial^2 \ell}{\partial \beta_j \partial \beta_{j'}} = \sum_{i=1}^{n} \Bigg[\left(\left(-exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)(x_{ji})(x_{j'i})\right) * \right.$$
$$\left(1 + \frac{exp^{exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)} exp^{(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)}}{\left(exp^{exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)} - 1\right)^2} - \right. \qquad (37)$$
$$\left. \left. \frac{1}{\left(exp^{exp(x_{1i}\hat{\beta}_1 + x_{2i}\hat{\beta}_2 + x_{3i}\hat{\beta}_3)} - 1\right)} \right) \right) \Bigg], \quad j, j' = 1, 2, 3 \;\; \forall j \neq j'$$

Using the statistical computing package *vglm* in R to obtain the MLE's of the parameters, the results obtained for complete sample data are displayed in Tables (2.1) to (2.9).

6. Simulation Study

The Monte-Carlo simulation study is set up to study the performance of the ML estimators of the unknown parameters for both Poisson and zero-truncated Poisson regression models for count data. This study was done through the bias (*SAB*), the mean square error (*MSE*), the scaled root mean square error (*SRMSE*) and the variance has been calculated. Also, the sample information matrix is obtained to give the (v-co) matrix. All computations are performed using R and Mathcad 15.

Simulating data for two types of models, it was assumed that the model contains two regressors X_2 and X_3 of different domains plus a constant regressor X_1. As for X_2 and X_3 it was assumed that X_2 has a gamma distribution with shape and scale parameters equal to one and two, and the third independent variable X_3 was assumed to follow the normal distribution with mean to equal two and standard deviation equal to one. These assumptions were kept for the two types of models. The simulation studies were carried out using different combination of values for the parameters of interest. These combinations were chosen to cover a wide and practical range of values of the parameters. Each parameter $(\beta_1, \beta_2, \beta_3)$ choice with different sample sizes $n = (20, 30, 50, 100)$ was repeated N $(= 5000)$ times.

- The simulation results for the Poisson regression model are obtained according to the following steps.
- Given a vector $\beta = (\beta_1, \beta_2, \beta_3)$ of parameters, for each sample sizes a number of random samples are generated from Poisson distribution with mean parameters $\lambda_i = exp(x'_i \beta)$ to obtain the dependent variable y_i by using the R program. The considered values of regression coefficients are $(\beta_1 = 0.6, 1.6)$, $(\beta_2 = 0.3, 0.9, 1.4, 1.9, 2.2)$ and $(\beta_3 = 0.5, 0.7, 1.5, 1.9, 2.6)$.
- The maximum likelihood estimates $\hat{\beta}$ of the three parameters $\beta = (\beta_1, \beta_2, \beta_3)$ are calculated using the R program that includes a package called *mle*. The *mle* is a program written to be used with the R package to compute the maximum likelihood estimate of the parameters of a given model through the negative log-likelihood function. For more information about *mle* visit http://127.0.0.1:18932/library/stats4/html/mle.html.
- The results of N samples are summarized to calculate the mean of the estimates for each of the three parameters β_1, β_2 *and* β_3. The average bias, the scaled absolute bias (*SAB*), the mean square error (*MSE*) and the scaled root mean square error (*SRMSE*) have been calculated. The inverse of the observed information matrix has been calculated to get the variance.
- The estimated values obtained above are used to get the following measures of the three parameters ($\beta_j, j = 1, 2, 3$).

$$Mean\ (\hat{\beta}) = \frac{1}{N} \sum \hat{\beta}$$

$$Bias\ (\hat{\beta}) = Mean(\hat{\beta}) - \beta$$

$$SAB = \left|Bias(\widehat{\boldsymbol{\beta}})\right| / (\beta)$$

$$MSE\,(\hat{\beta}_j) = \frac{1}{N}\Sigma(\hat{\beta}_j - \beta_j)^2, \quad \forall \quad j = 1,2,3$$

$$RMSE(\hat{\beta}_j) = \sqrt{MSE(\hat{\beta}_j)}, \quad \forall \; j = 1,2,3$$

$$SRMSE(\hat{\beta}_j) = \sqrt{MSE(\hat{\beta}_j)}\,/\beta_j\,, \quad \forall \; j = 1,2,3$$

Finally, the result of MLE's for the Poisson regression of the $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ for the different sample sizes are displayed in tables (1.1 – 1.10).

The simulation procedure mentioned before is also were conducted to generate random samples from the zero-truncated Poisson regression model with mean parameter $\mu_i = exp(x_i'\boldsymbol{\beta})/(1 - exp(-exp(x_i'\boldsymbol{\beta})))$ to obtain the dependent variable y_i by the R software. The chosen values of coefficients for this model are $(\beta_1 = 0.5, 1.5), (\beta_2 = 0.04,\ 0.06,\ 0.1,\ 0.2)$ and $(\beta_3 = 0.005,\ 0.03,\ 0.07,\ 0.3,\ 0.4)$. The ml estimates $\widehat{\boldsymbol{\beta}}$ of the $\beta = (\beta_1, \beta_2, \beta_3)$ are calculated using the R program that includes a package called *vglm* (which have written for R for fitting vector generalized linear models). For more information about this package visit http://127.0.0.1:28881/library/VGAM/html/vglm-class.html. The results of N samples are summed up by computing the average of the estimates for each of the three parameters. Also, the criterion mentioned for the PRM were considered and computed for the zero-truncated Poisson regression. The result of MLE's in this model of the three parameters $\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3$ for the different sample sizes are obtained and displayed in from the tables (2.1 – 2.9).

Conclusion Remarks

Form the above discussion and tables (1.9), (2.6) summarizing the performance of the Poisson regression model, it is noticed, that all estimates are good in the sense of the small average bias and scaled absolute bias (*SAB*), see for example table (1.9) where $(n = 30)$. For values of β_3 less than one ($\beta_3 < 1$) the average bias in $\hat{\beta}_3$ was smaller than 0.0007 and

SAB was less than 0.001. On the other hand, for value of β_3 greater than one $(\beta_3 > 1)$ the average bias in $\hat{\beta}_3$ was smaller than 0.0002 and *SAB* was less than 0.00011. These remarks are enforced by looking at mean square error (*MSE*), the scaled root mean square error (*SRMSE*). Also, for values of β_3 less than one ($\beta_3 < 1$), the *MSE* of $\hat{\beta}_3$ was smaller than 0.0025 and *SRMSE* was less than 0.09. In addition, for values of β_3 greater than one ($\beta_3 > 1$), the *MSE* in $\hat{\beta}_3$ was smaller 0.00051 and *SRMSE* was less than 0.011.

For the truncated Poisson regression model it is observed that the estimates $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$ perform well in the average bias and scaled absolute bias (*SAB*). Looking at the results in table (2.6) where β_1, β_2 are fixed at 0.5 and 0.04 respectively. For values of β_3 less than 0.05 ($\beta_3 < 0.05$) the average bias in $\hat{\beta}_3$ was smaller than 0.003 and scaled absolute bias (*SAB*) was less than 0.6. In addition, for values of β_3 greater than 0.05 $(\beta_3 > 0.05)$ the average bias in $\hat{\beta}_3$ was smaller than 0.01 and *SAB* was less than 0.1. These remarks are enforced by looking at mean square error (*MSE*), the root mean square error (*RMSE*) and the variance (*Var*) of the estimates. For values of β_3 less than 0.05 ($\beta_3 < 0.05$) the *MSE* in $\hat{\beta}_3$ was smaller than 0.035, the *RMSE* was less than 0.413 and the *Var* was less than 0.0096. On the other hand, for values of β_3 greater than 0.05 ($\beta_3 > 0.05$) the *MSE* in $\hat{\beta}_3$ was smaller 0.026, *RMSE* was less than 0.162 and the *Var* was less than 0.01.

In general, the Monte-Carlo simulation study indicated a good performance of the ML estimators of the three parameters for both the Poisson and zero-truncated Poisson regression models. All estimates are good in the sense of average bias, scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*RMSE*), scaled root mean square error (*SRMSE*) and the variance (*Var*) (*i.e. each one of these criterion has recorded a small value for each estimated parameter*). The good performance has been noticed for each parameter ($\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$) with the different sample sizes $n = (20, 30, 50, 100)$. As it is usually, when the sample size n was increased, the mean square error and variances of the estimates decrease which consistent with theoretical results. The MLE's of the parameters are approximately unbiased and has minimum variance (very small biased).

7. Some Applications

7.1 Wave Damage

This application uses a real data set obtained from, J. Crilley and L.N. Heminway of Lloyd's Register of shipping, which deals with wave damage to Cargo Ships, [see McCullagh and Nelder (1989)]. The purpose of this analysis is to examine the risk of damage caused by waves to the forward section of certain cargo-carrying vessels associated with the type of vessel, year of construction and period of operation. The real data set give the number of damage incidents (as distinct from the number of ships damaged) and the aggregate number of months in service by ship type, year of construction and period of operation. Note that a single ship may be damaged more than once and that some ships will have been operating in both periods. The data set has 40 observations; each observation has information on five variables as follows:

- ship type, coded 1-5 for A, B, C, D and E,
- year of construction (1=1960-64, 2=1965-70, 3=1970-74, 4=1975-79),
- period of operation (1=1960-74, 2=1975-79),
- months of service, ranging from 63 to 44882, and
- damage incidents, ranging from 0 to 53.

Some Observations in this data set have missing values for the explanatory variables or the response variable, so the available of this data which have been used for fitting the model is 34 observations.

Displaying the Poisson Regression Analysis:

Studying the relationship between the number of damage incidents and the ship type, year of construction and period of operation is hindered by the fact that data is collected from the different period at risk (exposure) which varies greatly. Count data models account with the difference by including the log of the exposure variable in the model with coefficient fixed to be one. An effect such as this is commonly referred to as an *offset*.

The link function of Poisson regression that relates the mean of the response variable to the linear function of explanatory variables for this real data set is given by:

$$log(expected\ number\ of\ damage\ incidents\ ) = \beta_0 + \beta_1 log(months) + \gamma_i + \tau_j + \delta_k \qquad (38)$$

where $log(months)$ is a variable whose coefficient β_1 is known to be one as mention above. The coefficient γ_i is the effect of the ith level of ship type, the coefficient τ_j is the effect of the jth level of years of construction, and the coefficient δ_k is the effect of the kth level of Period of operation.

Table (3.1) provides the parameters estimates to study the effect of each predictor. In addition, the Akaike's Information Criterion (AIC) has been calculated ($AIC = 154.56$), [The Akaike information criterion (AIC) is a measure of the relative quality of statistical models for a given set of data].

Remarks

The signs of the relative values of the coefficients for factor levels can give important information into the effects of the predictors in the model.

- An increasing value of coefficients with a positive coefficient corresponds to an increasing rate of damage incidents.
- Ships of type D and E have damage rate that do not different significantly from damage for type A. on the other hand, Ships type B have damage rate that is significantly lower than ships type A (p value of 0.001) and ships of type C have damage rate that is significantly lower than type B.
- Ships constructed between (1965 to 69) and (1970 to 74) have damage rate that is significantly higher than ships constructed (1960 to 1964). While ships constructed between 1975 to79 have damage rate that is significantly lower than ships constructed between (1965 to 69) and (1970 to 74). The oldest ships built between 1960 to 1964 have the lowest risk and safest rather than other ships.
- Ships in operation between 1975-79 have damage rate that is significantly higher (estimated coefficient of 0.384) than those in

operation between 1960–1974. Nevertheless, the ships in period of operation between 1960 to 1974 have a lower risk.

7.2 Hospital Stay

This application is based on a data set obtained from Institute for Digital Research and Education, for more information about data visit https://www.idre.ucla.edu. The data set contains with the length of hospital stay in days. The Length of hospital stay is recorded as a minimum of at least one day. The zero-truncated Poisson regression is used to model this count data for which the value zero cannot occur. The Predictor variables in this study are the age of the patient, type of health insurance and whether or not the patient died while in the hospital. The dataset contains 1,493 observations. The length of hospital stay variable is *stay* which is considered as a count variable. The variable *age* gives the age group from one to nine, which will be treated as interval in this study. The variables *hmo* and *died* are binary indicator variables for *Health Maintenance Organization* (*hmo's*) insured patients and patients who died while in the hospital, respectively. Figure (1) shows the histogram of length of hospital stay of data set in truncation-at-zero problem.

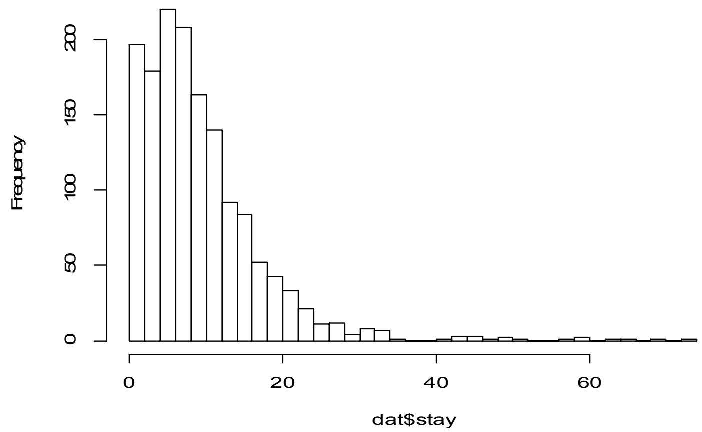

Figure 1: Histogram of the response variable

Table (3.2) showed that the variance of our outcome variable *stay* is greater than the mean, so the zero-truncated Poisson model well be more flexible for fitting these data. The frequency and the percentage for the levels of the covariates (*age*, *hmo and died*) are displayed in tables (3.3), (3.4) and (3.5), respectively. The R function *vglm* was used to fit the zero-truncated Poisson model. This function fits a very flexible class of models called vector generalized linear models to a wide range of assumed distributions. In our case, we believe the data are Poisson, but without zeros.

The linear function of covariates which affect the length of hospital stay for this data is given by:

$$\lambda = exp(\beta_0 + \upsilon_i + \varrho_j + \eta_k), \tag{39}$$

where the coefficient υ_i is the effect of the ith level of *age* group, the coefficient ϱ_j is the effect of the jth level of *hmo* and the coefficient η_k is the effect of the kth level of *died*.

Remarks

Table (3.6) displayed the parameters estimation for the hospital stay. It is a count variable that cannot be zero because the value of zero is almost impossible to be observed due to the nature of study. Let's look at the coefficients of the regression. These coefficients are interpreted as you would interpret coefficients from a standard Poisson model. The expected length of the stay changes by a factor of $exp\ (Coef.)$ for each unit increase in the corresponding predictor.

- The value of the coefficient for *age* equal to -0.0144, that the log count of stay decreases by 0.0144 for each year increase in *age*. Thus, if two patients have the same values for *hmo* and *died* (for example, both *died* while in the hospital and both were insured by *hmo's*) and one fell into age group 4 and the other into age group 5, the patient in *age* group 5 would have a predicted hospital stay of $exp\ (-0.0144) = 0.986$ times that of the patient in *age* group 4.
- The coefficient for *hmo* equal to -0.1359, indicates that the log count of stay for *hmo* patient is 0.1359 less than for non-*hmo* patients. Thus, if two patients have the same values for *hmo* and *died* (for example, if two patients both *died* while in the hospital and both

were in *age* group 8) and one was insured by an *hmo* and one was not, the patient insured by an *hmo* would have a predicted hospital stay of $exp(-0.1359) = 0.873$ times that of the patient not insured by an *hmo*.

- The coefficient for *died* equal to -0.203, that the log count of stay for patients who *died* while in the hospital was 0.2038 less than those patients who did not die. Thus, if two patients have the same values for *age* and *hmo* (for example, both were in *age* group 8 and both were insured by an *hmo*) and one *died* while in the hospital and one did not, then the patient who *died* would have a predicted hospital stay of $exp(-0.204) = 0.816$ times that of the patient who did not die.
- The value of the Intercept equal to 2.4358, denoted that the log count of the stay when all of the predictors equal zero.
- $Pr > |z|$ of coefficients of the regression, appear that the predictor *age* is $(-0.0144/0.0050) = -2.87$ with an associated p-value of 0.004, the predictor *hmo* is $(-0.1359/0.0237) = -5.72$ with an associated p-value of < 0.001 and the predictor *died* is $(-0.2037/0.0183) = -11.09$ with an associated p-value of < 0.001. If we set our alpha level at 0.05, we would reject the null hypothesis and conclude that the regression coefficient for *age*, *hmo* and *died* has been found to be statistically different from zero.
- $Pr > |z|$ of Intercept, denoted that the z test statistic for the intercept is $(2.436/0.0273) = 89.12$ with an associated p-value of < 0.001. If we set our alpha level at 0.05, we would reject the null hypothesis.
- The Confidence Interval (CI) for an individual coefficient given that the other predictors are in the model. For a given predictor with a level of 95% confidence, that we are 95% confident that the "true" coefficient between the lower and upper limit of the interval. It is calculated as the $Coef. \pm z_{\alpha/2} * (Std.Err.)$, where $z_{\alpha/2}$ is a critical value on the standard normal distribution equal 1.96.

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Numerical Results:

A.1 Results of Poisson Regression Model

Table (1.1) The MLE of Poisson regression parameters when $\beta_1 = 0.3, \beta_2 = 0.9, \beta_3 = 1.9$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	**0.3005**	**0.00051**	**0.00170**	**0.00593**	**0.25668**	**0.09616**
	$\hat{\beta}_2$	**0.8993**	**- 0.00070**	**0.00078**	**0.00154**	**0.04360**	**0.06480**
	$\hat{\beta}_3$	**1.9000**	**- 0.00018**	**0.00010**	**0.00056**	**0.01245**	**0.01338**
30	$\hat{\beta}_1$	**0.3000**	**0.00001**	**0.00005**	**0.00310**	**0.18559**	**0.05893**
	$\hat{\beta}_2$	**0.8998**	**-0.00018**	**0.00020**	**0.00065**	**0.02832**	**0.03392**
	$\hat{\beta}_3$	**1.9000**	**-0.00008**	**0.00004**	**0.00026**	**0.00848**	**0.00797**
50	$\hat{\beta}_1$	**0.2999**	**-0.00009**	**0.00031**	**0.00144**	**0.12649**	**0.03302**
	$\hat{\beta}_2$	**0.9000**	**-0.00004**	**0.00005**	**0.00025**	**0.01756**	**0.01738**
	$\hat{\beta}_3$	**1.9000**	**0.00003**	**0.00002**	**0.00011**	**0.00552**	**0.00446**
100	$\hat{\beta}_1$	**0.2998**	**-0.00022**	**0.00073**	**0.00057**	**0.07958**	**0.01553**
	$\hat{\beta}_2$	**0.9001**	**0.00007**	**0.00008**	**0.00008**	**0.00993**	**0.00765**
	$\hat{\beta}_3$	**1.9000**	**0.00004**	**0.00002**	**0.00004**	**0.00332**	**0.00207**

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.2) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 1.4, \beta_3 = 0.5$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	**0.5947**	**- 0.00527**	**0.00878**	**0.05067**	**0.37516**	**0.16178**
	$\hat{\beta}_2$	**1.3995**	**- 0.00051**	**0.00036**	**0.02110**	**0.10375**	**0.10640**
	$\hat{\beta}_3$	**0.5000**	**- 0.00002**	**0.00004**	**0.00610**	**0.15620**	**0.02449**
30	$\hat{\beta}_1$	**0.5942**	**- 0.00577**	**0.00963**	**0.02884**	**0.28303**	**0.09961**
	$\hat{\beta}_2$	**1.4019**	**0.00190**	**0.00136**	**0.00948**	**0.06954**	**0.05856**
	$\hat{\beta}_3$	**0.5005**	**0.00051**	**0.00102**	**0.00336**	**0.11593**	**0.01490**
50	$\hat{\beta}_1$	**0.5971**	**- 0.00291**	**0.00486**	**0.01549**	**0.20743**	**0.05551**
	$\hat{\beta}_2$	**1.4000**	**0.00010**	**0.00007**	**0.00410**	**0.04573**	**0.02873**
	$\hat{\beta}_3$	**0.5005**	**0.00052**	**0.00105**	**0.00172**	**0.08294**	**0.00824**
100	$\hat{\beta}_1$	**0.6001**	**0.00013**	**0.00021**	**0.00675**	**0.13693**	**0.02619**
	$\hat{\beta}_2$	**1.3990**	**- 0.00053**	**0.00038**	**0.00128**	**0.02555**	**0.01210**
	$\hat{\beta}_3$	**0.4998**	**- 0.00020**	**0.00041**	**0.00074**	**0.05440**	**0.00388**

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.3) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 0.3, \beta_3 = 0.7$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	0.5865	- 0.01347	0.02245	0.06185	0.41449	0.17210
	$\hat{\beta}_2$	0.3026	0.00264	0.00880	0.03101	0.58698	0.12281
	$\hat{\beta}_3$	0.7024	0.00243	0.00347	0.00729	0.12197	0.02558
30	$\hat{\beta}_1$	0.5933	- 0.00673	0.01122	0.03730	0.32188	0.10646
	$\hat{\beta}_2$	0.2961	- 0.00392	0.01309	0.01558	0.41606	0.06838
	$\hat{\beta}_3$	0.7016	0.00161	0.00230	0.00431	0.09378	0.01553
50	$\hat{\beta}_1$	0.5973	- 0.00273	0.00456	0.02019	0.23681	0.06031
	$\hat{\beta}_2$	0.2982	-0.00183	0.00610	0.00773	0.29306	0.03604
	$\hat{\beta}_3$	0.7009	0.00091	0.00130	0.00228	0.06821	0.00875
100	$\hat{\beta}_1$	0.6003	0.00030	0.00050	0.00946	0.16210	0.02874
	$\hat{\beta}_2$	0.2996	- 0.0003	0.00128	0.00333	0.19235	0.01626
	$\hat{\beta}_3$	0.6996	- 0.00035	0.00051	0.00104	0.04607	0.00414

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.4) The MLE of Poisson regression parameters when $\beta_1 = 0.7, \beta_2 = 1.9, \beta_3 = 1.5$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	0.6980	- 0.00199	0.00285	0.00608	0.11139	0.09405
	$\hat{\beta}_2$	1.9000	- 0.00003	0.00001	0.00142	0.01983	0.05768
	$\hat{\beta}_3$	1.5000	0.00044	0.00029	0.00054	0.01549	0.01338
30	$\hat{\beta}_1$	0.7004	0.00039	0.00057	0.00527	0.10370	0.05764
	$\hat{\beta}_2$	1.9000	- 0.00008	0.00004	0.00061	0.01299	0.03126
	$\hat{\beta}_3$	1.4998	- 0.00020	0.00013	0.00047	0.01445	0.00813
50	$\hat{\beta}_1$	0.7012	0.00121	0.00172	0.00147	0.05477	0.03167
	$\hat{\beta}_2$	1.9000	- 0.00039	0.00020	0.00020	0.00744	0.01572
	$\hat{\beta}_3$	1.5000	- 0.00024	0.00016	0.00011	0.00699	0.00437
100	$\hat{\beta}_1$	0.7006	0.00064	0.00092	0.00054	0.03319	0.01486
	$\hat{\beta}_2$	1.9000	- 0.00010	0.00005	0.00005	0.00372	0.00677
	$\hat{\beta}_3$	1.5000	- 0.00012	0.00008	0.00003	0.00365	0.02048

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.5) The MLE of Poisson regression parameters when $\beta_1 = 1.6, \beta_2 = 2.2, \beta_3 = 2.6$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	1.6000	- 0.00003	0.00002	0.00354	0.03718	0.05556
	$\hat{\beta}_2$	2.2000	0.00017	0.00007	0.00005	0.00321	0.03553
	$\hat{\beta}_3$	2.6000	- 0.00001	0.00001	0.00028	0.00643	0.00784
30	$\hat{\beta}_1$	1.6000	0.00021	0.00013	0.00012	0.00684	0.03406
	$\hat{\beta}_2$	2.2000	0.00003	0.00001	0.00002	0.00203	0.01978
	$\hat{\beta}_3$	2.6000	- 0.00008	0.00003	0.00001	0.00121	0.00480
50	$\hat{\beta}_1$	1.5980	0.00058	0.00036	0.00011	0.00655	0.01897
	$\hat{\beta}_2$	2.2010	- 0.00002	0.00001	0.000005	0.00101	0.01024
	$\hat{\beta}_3$	2.6000	- 0.00015	0.00005	0.000009	0.00115	0.00260
100	$\hat{\beta}_1$	1.6000	0.00033	0.00020	0.00004	0.00395	0.00887
	$\hat{\beta}_2$	2.2000	- 0.00006	0.00002	0.000001	0.00045	0.00433
	$\hat{\beta}_3$	2.6000	- 0.00006	0.00002	0.000002	0.00054	0.00121

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Looking the tables from the point view of β_3 by fixing β_1 and β_2 we get the following tables:

Table (1.6) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 0.3$ and different value of β_3 at $n = 30$

		β_3				
		0.5	**0.7**	**1.5**	**1.9**	**2.6**
Mean	$\hat{\beta}_1$	0.5967	0.5933	0.5994	0.6003	0.6002
	$\hat{\beta}_2$	0.2956	0.2961	0.2995	0.3001	0.2998
	$\hat{\beta}_3$	0.4996	0.7016	1.5000	1.9000	2.6000
Bias	$\hat{\beta}_1$	- 0.00326	-0.00673	-0.00057	0.00032	0.00018
	$\hat{\beta}_2$	- 0.00441	-0.00392	-0.00043	0.00010	-0.00021
	$\hat{\beta}_3$	- 0.00041	0.00161	0.00011	- 0.00013	-0.000001
SAB	$\hat{\beta}_1$	0.00544	0.01122	0.00095	0.00053	0.00031
	$\hat{\beta}_2$	0.01468	0.01309	0.00143	0.00034	0.00070
	$\hat{\beta}_3$	0.00083	0.00230	0.00007	0.00007	0.000001
MSE	$\hat{\beta}_1$	0.05451	0.03730	0.00736	0.00316	0.00089
	$\hat{\beta}_2$	0.02621	0.01558	0.00228	0.00077	0.00028
	$\hat{\beta}_3$	0.00704	0.00431	0.00069	0.00028	0.00007
SRMSE	$\hat{\beta}_1$	0.38915	0.32188	0.14304	0.09380	0.04976
	$\hat{\beta}_2$	0.53968	0.41606	0.15949	0.09273	0.05597
	$\hat{\beta}_3$	0.16782	0.09378	0.01755	0.00890	0.00335
Var	$\hat{\beta}_1$	0.12086	0.10646	0.07019	0.05873	0.04503
	$\hat{\beta}_2$	0.07959	0.06838	0.04433	0.03597	0.02759
	$\hat{\beta}_3$	0.01814	0.01553	0.00976	0.00807	0.00610

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance ($\hat{\beta}'S$)

Table (1.7) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 0.9$ and different value of β_3 at $n = 30$

		β_3				
		0.5	**0.7**	**1.5**	**1.9**	**2.6**
Mean	$\hat{\beta}_1$	**0.5955**	**0.5972**	**0.5995**	**0.5998**	**0.6002**
	$\hat{\beta}_2$	**0.8990**	**0.9021**	**0.8994**	**0.9001**	**0.9000**
	$\hat{\beta}_3$	**0.5002**	**0.6990**	**1.5000**	**1.9000**	**2.6000**
Bias	$\hat{\beta}_1$	**-0.00446**	**-0.00277**	**-0.00052**	**-0.00024**	**0.00016**
	$\hat{\beta}_2$	**-0.00100**	**0.00205**	**-0.00061**	**0.00006**	**0.00001**
	$\hat{\beta}_3$	**0.00021**	**-0.00099**	**0.00025**	**0.00003**	**-0.00003**
SAB	$\hat{\beta}_1$	**0.00744**	**0.00462**	**0.00087**	**0.00040**	**0.00028**
	$\hat{\beta}_2$	**0.00111**	**0.00228**	**0.00067**	**0.00007**	**0.000014**
	$\hat{\beta}_3$	**0.00041**	**0.00142**	**0.00016**	**0.00001**	**0.000012**
MSE	$\hat{\beta}_1$	**0.03741**	**0.02686**	**0.00566**	**0.00246**	**0.00054**
	$\hat{\beta}_2$	**0.01419**	**0.00914**	**0.00131**	**0.00047**	**0.00013**
	$\hat{\beta}_3$	**0.00459**	**0.00300**	**0.00052**	**0.00021**	**0.00004**
SRMSE	$\hat{\beta}_1$	**0.32237**	**0.27316**	**0.12549**	**0.08279**	**0.03888**
	$\hat{\beta}_2$	**0.13239**	**0.10624**	**0.04024**	**0.02416**	**0.01295**
	$\hat{\beta}_3$	**0.13561**	**0.07836**	**0.01525**	**0.00770**	**0.00249**
Var	$\hat{\beta}_1$	**0.10753**	**0.09714**	**0.06559**	**0.05525**	**0.04368**
	$\hat{\beta}_2$	**0.06465**	**0.05834**	**0.03887**	**0.03216**	**0.02500**
	$\hat{\beta}_3$	**0.01614**	**0.01416**	**0.00914**	**0.00764**	**0.00592**

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.8) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 1.4$ and different value of β_3 at $n = 30$

		β_3				
		0.5	**0.7**	**1.5**	**1.9**	**2.6**
Mean	$\hat{\beta}_1$	**0.5978**	**0.5969**	**0.5984**	**0.5995**	**0.6008**
	$\hat{\beta}_2$	**1.3986**	**1.4021**	**1.4000**	**1.4000**	**1.4000**
	$\hat{\beta}_3$	**0.4999**	**0.6999**	**1.5000**	**1.9000**	**2.6000**
Bias	$\hat{\beta}_1$	**-0.00222**	**-0.00312**	**-0.00312**	**-0.00051**	**0.00078**
	$\hat{\beta}_2$	**-0.00136**	**0.00214**	**0.00214**	**0.00023**	**0.00020**
	$\hat{\beta}_3$	**-0.00012**	**-0.00005**	**-0.80005**	**0.00019**	**-0.00019**
SAB	$\hat{\beta}_1$	**0.00370**	**0.00521**	**0.00521**	**0.00085**	**0.00131**
	$\hat{\beta}_2$	**0.00097**	**0.00153**	**0.00153**	**0.00016**	**0.00014**
	$\hat{\beta}_3$	**0.00024**	**0.00007**	**0.53330**	**0.00010**	**0.00007**
MSE	$\hat{\beta}_1$	**0.02904**	**0.02072**	**0.00436**	**0.00176**	**0.00189**
	$\hat{\beta}_2$	**0.00943**	**0.00609**	**0.00094**	**0.00031**	**0.00009**
	$\hat{\beta}_3$	**0.00346**	**0.00222**	**0.00038**	**0.00014**	**0.00013**
SRMSE	$\hat{\beta}_1$	**0.28403**	**0.23993**	**0.11009**	**0.07002**	**0.07261**
	$\hat{\beta}_2$	**0.06938**	**0.05578**	**0.02195**	**0.01268**	**0.00702**
	$\hat{\beta}_3$	**0.11771**	**0.06732**	**0.01305**	**0.00634**	**0.00454**
Var	$\hat{\beta}_1$	**0.09834**	**0.08970**	**0.06203**	**0.05207**	**0.04143**
	$\hat{\beta}_2$	**0.05744**	**0.05113**	**0.03468**	**0.02959**	**0.02389**
	$\hat{\beta}_3$	**0.01476**	**0.01301**	**0.00861**	**0.00714**	**0.00568**

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.9) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 1.9$ and different value of β_3 at $n = 30$

		β_3				
		0.5	0.7	1.5	1.9	2.6
Mean	$\hat{\beta}_1$	0.5979	0.5993	0.6003	0.5997	0.5998
	$\hat{\beta}_2$	1.9000	1.9010	1.9000	1.9000	1.9000
	$\hat{\beta}_3$	0.4999	0.6993	1.5000	1.9000	2.6000
Bias	$\hat{\beta}_1$	-0.00201	-0.00069	0.00026	-0.00031	-0.00021
	$\hat{\beta}_2$	-0.00021	0.00108	0.00018	-0.00024	0.00031
	$\hat{\beta}_3$	-0.00012	-0.00069	-0.00012	0.00020	-0.00003
SAB	$\hat{\beta}_1$	0.00335	0.00115	0.00044	0.00053	0.00035
	$\hat{\beta}_2$	0.00011	0.00057	0.00009	0.00012	0.00016
	$\hat{\beta}_3$	0.00024	0.00098	0.00008	0.00011	0.00001
MSE	$\hat{\beta}_1$	0.02181	0.01502	0.00325	0.00195	0.00419
	$\hat{\beta}_2$	0.00633	0.00392	0.00067	0.00027	0.00027
	$\hat{\beta}_3$	0.00240	0.00157	0.00027	0.00023	0.00051
SRMSE	$\hat{\beta}_1$	0.24611	0.20427	0.09514	0.07377	0.10798
	$\hat{\beta}_2$	0.04187	0.03298	0.01363	0.00881	0.00875
	$\hat{\beta}_3$	0.09809	0.05671	0.01102	0.00802	0.00867
Var	$\hat{\beta}_1$	0.09115	0.08200	0.05815	0.00694	0.03987
	$\hat{\beta}_2$	0.05102	0.04531	0.03245	0.02838	0.02191
	$\hat{\beta}_3$	0.01352	0.01193	0.00808	0.05056	0.00543

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (1.10) The MLE of Poisson regression parameters when $\beta_1 = 0.6, \beta_2 = 2.2$ and different value of β_3 at $n = 30$

		β_3				
		0.5	**0.7**	**1.5**	**1.9**	**2.6**
Mean	$\hat\beta_1$	**0.5945**	**0.5957**	**0.6007**	**0.6013**	**0.5998**
	$\hat\beta_2$	**2.2010**	**2.2010**	**2.2000**	**2.2000**	**2.2000**
	$\hat\beta_3$	**0.5013**	**0.7009**	**1.5000**	**1.8996**	**2.6000**
Bias	$\hat\beta_1$	**-0.00547**	**-0.00423**	**0.00075**	**0.00128**	**-0.00021**
	$\hat\beta_2$	**0.00098**	**0.00136**	**-0.00039**	**-0.00015**	**0.00003**
	$\hat\beta_3$	**0.00130**	**0.00088**	**-0.00020**	**-0.00041**	**0.00007**
SAB	$\hat\beta_1$	**0.00913**	**0.00706**	**0.00126**	**0.00214**	**0.00035**
	$\hat\beta_2$	**0.00044**	**0.00061**	**0.00017**	**0.00007**	**0.00001**
	$\hat\beta_3$	**0.00260**	**0.00125**	**0.00013**	**0.00021**	**0.00002**
MSE	$\hat\beta_1$	**0.01832**	**0.01335**	**0.00291**	**0.00426**	**0.00045**
	$\hat\beta_2$	**0.00532**	**0.00331**	**0.00051**	**0.00022**	**0.00006**
	$\hat\beta_3$	**0.00203**	**0.00131**	**0.00024**	**0.00051**	**0.00002**
SRMSE	$\hat\beta_1$	**0.22563**	**0.19258**	**0.09001**	**0.10882**	**0.03562**
	$\hat\beta_2$	**0.03316**	**0.02617**	**0.01023**	**0.00687**	**0.00370**
	$\hat\beta_3$	**0.09025**	**0.05179**	**0.01039**	**0.01188**	**0.00202**
Var	$\hat\beta_1$	**0.08609**	**0.07877**	**0.05611**	**0.04900**	**0.03949**
	$\hat\beta_2$	**0.04815**	**0.04331**	**0.03087**	**0.02710**	**0.02177**
	$\hat\beta_3$	**0.01278**	**0.01148**	**0.00786**	**0.00678**	**0.00540**

The mean of the estimates for Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), scaled root mean square error (*SRMSE*), Variance (*Var* ($\hat\beta$'S))

A.2 Results of Zero-Truncated Poisson Regression Model

Table (2.1) The MLE of three parameters when $\beta_1 = 1.3,\ \beta_2 = 0.2,\ \beta_3 = 0.03$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	1.2907	-0.00934	0.00718	0.10143	0.31848	0.07295
	$\hat{\beta}_2$	0.1837	-0.01630	0.08153	0.07952	0.28199	0.06348
	$\hat{\beta}_3$	0.0281	-0.00182	0.06067	0.01630	0.12767	0.01224
30	$\hat{\beta}_1$	1.2953	-0.00472	0.00363	0.06048	0.24592	0.04505
	$\hat{\beta}_2$	0.1952	-0.00477	0.02388	0.04190	0.20469	0.03534
	$\hat{\beta}_3$	0.0272	-0.00270	0.09024	0.00972	0.09859	0.00755
50	$\hat{\beta}_1$	1.2970	-0.00295	0.00227	0.03265	0.18069	0.02571
	$\hat{\beta}_2$	0.1921	-0.00787	0.03935	0.02094	0.14471	0.01870
	$\hat{\beta}_3$	0.0293	-0.00062	0.02090	0.00537	0.07328	0.00428
100	$\hat{\beta}_1$	1.2972	-0.00277	0.00213	0.01539	0.12405	0.01222
	$\hat{\beta}_2$	0.1956	-0.00438	0.02191	0.00899	0.09481	0.00845
	$\hat{\beta}_3$	0.0307	0.00074	0.02482	0.00248	0.04979	0.00203

The average of the estimates of zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*RMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (2.2) The MLE of three parameters when $\beta_1 = 0.3,\ \beta_2 = 0.1,\ \beta_3 = 0.07$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	0.2537	-0.04634	0.15449	0.39426	0.62790	0.09705
	$\hat{\beta}_2$	0.0057	-0.09427	0.94273	0.37179	0.60974	0.08203
	$\hat{\beta}_3$	0.0752	0.00520	0.07439	0.06099	0.24696	0.01604
30	$\hat{\beta}_1$	0.2656	-0.03440	0.11466	0.22050	0.46957	0.05921
	$\hat{\beta}_2$	0.0635	-0.03646	0.36464	0.16567	0.40702	0.04570
	$\hat{\beta}_3$	0.0681	-0.00187	0.02678	0.03445	0.18560	0.00977
50	$\hat{\beta}_1$	0.2757	-0.02428	0.08096	0.11661	0.34148	0.03336
	$\hat{\beta}_2$	0.0738	-0.02612	0.26126	0.07859	0.28033	0.02363
	$\hat{\beta}_3$	0.0723	0.00239	0.03417	0.01836	0.13549	0.00544
100	$\hat{\beta}_1$	0.2893	-0.01072	0.03575	0.05516	0.23486	0.01587
	$\hat{\beta}_2$	0.0866	-0.01337	0.13373	0.03419	0.18490	0.01062
	$\hat{\beta}_3$	0.0702	0.00022	0.00326	0.00861	0.09279	0.00260

The average of the estimates of zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*RMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (2.3) The MLE of three parameters when $\beta_1 = 1.5,\ \beta_2 = 0.04,\ \beta_3 = 0.3$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	1.4918	-0.00818	0.00545	0.05273	0.22963	0.06810
	$\hat{\beta}_2$	0.0347	-0.00522	0.13057	0.03852	0.19626	0.05707
	$\hat{\beta}_3$	0.3007	0.00078	0.00261	0.00768	0.08763	0.01105
30	$\hat{\beta}_1$	1.4895	-0.01048	0.00699	0.03378	0.18379	0.04196
	$\hat{\beta}_2$	0.0353	-0.00466	0.11658	0.02112	0.14532	0.03232
	$\hat{\beta}_3$	0.3026	0.00262	0.00874	0.00477	0.06906	0.00682
50	$\hat{\beta}_1$	1.4988	-0.00112	0.00074	0.01767	0.13292	0.02390
	$\hat{\beta}_2$	0.0363	-0.00361	0.09034	0.01147	0.10709	0.01738
	$\hat{\beta}_3$	0.2993	-0.00061	0.00204	0.00246	0.04959	0.00386
100	$\hat{\beta}_1$	1.4978	-0.00218	0.00145	0.00912	0.09549	0.01142
	$\hat{\beta}_2$	0.0402	0.00021	0.00549	0.00499	0.07063	0.00786
	$\hat{\beta}_3$	0.2998	-0.00015	0.00053	0.00125	0.03535	0.00183

The average of the estimates of zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*RMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (2.4) The MLE of three parameters when $\beta_1 = 1.5,\ \beta_2 = 0.04,\ \beta_3 = 0.005$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	1.4858	-0.01416	0.00245	0.01494	0.12222	0.07225
	$\hat{\beta}_2$	0.0241	-0.01585	0.39639	0.07809	0.27944	0.06095
	$\hat{\beta}_3$	0.0050	0.00001	0.00944	0.09133	0.30221	0.01216
30	$\hat{\beta}_1$	1.4956	-0.00442	0.00294	0.05293	0.23006	0.04481
	$\hat{\beta}_2$	0.0259	-0.01406	0.35166	0.04394	0.20961	0.03548
	$\hat{\beta}_3$	0.0042	-0.00075	0.15188	0.00861	0.09279	0.00749
50	$\hat{\beta}_1$	1.4960	-0.00419	0.00279	0.02991	0.17294	0.02529
	$\hat{\beta}_2$	0.0292	-0.01074	0.26864	0.02190	0.14798	0.01875
	$\hat{\beta}_3$	0.0054	0.00043	0.08736	0.00487	0.06978	0.00422
100	$\hat{\beta}_1$	1.4980	-0.00205	0.00136	0.01339	0.11571	0.01210
	$\hat{\beta}_2$	0.0350	-0.00496	0.12415	0.00988	0.09939	0.00849
	$\hat{\beta}_3$	0.0054	0.00046	0.09243	0.00228	0.04774	0.00202

The average of the estimates of zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*RMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (2.5) The MLE of three parameters when $\beta_1 = 0.5, \beta_2 = 0.06, \beta_3 = 0.4$

n		*Mean*	*Bias*	*SAB*	*MSE*	*SRMSE*	*Var*
20	$\hat{\beta}_1$	**0.4644**	**-0.03554**	**0.07108**	**0.15063**	**0.38811**	**0.07893**
	$\hat{\beta}_2$	**0.0393**	**-0.02061**	**0.34362**	**0.09638**	**0.31045**	**0.06438**
	$\hat{\beta}_3$	**0.4093**	**0.00936**	**0.02341**	**0.01987**	**0.14096**	**0.01293**
30	$\hat{\beta}_1$	**0.4808**	**-0.01919**	**0.03838**	**0.08766**	**0.29607**	**0.04870**
	$\hat{\beta}_2$	**0.0404**	**-0.01959**	**0.32662**	**0.05250**	**0.22912**	**0.03666**
	$\hat{\beta}_3$	**0.4050**	**0.00500**	**0.01251**	**0.01157**	**0.10756**	**0.00788**
50	$\hat{\beta}_1$	**0.4896**	**-0.01039**	**0.02078**	**0.04919**	**0.22178**	**0.02761**
	$\hat{\beta}_2$	**0.0517**	**-0.00824**	**0.13738**	**0.02701**	**0.16434**	**0.01936**
	$\hat{\beta}_3$	**0.4021**	**0.00215**	**0.00539**	**0.00632**	**0.07949**	**0.00446**
100	$\hat{\beta}_1$	**0.4966**	**-0.00338**	**0.00677**	**0.02323**	**0.15241**	**0.01326**
	$\hat{\beta}_2$	**0.0555**	**-0.00444**	**0.07402**	**0.01109**	**0.10531**	**0.00875**
	$\hat{\beta}_3$	**0.4000**	**0.00007**	**0.00017**	**0.00290**	**0.05385**	**0.00213**

The average of the estimates of zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*RMSE*), Variance (*Var* ($\hat{\beta}'S$))

Looking the tables from the point view of β_3 by fixing β_1 and β_2 we get the following tables:

Table (2.6) The MLE of ZTP regression parameters when $\beta_1 = 0.5, \beta_2 = 0.04$ and different value of β_3 at $n = 30$

		β_3				
		0.5	**0.7**	**1.5**	**1.9**	**2.6**
Mean	$\hat{\beta}_1$	**0.4611**	**0.4644**	**0.4667**	**0.4869**	**0.4861**
	$\hat{\beta}_2$	**-0.0063**	**-0.0045**	**-0.0133**	**0.0252**	**0.0253**
	$\hat{\beta}_3$	**0.0084**	**0.0327**	**0.0768**	**0.3004**	**0.4017**
Bias	$\hat{\beta}_1$	**-0.03889**	**-0.03562**	**-0.03331**	**-0.01305**	**-0.01388**
	$\hat{\beta}_2$	**-0.04634**	**-0.04458**	**-0.05339**	**-0.01476**	**-0.01462**
	$\hat{\beta}_3$	**0.00346**	**0.00278**	**0.00683**	**0.00041**	**0.00178**
SAB	$\hat{\beta}_1$	**0.07778**	**0.07125**	**0.06661**	**0.02610**	**0.02777**
	$\hat{\beta}_2$	**1.15860**	**1.11458**	**1.33481**	**0.36912**	**0.36571**
	$\hat{\beta}_3$	**0.69381**	**0.09276**	**0.09767**	**0.00139**	**0.00446**
MSE	$\hat{\beta}_1$	**0.19944**	**0.19062**	**0.17203**	**0.10795**	**0.08812**
	$\hat{\beta}_2$	**0.17129**	**0.16233**	**0.14354**	**0.06851**	**0.05223**
	$\hat{\beta}_3$	**0.03244**	**0.03029**	**0.02656**	**0.01446**	**0.01144**
SRMSE	$\hat{\beta}_1$	**0.07409**	**0.43660**	**0.41476**	**0.32856**	**0.29686**
	$\hat{\beta}_2$	**0.44659**	**0.40290**	**0.37887**	**0.26174**	**0.22855**
	$\hat{\beta}_3$	**0.41388**	**0.17404**	**0.16297**	**0.12028**	**0.10699**
Var	$\hat{\beta}_1$	**0.05720**	**0.05642**	**0.05492**	**0.05006**	**0.04897**
	$\hat{\beta}_2$	**0.04512**	**0.04397**	**0.04305**	**0.03781**	**0.03653**
	$\hat{\beta}_3$	**0.00963**	**0.00931**	**0.00910**	**0.00811**	**0.00794**

The mean of the estimates for zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*SRMSE*), Var ($\hat{\beta}'S$))

Table (2.7) The MLE of ZTP regression parameters when $\beta_1 = 0.5,\ \beta_2 = 0.06$ and different value of β_3 at $n = 30$

		β_3				
		0.5	0.7	1.5	1.9	2.6
Mean	$\hat\beta_1$	0.4748	0.4683	0.4729	0.4824	0.4742
	$\hat\beta_2$	0.0090	0.0048	0.0169	0.0365	0.0471
	$\hat\beta_3$	0.0044	0.0363	0.0717	0.3038	0.4057
Bias	$\hat\beta_1$	-0.02519	-0.03168	-0.02714	-0.01762	-0.02583
	$\hat\beta_2$	-0.05095	-0.05517	-0.04309	-0.02346	-0.01285
	$\hat\beta_3$	-0.00053	0.00632	0.00178	0.00383	0.00571
SAB	$\hat\beta_1$	0.05039	0.06336	0.05429	0.03524	0.05167
	$\hat\beta_2$	0.84923	0.91951	0.71820	0.39100	0.21417
	$\hat\beta_3$	0.10625	0.21071	0.02549	0.01278	0.01428
MSE	$\hat\beta_1$	0.19418	0.18927	0.17545	0.10569	0.08764
	$\hat\beta_2$	0.17816	0.14949	0.14020	0.06821	0.05378
	$\hat\beta_3$	0.03176	0.02978	0.02693	0.01469	0.01095
SRMSE	$\hat\beta_1$	0.44066	0.43505	0.41887	0.32510	0.29605
	$\hat\beta_2$	0.42209	0.38664	0.37443	0.26117	0.23191
	$\hat\beta_3$	0.17821	0.17259	0.16411	0.12121	0.10464
Var	$\hat\beta_1$	0.05668	0.05595	0.05444	0.05008	0.04921
	$\hat\beta_2$	0.04522	0.04325	0.04212	0.03723	0.03657
	$\hat\beta_3$	0.00951	0.00930	0.00901	0.00815	0.00793

The mean of the estimates for zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*SRMSE*), Variance (*Var* ($\hat\beta'S$))

Table (2.8) Table (2.8) The MLE of ZTP regression parameters when $\beta_1 = 0.5,\ \beta_2 = 0.1$ and different value of β_3 at $n = 30$

		β_3				
		0.5	0.7	1.5	1.9	2.6
Mean	$\hat{\beta}_1$	**0.005**	**0.03**	**0.07**	**0.3**	**0.4**
	$\hat{\beta}_2$	**0.4729**	**0.4718**	**0.4693**	**0.4903**	**0.4817**
	$\hat{\beta}_3$	**0.0533**	**0.0618**	**0.0560**	**0.0795**	**0.0837**
Bias	$\hat{\beta}_1$	**0.0040**	**0.0295**	**0.0741**	**0.3005**	**0.4053**
	$\hat{\beta}_2$	**-0.02710**	**-0.02822**	**-0.03065**	**-0.00965**	**-0.01825**
	$\hat{\beta}_3$	**-0.04666**	**-0.03818**	**-0.04398**	**-0.02041**	**-0.01623**
SAB	$\hat{\beta}_1$	**-0.00098**	**-0.00042**	**0.00413**	**0.00047**	**0.00534**
	$\hat{\beta}_2$	**0.05421**	**0.05644**	**0.06131**	**0.01931**	**0.03651**
	$\hat{\beta}_3$	**0.46668**	**0.38180**	**0.43982**	**0.20413**	**0.16233**
MSE	$\hat{\beta}_1$	**0.19779**	**0.01431**	**0.05903**	**0.00156**	**0.01336**
	$\hat{\beta}_2$	**0.18437**	**0.19193**	**0.17163**	**0.10242**	**0.08637**
	$\hat{\beta}_3$	**0.15885**	**0.14645**	**0.13183**	**0.06527**	**0.05139**
SRMSE	$\hat{\beta}_1$	**0.03013**	**0.03009**	**0.02675**	**0.01395**	**0.01087**
	$\hat{\beta}_2$	**0.42938**	**0.43810**	**0.41429**	**0.32004**	**0.29389**
	$\hat{\beta}_3$	**0.39857**	**0.38269**	**0.36308**	**0.25548**	**0.22669**
Var	$\hat{\beta}_1$	**0.17360**	**0.17348**	**0.16358**	**0.11811**	**0.10429**
	$\hat{\beta}_2$	**0.05573**	**0.05531**	**0.05443**	**0.04974**	**0.04846**
	$\hat{\beta}_3$	**0.04381**	**0.04275**	**0.04173**	**0.03718**	**0.03598**

The mean of the estimates for zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*SRMSE*), Variance (*Var* ($\hat{\beta}'S$))

Table (2.9) The MLE of ZTP regression parameters when $\beta_1 = 0.5, \beta_2 = 0.2$ and different value of β_3 at $n = 30$

		β_3				
		0.5	**0.7**	**1.5**	**1.9**	**2.6**
Mean	$\hat{\beta}_1$	**0.4767**	**0.4823**	**0.4786**	**0.4907**	**0.4805**
	$\hat{\beta}_2$	**0.1665**	**0.1590**	**0.1772**	**0.1902**	**0.1921**
	$\hat{\beta}_3$	**0.0035**	**0.0280**	**0.0684**	**0.2977**	**0.4039**
Bias	$\hat{\beta}_1$	**-0.02331**	**-0.01767**	**-0.02135**	**-0.00928**	**-0.01948**
	$\hat{\beta}_2$	**-0.03343**	**-0.04100**	**-0.02277**	**-0.00975**	**-0.00787**
	$\hat{\beta}_3$	**-0.00148**	**-0.00198**	**-0.00155**	**-0.00228**	**0.00390**
SAB	$\hat{\beta}_1$	**0.04662**	**0.03534**	**0.04271**	**0.01857**	**0.03896**
	$\hat{\beta}_2$	**0.16718**	**0.20501**	**0.11386**	**0.04875**	**0.03936**
	$\hat{\beta}_3$	**0.29677**	**0.06624**	**0.02224**	**0.00760**	**0.00975**
MSE	$\hat{\beta}_1$	**0.18201**	**0.16728**	**0.15192**	**0.09776**	**0.08319**
	$\hat{\beta}_2$	**0.12926**	**0.12486**	**0.10721**	**0.05707**	**0.04336**
	$\hat{\beta}_3$	**0.03049**	**0.02750**	**0.02426**	**0.01329**	**0.01059**
SRMSE	$\hat{\beta}_1$	**0.42663**	**0.40900**	**0.38977**	**0.31267**	**0.28844**
	$\hat{\beta}_2$	**0.35953**	**0.35336**	**0.32744**	**0.23891**	**0.20825**
	$\hat{\beta}_3$	**0.17461**	**0.16584**	**0.15576**	**0.11529**	**0.10292**
Var	$\hat{\beta}_1$	**0.05536**	**0.05462**	**0.05346**	**0.04947**	**0.04779**
	$\hat{\beta}_2$	**0.04201**	**0.04147**	**0.04053**	**0.03657**	**0.03552**
	$\hat{\beta}_3$	**0.00921**	**0.00908**	**0.00882**	**0.00801**	**0.00774**

The mean of the estimates for zero truncated Poisson regression model, Bias, the scaled absolute bias (*SAB*), mean square error (*MSE*), root mean square error (*SRMSE*), Var ($\hat{\beta}'S$)

A.3 Results of Applications

Table (3.1) Parameter Estimation for the Ship Damage

Coefficients	Estimate	Std. Error	z value	Pr(>|z|)
(Intercept)	**-6.40590**	**0.21744**	**-29.460**	**0.0000**
Type B	**-0.54334**	**0.17759**	**-3.060**	**0.0022**
Type C	**-0.68740**	**0.32904**	**-2.089**	**0.0367**
Type D	**-0.07596**	**0.29058**	**-0.261**	**0.7938**
Type E	**0.32558**	**0.23588**	**1.380**	**0.1675**
Construction 1965-69	**0.69714**	**0.14964**	**4.659**	**0.0000**
Construction 1970-74	**0.81843**	**0.16977**	**4.821**	**0.0000**
Construction 1975-79	**0.45343**	**0.23317**	**1.945**	**0.0518**
Operation 1975-79	**0.38447**	**0.11827**	**3.251**	**0.0012**

Model: (intercept), type, construction, operation, offset = ***Log*** **(months)**

Table (3.2) The Summarize of the hospital stay Variable

variable	Obs.	Mean	Std. Dev.	Min	Max
stay	1493	9.72873	8.132908	1.000	74.000

Table (3.3) The frequency and percentage of age Variable

Age Group	1	2	3	4	5	6	7	8	9	Total
percentage	0.4	4.02	10.92	19.94	21.23	21.90	12.73	6.23	3.08	100.0
Freq.	6	60	163	291	317	327	190	93	46	1493

Table (3.4) The frequency and percentage of hmo Variable

hmo	Freq.	percentage
0	1,254	83.99
1	239	16.01
Total	1493	100.00

Table (3.5) The frequency and percentage of died Variable

died	Freq.	percent
0	981	65.71
1	512	34.29
Total	1493	100.00

Table (3.6) Parameter Estimation for the Hospital Stay

Coefficients	Estimate	Std. Error	z value	$Pr(>\|z\|)$	[95% Conf. Interval]
Intercept	2.435	0.02733	89.1181	0.00	- 0.0243099 - 0.0045742
age	- 0.014	0.00503	- 2.8685	0.004	- 0.1824365 - 0.0893701
hmo1	- 0.135	0.02374	- 5.7242	0.000	- 0.239781 - 0.167760
died1	- 0.203	0.01837	- 11.0909	0.000	2.382238 2.489379

Response Variable: length of hospital stay
Model: (intercept), age, hmo, died